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SCHRÖDINGER OPERATORS WITH PERIODIC POTENTIALS AND CONSTANT MAGNETIC FIELDS WITH INTEGER FLUX

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1. Introduction and main results

In this paper we study the spectral property of the 2-dimensional Schrödinger operators with periodic potentials and constant magnetic fields :

$$H(\lambda)=(D_{x_1}+bx_2)^2+(D_{x_2}-bx_1)^2+\lambda^2V(x) \quad \text{in} \quad L^2(\mathbf{R}^2), \tag{1.1}$$

where $D_{x_j}=-i\partial/\partial x_j$ $(j=1,2)$, $b\in\mathbf{R}$, $V(x)$ is a real-valued function on $\mathbf{R}^2$, and λ is a positive parameter. The corresponding magnetic field is defined by the 2-form $B=-2bdx_1\wedge dx_2$. We assume that $V(x)$ satisfies the following conditions :
$(H.1)$ $V(x)\in C^\infty(\mathbf{R}^2;\mathbf{R})$.
$(H.2)$ $V(x+\gamma)=V(x)$ on $\mathbf{R}^2$ for any $\gamma\in\Gamma=2\pi\mathbf{Z}\oplus 2\pi\mathbf{Z}$.
$(H.3)$ $V(x)\geq 0$ on $\mathbf{R}^2$.
$(H.4)$ $V(x)=0$ if and only if $x\in\Gamma$.
$(H.5)$ $V''(0)=2\begin{pmatrix}\mu_1 & 0\\ 0 & \mu_2\end{pmatrix}$, $\mu_1,\mu_2>0$.

The spectral property of $H(\lambda)$ depends largely on number theoretical properties of B and Γ. In this paper we assume that
$(H.6)$ $b\in(1/4\pi)\mathbf{Z}$.

Under the assumption $(H.6)$, the spectrum of $H(\lambda)$ has a band structure. Our main purpose is to study the asymptotic behavior of the spectrum of $H(\lambda)$ when λ tends to infinity. When the magnetic field is absent (i.e. b=0), B. Simon [5] and A. Outassout [4] proved that the width of the lowest band (*the ground state band*) decreases in exponential order when $\lambda\to\infty$. Simon used the theory of Brownian motion in the proof, while Outassout employed the W.K.B. type analysis developed by B. Helffer-J. Sjöstrand [2]. In this paper we prove similar estimates in the presence of the magnetic field B.

For x, $y\in\mathbf{R}^2$, we denote by $d_V(x,y)$ the Agmon distance associated with $V(x)$

(see §3), and we set

$$s_0 = \min_{\gamma \in \Gamma \setminus \{0\}} d_V(0, \gamma).$$

The hypotheses $(H.3)$ and $(H.4)$ imply that $s_0 > 0$. Then we have the following theorem.

Theorem A. *Assume* $(H.1) \sim (H.6)$. *Let* $L(\lambda)$ *be the width of the ground state band. Then, for any* $\eta > 0$, *there exists a constant* $C_\eta > 0$ *such that*

$$L(\lambda) \leq C_\eta e^{-(s_0 - 2\eta)\lambda} \quad as \quad \lambda \to \infty. \tag{1.2}$$

We improve the estimate (2) under an additional geometrical assumption. Let

$$\Lambda = \{\gamma \in \Gamma; d_V(0, \gamma) = s_0\}.$$

For $x_0 \in \mathbf{R}^2$ and $r > 0$, we set

$$B_V(x_0, r) = \{x \in \mathbf{R}^2; d_V(x_0, x) < r\}.$$

For each $\gamma \in \Lambda$, we assume the following.

$(H.7)$ There is a unique geodesic κ of length s_0 joining 0 and γ.

$(H.8)$ Let $x_0 \in \kappa \cap B_V(0, s_0) \cap B_V(\gamma, s_0)$ and let $\Gamma_0 \subset\subset B_V(0, s_0) \cap B_V(\gamma, s_0)$ be any smooth curve such that $\overline{\Gamma_0} \cap \kappa = \{x_0\}$ and Γ_0 intersects κ transversally at x_0. Then there exists a constant $C = C(x_0, \Gamma_0) > 0$ such that

$$d_V(x, 0) + d_V(x, \gamma) \geq s_0 + C d_V(x, x_0)^2 \quad \text{for any} \quad x \in \Gamma_0.$$

Theorem B. *Under the hypotheses* $(H.1) \sim (H.8)$, *the width of the ground state band of* $H(\lambda)$ *is*

$$(b_0 \lambda^{3/2} + O(\lambda^{1/2})) e^{-s_0 \lambda} \quad as \quad \lambda \to \infty,$$

where $b_0 > 0$ *is a constant depending only on* $V(x)$ *and* B.

We owe the basic ideas of the proof of these theorems to the work of Helffer-Sjöstrand [2] on the tunneling effect of Schrödinger operators and to that of Outassourt [4] which applied the technique of Helffer-Sjöstrand to periodic potentials and the tight-binding approximation. The assumption $(H.6)$ allows us to generalize this idea to the magnetic Schrödinger operators with small modification. In §2, we introduce a differential operator on a torus, and estimate its eigenvalues by using harmonic approximation. In §3, we prove Theorem A by slightly deforming the periodic potential and comparing the first eigenvalue of the resulting Schrödinger operator with the one introduced in §2. In order to prove Theorem B, we shall introduce in §4 a W.K.B. solution of the magnetic Schrödinger operator and approximate the eigenfunctions of the reference problem.

2. Preliminaries

First we introduce various function spaces and magnetic translations which reduce our problem to that of a differential operators on a torus. For details see Sjöstrand [6] p. 247.

Let E be the fundamental domain of $\Gamma = 2\pi\mathbf{Z} \oplus 2\pi\mathbf{Z}$, Γ^* be the dual lattice of Γ, and E^* be the fundamental domain of Γ^*. Namely,

$$E = [0, 2\pi) \times [0, 2\pi),$$
$$\Gamma^* = \{\gamma^* \in \mathbf{R}^2; \gamma \cdot \gamma^* \in 2\pi\mathbf{Z}, {}^\forall\gamma \in \Gamma\} = \mathbf{Z} \oplus \mathbf{Z},$$
$$E^* = [0, 1) \times [0, 1).$$

Let $H^2_B(\mathbf{R}^2) = \{u \in L^2(\mathbf{R}^2); T_i u, T_i T_j u \in L^2(\mathbf{R}^2), {}^\forall i, j \in \{1, 2\}\}$, where

$$T_1 = D_{x_1} + bx_2, T_2 = D_{x_2} - bx_1.$$

We define the inner product of $H^2_B(\mathbf{R}^2)$ by

$$(u, v)_{H^2_B(\mathbf{R}^2)} = (u, v)_{L^2(\mathbf{R}^2)} + \sum_{i=1}^{2}(T_i u, T_i v)_{L^2(\mathbf{R}^2)} + \sum_{i,j=1}^{2}(T_i T_j u, T_i T_j v)_{L^2(\mathbf{R}^2)}.$$

Then, $H(\lambda)$ is self-adjoint with domain $H^2_B(\mathbf{R}^2)$.

For $\gamma = (\gamma_1, \gamma_2) \in \Gamma$ and $u \in L^2_{loc}(\mathbf{R}^2)$, we define the magnetic translation $\mathbf{T}^B_\gamma$ by

$$(\mathbf{T}^B_\gamma u)(x) = e^{ib\gamma_1\gamma_2} e^{-ib(x_1\gamma_2 - x_2\gamma_1)} u(x - \gamma).$$

$\{\mathbf{T}^B_\gamma\}_{\gamma\in\Gamma}$ is an Abelian group, and each $\mathbf{T}^B_\gamma$ commutes with the differential operator $H(\lambda)$ defined by (1).

For $u \in \mathcal{S}(\mathbf{R}^2)$ and $\theta \in E^*$, we define

$$(\mathcal{U}u)(x; \theta) = \sum_{\gamma\in\Gamma} e^{i\gamma\cdot\theta}(\mathbf{T}^B_\gamma u)(x), \quad x \in \mathbf{R}^2.$$

For $\theta \in E^*$, we define

$$\mathcal{H}_{B,\theta} = \{v \in L^2_{loc}(\mathbf{R}^2); \mathbf{T}^B_\gamma v = e^{-i\gamma\cdot\theta} v \quad \text{a.e. in} \quad \mathbf{R}^2, {}^\forall\gamma \in \Gamma\}$$

equipped with the inner product $(u,v)_{\mathcal{H}_{B,\theta}} = \int_E u(x)\overline{v(x)}dx$.

Let $\mathcal{H} = \int_{E^*}^{\oplus} \mathcal{H}_{B,\theta} d\theta$ equipped with the inner product

$$(u,v)_{\mathcal{H}} = (\operatorname{vol} E^*)^{-1} \int_{E^*} d\theta \int_E u(x,\theta)\overline{v(x,\theta)}dx.$$

For $\theta \in E^*$, we define

$$H(\lambda;\theta) = (D_{x_1} + bx_2)^2 + (D_{x_2} - bx_1)^2 + \lambda^2 V(x) \quad \text{in} \quad \mathcal{H}_{B,\theta} \tag{2.1}$$

with domain

$$\mathcal{H}^2_{B,\theta} = \{v \in \mathcal{H}_{B,\theta}; T_i v, T_i T_j v \in \mathcal{H}_{B,\theta}, {}^{\forall} i,j \in \{1,2\}\}.$$

We then have the following fundamental proposition (cf. [6] p. 255).

Proposition 2.1. *$\mathcal{U}$ is uniquely extended to a unitary operator from $L^2(\mathbf{R}^2)$ to $\mathcal{H}$, and the following equality holds:*

$$\mathcal{U} H(\lambda) \mathcal{U}^{-1} = \int_{E^*}^{\oplus} H(\lambda;\theta) d\theta. \tag{2.2}$$

Because $H(\lambda;\theta)$ has a compact resolvent, the spectum of $H(\lambda;\theta)$ is discrete. We denote by $\mathcal{E}_j(\lambda;\theta)$ the j-th eigenvalue of $H(\lambda;\theta)$ counted with multiplicity. By the min-max principle, $\mathcal{E}_j(\lambda;\theta)$ is a continuous function of $\theta \in E^*$. So, we have

$$\sigma(H(\lambda)) = \bigcup_{j=1}^{\infty} \mathcal{E}_j(\lambda;E^*), \quad \text{where} \quad \mathcal{E}_j(\lambda;E^*) = \{\mathcal{E}_j(\lambda;\theta); \theta \in E^*\}. \tag{2.3}$$

$\mathcal{E}_j(\lambda;E^*)$ is either a closed interval or a one-point set. We call $\mathcal{E}_j(\lambda;E^*)$ the j-th band, and $\mathcal{E}_1(\lambda;E^*)$ the ground state band.

Before going into the precise analysis of the ground state band, we first get the asymptotic expansion of first order of each eigenvalue. Let $\mathbf{N}$ be the set of non-negative integers and $\mathbf{N}_+$ the set of positive integers. Let

$$\Lambda_0 = \{(2j+1)\sqrt{\mu_1} + (2k+1)\sqrt{\mu_2}; j,k \in \mathbf{N}\}$$

(where μ_1, μ_2 are defined in $(H.5)$) and let v_n be the n-th smallest element of Λ_0 counted with multiplicity. Then we have the following theorem.

Theorem 2.2. *For each $n \in \mathbf{N}_+$, we have*

$$\mathcal{E}_n(\lambda;\theta) = v_n \lambda + o(\lambda) \quad (\lambda \to \infty), \tag{2.4}$$

where the error term is uniform with respect to $\theta \in E^*$.

Proof. The proof is done along the line of Theorem 1 of Simon [5]. We prove the following two inequalities.

$$\mathcal{E}_n(\lambda;\theta) \geq v_n\lambda - O(\lambda^{4/5}) \quad (\lambda \to \infty), \tag{2.5}$$

$$\mathcal{E}_n(\lambda;\theta) \leq v_n\lambda + O(\lambda^{1/2}) \quad (\lambda \to \infty), \tag{2.6}$$

where the error term is uniform with respect to $\theta \in E^*$.

As was done in Simon [5], (5) is proved by using the I.M.S. localization formula and the min-max principle. The presense of the magnetic fields requires no essential change.

Next we prove (6). To show this, we use the harmonic approximation (cf. [5]). $(H.5)$ implies that

$$V(x) = \mu_1 {x_1}^2 + \mu_2 {x_2}^2 + O(|x|^3) \quad (|x| \to 0).$$

Let us introduce the following approximate operator :

$$H_0(\lambda) = (D_{x_1} + bx_2)^2 + (D_{x_2} - bx_1)^2 + \lambda^2(\mu_1 {x_1}^2 + \mu_2 {x_2}^2) \quad \text{in} \quad L^2(\mathbf{R}^2). \tag{2.7}$$

We use the eigenvalues and eigenfunctions of $H_0(\lambda)$ to approximate $\mathcal{E}_j(\lambda;\theta)$. By the symplectic invariance of Weyl operators, $H_0(\lambda)$ is unitarily equivalent to the following harmonic oscillator (see Appendix) :

$$-\Delta + m_1(\lambda){x_1}^2 + m_2(\lambda){x_2}^2 \quad \text{in} \quad L^2(\mathbf{R}^2), \tag{2.8}$$

where $m_1(\lambda)$ and $m_2(\lambda)$ are the roots of

$$t^2 - ((\mu_1 + \mu_2)\lambda^2 + 4b^2)t + \mu_1\mu_2\lambda^4 = 0, m_1(\lambda) < m_2(\lambda).$$

Therefore, the eigenvalues of $H_0(\lambda)$ are

$$\widetilde{\mathcal{E}}_{j,k}(\lambda) = (2j+1)\sqrt{m_1(\lambda)} + (2k+1)\sqrt{m_2(\lambda)}, \quad j,k \in \mathbf{N}.$$

Let

$$v_{j,k} = (2j+1)\sqrt{\min(\mu_1,\mu_2)} + (2k+1)\sqrt{\max(\mu_1,\mu_2)}.$$

In Appendix, we shall show that

$$\widetilde{\mathcal{E}}_{j,k}(\lambda) = v_{j,k}\lambda + O(1) \quad (\lambda \to \infty). \tag{2.9}$$

Let $\{\psi_{j,k}\}_{j,k\in\mathbf{N}}$ be the complete orthonormal system of $L^2(\mathbf{R}^2)$, where $\psi_{j,k}(\lambda;x)$ is the eigenfunction of $H_0(\lambda)$ associated with the eigenvalue $(2j+1)\sqrt{m_1(\lambda)} + (2k+$

$1)\sqrt{m_2(\lambda)}$. Each $\psi_{j,k}$ can be computed explicitly, and the following estimate holds (see Appendix) :

$$|\psi_{j,k}(\lambda;x)| \leq C_{j,k}\lambda^{1/2}\exp(-c\lambda|x|^2), \tag{2.10}$$

where $C_{j,k} > 0$ and $c > 0$ are constants independent of $\lambda > 1$. We can choose $\{(j_n, k_n)\}_{n\geq 1}$ $(j_n, k_n \in \mathbf{N})$ such that

$$v_n = v_{j_n,k_n}(n = 1, 2, \cdots), (j_n, k_n) \neq (j_m, k_m) \quad \text{if} \quad n \neq m.$$

Let $\psi_n = \psi_{j_n,k_n}$, $C_n = C_{j_n,k_n}$, $\widetilde{\mathcal{E}}_n(\lambda) = \widetilde{\mathcal{E}}_{j_n,k_n}(\lambda)$ $(n = 1, 2, \cdots)$, and

$$\varphi_n(\lambda;x;\theta) = \sum_{\gamma\in\Gamma} e^{i\gamma\cdot\theta}(\mathbf{T}^B_\gamma\psi_n)(\lambda;x) \ (\theta \in E^*). \tag{2.11}$$

We prove the following estimates :

$$(\varphi_n(\lambda;x;\theta), \varphi_m(\lambda;x;\theta))_{\mathcal{H}_{B,\theta}} = \delta_{nm} + O(e^{-k\lambda}) \ (\lambda \to \infty), \tag{2.12}$$

$$(H(\lambda;\theta)\varphi_n(\lambda;x;\theta), \varphi_m(\lambda;x;\theta))_{\mathcal{H}_{B,\theta}} = v_n\lambda\delta_{nm} + O(\lambda^{1/2}) \ (\lambda \to \infty), \tag{2.13}$$

where $k > 0$ is a constant independent of λ and each error term is uniform with respect to $\theta \in E^*$. The inequality (6) then follows from (12) and (13) by Schmidt's orthogonalization process and the Rayleigh-Ritz Principle.

First we show (12). For $\gamma = (\gamma_1, \gamma_2) \in \Gamma$, we set

$$\theta(\gamma) = e^{ib\gamma_1\gamma_2}. \tag{2.14}$$

Then $(H.6)$ implies that

$$\theta(\gamma) \in \{1, -1\}. \tag{2.15}$$

From (11) and (15), we have

$$\begin{aligned} &(\varphi_n(\lambda;x;\theta), \varphi_m(\lambda;x;\theta))_{\mathcal{H}_{B,\theta}} \\ &= \int_E \sum_{\gamma\in\Gamma} \psi_n(\lambda;x-\gamma)\overline{\psi_m(\lambda;x-\gamma)}dx \\ &+ \sum_{\gamma\in\Gamma} \sum_{\substack{\gamma'\in\Gamma \\ \gamma'\neq\gamma}} \int_E e^{i(\gamma-\gamma')\cdot\theta}\theta(\gamma)\theta(\gamma') \\ &\times e^{-ib(x_1\gamma_2 - x_1\gamma_1)}e^{ib(x_1\gamma'_2 - x_2\gamma'_1)}\psi_n(\lambda;x-\gamma)\overline{\psi_m(\lambda;x-\gamma')}dx \end{aligned} \tag{2.16}$$

where $\gamma = (\gamma_1, \gamma_2)$, $\gamma' = (\gamma'_1, \gamma'_2)$.

The first term of the right-hand side equals

$$\int_{\mathbf{R}^2} \psi_n(\lambda; x)\overline{\psi_m(\lambda; x)}dx = \delta_{nm}.$$

We denote the second term by $R_{n,m}(\lambda)$. Then (10) implies that

$$|R_{n,m}(\lambda)| \le C_n C_m \lambda \sum_{\gamma\in\Gamma} \sum_{\substack{\gamma'\in\Gamma \\ \gamma'\neq\gamma}} \int_E \exp(-c\lambda(|x-\gamma|^2 + |x-\gamma'|^2))dx.$$

A simple calculation shows that

$$|x-\gamma|^2 + |x-\gamma'|^2 \ge 2\pi^2 \quad \text{in} \quad \mathbf{R}^2 \quad \text{for} \quad {}^\forall\gamma, \gamma' \in \Gamma, \gamma \neq \gamma'.$$

Let $k = \pi^2 c$ (> 0). Using the above inequality, we have

(2.17)

$$\begin{aligned}
&|R_{n,m}(\lambda)| \\
&\le C_n C_m \lambda e^{-k\lambda} \sum_{\gamma\in\Gamma} \sum_{\substack{\gamma'\in\Gamma \\ \gamma'\neq\gamma}} \int_E \exp\left(-\frac{1}{2}c\lambda(|x-\gamma|^2 + |x-\gamma'|^2)\right) dx \\
&\le C_n C_m \lambda e^{-k\lambda} \left(\sum_{\gamma\in\Gamma} \exp\left(-\frac{1}{2}c\lambda \min_{x\in E}|x-\gamma|^2\right)\right) \sum_{\gamma'\in\Gamma} \int_E \exp\left(-\frac{1}{2}c\lambda|x-\gamma'|^2\right) dx \\
&= C_n C_m \lambda e^{-k\lambda} \left(\sum_{\gamma\in\Gamma} \exp\left(-\frac{1}{2}c\lambda \min_{x\in E}|x-\gamma|^2\right)\right) \int_{\mathbf{R}^2} \exp\left(-\frac{1}{2}c\lambda|x|^2\right) dx \\
&= C_n C_m e^{-k\lambda} \left(\int_{\mathbf{R}^2} \exp\left(-\frac{1}{2}c|x|^2\right) dx\right) \sum_{\gamma\in\Gamma} \exp\left(-\frac{1}{2}c\lambda \min_{x\in E}|x-\gamma|^2\right),
\end{aligned}$$

where we have used the scale change $\sqrt{\lambda}x \to x$ in the third line.

For $\gamma \in \Gamma$, $|\gamma| \ge 4\sqrt{2}\pi$, we have

$$\min_{x\in E}|x-\gamma|^2 \ge \frac{1}{4}|\gamma|^2.$$

So, there exists a constant $C' > 0$ independent of $\lambda > 1$ such that

$$\sum_{\gamma\in\Gamma} \exp\left(-\frac{1}{2}c\lambda \min_{x\in E}|x-\gamma|^2\right) \le C' \quad \text{for any} \quad \lambda \ge 1.$$

Therefore, we get (12).

Next we show (13). We denote by $\widetilde{H}^{\circ}(\lambda)$ and $H^{\circ}(\lambda)$ the formal differential operators

$$(D_{x_1}+bx_2)^2+(D_{x_2}-bx_1)^2+\lambda^2(\mu_1{x_1}^2+\mu_2{x_2}^2)$$

and

$$(D_{x_1}+bx_2)^2+(D_{x_2}-bx_1)^2+\lambda^2V(x)$$

respectively. Let $E_0=[-\pi,\pi)\times[-\pi,\pi)$. Because each $\mathbf{T}^B_\gamma(\gamma\in\Gamma)$ commutes with $H^{\circ}(\lambda)$, we have by using (11)

$$\begin{aligned}
(2.18)\qquad & (H(\lambda;\theta)\varphi_n(\lambda;x;\theta),\varphi_m(\lambda;x;\theta))_{\mathcal{H}_{B,\theta}}\\
&=\int_{E_0}(H^{\circ}(\lambda)\varphi_n(\lambda;x;\theta))\overline{\varphi_m(\lambda;x;\theta)}dx\\
&=\int_{E_0}\sum_{\gamma\in\Gamma}e^{i\gamma\cdot\theta}(H^{\circ}(\lambda)\mathbf{T}^B_\gamma\psi_n)(\lambda;x)\overline{\varphi_m(\lambda;x;\theta)}dx\\
&=\int_{E_0}\sum_{\gamma\in\Gamma}e^{i\gamma\cdot\theta}(\mathbf{T}^B_\gamma H^{\circ}(\lambda)\psi_n)(\lambda;x)\overline{\varphi_m(\lambda;x;\theta)}dx\\
(2.19)\qquad &=\int_{E_0}\sum_{\gamma\in\Gamma}e^{i\gamma\cdot\theta}(\mathbf{T}^B_\gamma(H^{\circ}(\lambda)-\widetilde{H}^{\circ}(\lambda))\psi_n)(\lambda;x)\overline{\varphi_m(\lambda;x;\theta)}dx\\
&\quad+\int_{E_0}\sum_{\gamma\in\Gamma}e^{i\gamma\cdot\theta}(\mathbf{T}^B_\gamma\widetilde{H}^{\circ}(\lambda)\psi_n)(\lambda;x)\overline{\varphi_m(\lambda;x;\theta)}dx.
\end{aligned}$$

Let us recall that

$$\widetilde{H}^{\circ}(\lambda)\psi_n=\widetilde{\mathcal{E}}_n(\lambda)\psi_n.$$

This together with (9) implies that

$$\widetilde{H}^{\circ}(\lambda)\psi_n=(v_n\lambda+O(1))\psi_n\ (\lambda\to\infty).$$

So, the second term of (19) equals

$$\begin{aligned}
&(v_n\lambda+O(1))\int_{E_0}\sum_{\gamma\in\Gamma}e^{i\gamma\cdot\theta}(\mathbf{T}^B_\gamma\psi_n)(\lambda;x)\overline{\varphi_m(\lambda;x;\theta)}dx\\
&=(v_n\lambda+O(1))\int_{E_0}\varphi_n(\lambda;x;\theta)\overline{\varphi_m(\lambda;x;\theta)}dx\\
&=(v_n\lambda+O(1))(\delta_{nm}+O(e^{-k\lambda}))\\
&=v_n\delta_{nm}\lambda+O(1),
\end{aligned}$$

where we used (12).

We denote by $R'_{n,m}(\lambda)$ the first term of (19). We have

$$\begin{aligned}
&\mathbf{T}^B_\gamma((H^\circ(\lambda) - \widetilde{H}^\circ(\lambda))\psi_n)(\lambda; x) \\
&= \lambda^2 \mathbf{T}^B_\gamma\{(V(x) - ({\mu_1}^2{x_1}^2 + {\mu_2}^2{x_2}^2))\psi_n\} \\
&= \lambda^2\theta(\gamma)e^{-ib(x_1\gamma_2 - x_2\gamma_1)}\{V(x-\gamma) - (\mu_1(x_1-\gamma_1)^2 + \mu_2(x_2-\gamma_2)^2)\}\psi_n(\lambda; x-\gamma).
\end{aligned}$$

By $(H.5)$, there exists a constant $C_0 > 0$ such that

$$|V(x) - (\mu_1{x_1}^2 + \mu_2{x_2}^2)| \leq C_0|x|^3 \quad \text{in} \quad E_0.$$

Because $V(x)$ is bounded in $\mathbf{R}^2$ and $\mathrm{dis}(\Gamma\backslash\{0\}, E_0) > 0$, there exists a constant $C'_0 > 0$ independent of $\gamma \in \Gamma\backslash\{0\}$ such that

$$|V(x-\gamma)| + \mu_1(x_1-\gamma_1)^2 + \mu_2(x_2-\gamma_2)^2 \leq C'_0|x-\gamma|^3 \text{ in } E_0 \text{ for any } \gamma \in \Gamma\backslash\{0\}.$$

Using (10), we have for any $\gamma \in \Gamma$

$$|\mathbf{T}^B_\gamma((H^\circ(\lambda) - \widetilde{H}^\circ(\lambda))\psi_n)(\lambda; x)| \leq C''_0\lambda^{5/2}|x-\gamma|^3\exp(-c\lambda|x-\gamma|^2) \quad \text{in} \quad E_0,$$

where $C''_0 > 0$ is a constant independent of $\lambda > 1$ and $\gamma \in \Gamma$.

Using (10) again, we have

$$\begin{aligned}
|R'_{n,m}(\lambda)| &\leq C''_0 C_m \lambda^3 \sum_{\gamma\in\Gamma}\sum_{\gamma'\in\Gamma}\int_{E_0}|x-\gamma|^3\exp(-c\lambda|x-\gamma|^2)\exp(-c\lambda|x-\gamma'|^2)dx \\
&\leq C''_0 C_m\lambda^3\left(\sum_{\gamma'\in\Gamma}\exp(-c\lambda\min_{x\in E_0}|x-\gamma'|^2)\right) \\
&\quad\times\sum_{\gamma\in\Gamma}\int_{E_0}|x-\gamma|^3\exp(-c\lambda|x-\gamma|^2)dx
\end{aligned}$$

As in the preceding caluculus, there exists a constant $C' > 0$ independent of $\lambda \geq 1$ such that

$$\sum_{\gamma'\in\Gamma}\exp(-c\lambda\min_{x\in E_0}|x-\gamma'|^2) \leq C' \quad (\lambda \geq 1).$$

So we have

$$\begin{aligned}
|R'_{n,m}(\lambda)| &\leq C'C''_0C_m\lambda^3\int_{\mathbf{R}^2}|x|^3\exp(-c\lambda|x|^2)dx \\
&= O(\lambda^{1/2})
\end{aligned}$$

and we get (13). $\square$

3. Proof of Theorem A

In this section, we give the proof of Theorem A. The most important part of the proof is the exponential decay of eigenfunctions of approximate operators (cf. [1] §3.3).

First we introduce the Agmon distance. For x, $y \in \mathbf{R}^2$, we define

$$d_V(x,y) = \inf_{\gamma} \int_0^1 \sqrt{V(\gamma(t))}|\dot{\gamma}(t)|dt \tag{3.1}$$

where $\gamma : [0,1] \to \mathbf{R}^2$ is a piecewise C^1 path satisfying $\gamma(0) = x$ and $\gamma(1) = y$. $d_V(x,y)$ has the following properties (see [1] §2.3, 2.4, and 3.1) : For any $y \in \mathbf{R}^2$,

$$|\nabla_x d_V(x,y)|^2 \leq V(x) \quad \text{a.e. in} \quad \mathbf{R}^2. \tag{3.2}$$

$d_V(x,0)$ is smooth in a neighborhood of 0 and satisfies

$$|\nabla_x d_V(x,0)|^2 = V(x) \quad \text{in a neighborhood of } 0.$$

For $x_0 \in \mathbf{R}^2$ and $r > 0$, we set

$$B_V(x_0, r) = \{x \in \mathbf{R}^2 : d_V(x_0, x) < r\}.$$

Let

$$s_0 = \min_{\gamma \in \Gamma \setminus \{0\}} d_V(0,\gamma) \quad (> 0).$$

For sufficiently small $\eta > 0$, we choose $W_\eta \in C_0^\infty(\mathbf{R}^2)$ such that

$$W_\eta = 1 \quad \text{on} \quad B_V\left(0, \frac{\eta}{4}\right), W_\eta \geq 0 \quad \text{in} \quad \mathbf{R}^2, \mathrm{supp} W_\eta \subset B_V\left(0, \frac{\eta}{2}\right).$$

Let

$$\widetilde{V}(x) = V(x) + \sum_{\gamma \in \Gamma \setminus \{0\}} W_\eta(x - \gamma).$$

To approximate $\mathcal{E}_1(\lambda;\theta) \quad (\theta \in E^*)$, we introduce the following approximate operator

$$\widetilde{H}(\lambda) = (D_{x_1} + bx_2)^2 + (D_{x_2} - bx_1)^2 + \lambda^2 \widetilde{V}(x) \tag{3.3}$$

in $L^2(\mathbf{R}^2)$ with domain $H_B^2(\mathbf{R}^2)$.

Since $\widetilde{V}(x)$ has a non-degenerate minimum only at the origin, one can argue as in §2 to show the following fact.

For any $n \in \mathbf{N}_+$ and sufficiently large λ, $\widetilde{H}(\lambda)$ has at least n eigenvalues below its essential spectrum, and the j-th eigenvalue counted with multiplicity has asymptotic expansion $v_j\lambda + o(\lambda) \quad (\lambda \to \infty)$.

Let $\widetilde{\mathcal{E}}(\lambda)$ be the first eigenvalue of $\widetilde{H}(\lambda)$ and let $\widetilde{\phi}(\lambda)(x)$ be the associated normalized eigenfunction. We have the following theorem which is analogous to Helffer-Sjöstrand (cf. [2] Lemma 2.4).

Lemma 3.1. *For sufficiently small $\epsilon > 0$ we have*

$$\|e^{\lambda(1-\epsilon)d_{\widetilde{V}}(x,0)}\widetilde{\phi}(\lambda)(x)\|_{H^1_B(\mathbf{R}^2)} = O_\epsilon(e^{\epsilon\lambda}) \quad (\lambda \to \infty), \tag{3.4}$$

where $(u,v)_{H^1_B(\mathbf{R}^2)} = (u,v)_{L^2(\mathbf{R}^2)} + \sum_{i=1}^2 (T_i u, T_i v)_{L^2(\mathbf{R}^2)}$.

Proof. First we show the following equality

$$\begin{aligned}&\int_{\mathbf{R}^2} \{|(D_{x_1} + bx_2)(e^{\lambda\varphi}\widetilde{\phi})|^2 + |(D_{x_2} - b_{x_1})(e^{\lambda\varphi}\widetilde{\phi})|^2\}dx \\ &+ \int_{\mathbf{R}^2} e^{2\lambda\varphi}(\lambda^2(\widetilde{V} - |\nabla\varphi|^2) - \widetilde{\mathcal{E}}(\lambda))|\widetilde{\phi}|^2 dx = 0,\end{aligned} \tag{3.5}$$

where φ is any $\mathbf{R}$-valued locally Lipschitzian function in $\mathbf{R}^2$, which is constant for sufficiently large $|x|$.

We choose $\chi \in C_0^\infty(\mathbf{R}^2)$ such that

$$\chi(x) = \begin{cases} 1 & (|x| \le 1) \\ 0 & (|x| \ge 2) \end{cases}, 0 \le \chi \le 1 \quad \text{in} \quad \mathbf{R}^2.$$

For $R > 0$, we set

$$u_R(\lambda; x) = \chi\left(\frac{x}{R}\right)\widetilde{\phi}(\lambda)(x).$$

Note that $\varphi \in H^1_{loc}(\mathbf{R}^2)$ and $u_R = 0$ for $|x| \ge 2R$. Then integrating by parts shows that

$$\begin{aligned}&\operatorname{Re}\int_{\mathbf{R}^2} e^{2\lambda\varphi}\{(\widetilde{H}(\lambda) - \widetilde{\mathcal{E}}(\lambda))u_R\}\overline{u_R}dx \\ &= \int_{\mathbf{R}^2}\{|(D_{x_1} + bx_2)(e^{\lambda\varphi}u_R)|^2 + |(D_{x_2} - bx_1)(e^{\lambda\varphi}u_R)|^2\}dx \\ &+ \int_{\mathbf{R}^2} e^{2\lambda\varphi}(\lambda^2(\widetilde{V} - |\nabla\varphi|^2) - \widetilde{\mathcal{E}}(\lambda))|u_R|^2 dx.\end{aligned} \tag{3.6}$$

Because φ is constant for sufficiently large $|x|$ and $u_R \to \widetilde{\phi}$ in $H^2_B(\mathbf{R}^2) \quad (R \to \infty)$, we get (5) by taking the limit $R \to \infty$ in (6).

Let

$$\chi_R(t) = \begin{cases} t & (0 \le t \le R) \\ R & (t > R). \end{cases}$$

We set

$$\varphi(x) = d_{\widetilde{V}}(x, 0) \quad \text{and} \quad \varphi_R(x) = (1-\delta)\chi_R(\varphi(x)) \quad (0 < \delta < 1).$$

It follows from (2) that if $\varphi(x) \le R$, we have

$$\begin{aligned} |\nabla \varphi_R|^2 &= (1-\delta)^2 |\nabla \varphi|^2 \\ &\le (1-\delta)^2 \widetilde{V}(x), \end{aligned} \tag{3.7}$$

and $\nabla \varphi_R = 0$ otherwise. So it follows that if $\widetilde{V}(x) \ge \delta$, we have

$$\widetilde{V} - |\nabla \varphi_R|^2 - \lambda^{-2} \widetilde{\mathcal{E}}(\lambda) \ge \delta^2 (2-\delta) - \lambda^{-2} \widetilde{\mathcal{E}}(\lambda). \tag{3.8}$$

Because $\widetilde{\mathcal{E}}(\lambda) = v_1 \lambda + o(\lambda) \quad (\lambda \to \infty)$, for any $\delta > 0$, there exists $\lambda(\delta) > 1$ such that

$$\widetilde{V} - |\nabla \varphi_R|^2 - \lambda^{-2} \widetilde{\mathcal{E}}(\lambda) \ge \delta^2 \quad \text{if} \quad \widetilde{V}(x) \ge \delta, \lambda > \lambda(\delta). \tag{3.9}$$

We set

$$Q_\delta^+ = \{x \in \mathbf{R}^2; \widetilde{V}(x) \ge \delta\}, Q_\delta^- = \{x \in \mathbf{R}^2; \widetilde{V}(x) < \delta\}.$$

Then, (5) and (9) imply that

$$\begin{aligned} &\lambda^{-2} \int_{\mathbf{R}^2} \{|(D_{x_1} + bx_2)(e^{\lambda \varphi_R} \widetilde{\phi})|^2 + |(D_{x_2} - bx_1)(e^{\lambda \phi_R} \widetilde{\phi})|^2\} dx \\ &+ \delta^2 \int_{Q_\delta^+} e^{2\lambda \varphi_R} |\widetilde{\phi}|^2 dx \\ \le &\sup_{Q_\delta^-} |\widetilde{V} - |\nabla \varphi_R|^2 - \lambda^{-2} \widetilde{\mathcal{E}}(\lambda)| \int_{Q_\delta^-} e^{2\lambda \varphi_R} |\widetilde{\phi}|^2 dx. \end{aligned} \tag{3.10}$$

Let

$$a(\delta) = 2 \sup_{x \in Q_\delta^-} \varphi_R(x).$$

$(H.4)$ and $(H.5)$ imply that

$$a(\delta) = O(\delta^2) \quad (\delta \to 0). \tag{3.11}$$

Besides, there exists a constant $C > 0$ such that for any $R > 0$ and $\lambda > \lambda(\delta)$

$$\sup_{Q_\delta^-} |\widetilde{V} - |\nabla\varphi_R|^2 - \lambda^{-2}\widetilde{\mathcal{E}}(\lambda)| \leq C.$$

So, it follows from (10) that

$$\begin{aligned}(3.12)\qquad & \lambda^{-2}\int_{\mathbf{R}^2}\{|(D_{x_1}+bx_2)(e^{\lambda\varphi_R}\widetilde{\phi})|^2+|(D_{x_2}-bx_1)(e^{\lambda\varphi_R}\widetilde{\phi})|^2\}dx\\ & +\delta^2\int_{\mathbf{R}^2} e^{2\lambda\varphi_R}|\widetilde{\phi}|^2dx\\ & \leq (C+1)e^{\lambda a(\delta)}\end{aligned}$$

Taking the limit $R \to \infty$, we have

$$\begin{aligned}(3.13)\qquad & \lambda^{-2}\int_{\mathbf{R}^2}\{|(D_{x_1}+bx_2)(e^{\lambda(1-\delta)\varphi}\widetilde{\phi})|^2+|(D_{x_2}-bx_1)(e^{\lambda(1-\delta)\varphi}\widetilde{\phi})|^2\}dx\\ & +\delta^2\int_{\mathbf{R}^2} e^{2\lambda(1-\delta)\varphi}|\widetilde{\phi}|^2dx\\ & \leq (C+1)e^{\lambda a(\delta)}\quad (\lambda > \lambda(\delta)).\end{aligned}$$

Plugging (11) to (13), we get (4). □

Lemma 3.2. *For any $\epsilon > 0$, $\alpha \in \mathbf{N}^2$, and $R > 0$, there exists a constant $C_{\alpha,\epsilon,R} > 0$ independent of λ such that*

$$(3.14)\qquad |\partial_x^\alpha\widetilde{\phi}(\lambda)(x)| \leq C_{\alpha,\epsilon,R}e^{-\lambda(d_{\widetilde{V}}(x,0)-\epsilon)}\quad in\quad B_{\widetilde{V}}(0,R).$$

Proof. Because $\widetilde{V}(x) = 0 \Longleftrightarrow x = 0$ and $\widetilde{V}(x)$ is non-degenerate at $x = 0$, we have

$$(3.15)\qquad \varphi(x) = d_{\widetilde{V}}(x,0) \in C^\infty(\mathbf{R}^2;\mathbf{R}).$$

Let

$$w(\lambda)(x) = e^{\lambda\varphi(x)}\widetilde{\phi}(x).$$

Let K, $\widetilde{K}$ be any bounded open set of $\mathbf{R}^2$ satisfying $K \subset\subset \widetilde{K}$. Lemma 3.1 implies

$$(3.16)\qquad \|e^{\lambda\varphi}\widetilde{\phi}\|_{H^1(\widetilde{K})} = O(e^{\epsilon\lambda}).$$

Because $\widetilde{H}(\lambda)\widetilde{\phi} = \widetilde{\mathcal{E}}(\lambda)\widetilde{\phi}$, we have

$$\begin{aligned}(3.17)\qquad -\Delta w =\ & (\widetilde{\mathcal{E}}(\lambda) - \lambda^2\widetilde{V}(x) - \lambda\Delta\varphi + \lambda^2|\nabla\varphi|^2 + b^2|x|^2)w\\ & + 2\lambda(D_{x_1}\varphi)(D_{x_1}+bx_2)w + 2\lambda(D_{x_2}\varphi)(D_{x_2}-bx_1)w\\ & - 2bx_2(D_{x_1}+bx_2)w + 2bx_1(D_{x_2}-bx_1)w.\end{aligned}$$

We denote by $f(\lambda)(x)$ the right-hand side of (17). Noting $\widetilde{\mathcal{E}}(\lambda) = v_1\lambda + o(\lambda)$ and (16), we have by an a-priori estimate for the Laplacian

$$\begin{aligned} \|w\|_{H^2(K)} &\le C_{K,\widetilde{K}}(\|f(\lambda)\|_{L^2(\widetilde{K})} + \|w(\lambda)\|_{L^2(\widetilde{K})}) \\ &\le C'_{K,\widetilde{K}}\lambda^2\|w(\lambda)\|_{H^1(\widetilde{K})} \\ &= O(e^{\epsilon\lambda}). \end{aligned} \tag{3.18}$$

Let K', $\widetilde{K'}$ be any bounded open set of $\mathbf{R}^2$ satisfying $K' \subset\subset \widetilde{K'}$. Then the above argument shows that

$$\begin{aligned} \|w\|_{H^3(K')} &\le C_{K',\widetilde{K'}}(\|f(\lambda)\|_{H^1(\widetilde{K'})} + \|w\|_{L^2(\widetilde{K'})}) \\ &\le C'_{K',\widetilde{K'}}\lambda^2\|w(\lambda)\|_{H^2(\widetilde{K'})} \\ &= O(e^{\epsilon\lambda}). \end{aligned} \tag{3.19}$$

Inductively, for any $m \in \mathbf{N}_+$ and $R > 0$, we have

$$\|w\|_{H^m(B_{\widetilde{V}}(0,R))} \le C_R e^{\epsilon\lambda}. \tag{3.20}$$

where C_R is a constant independent of λ.

By using Sobolev's imbedding theorem and (20), we get (14). $\square$

We turn to the proof of the Theorem A. First note that $V(x) = \widetilde{V}(x)$ in $B_V(0, s_0 - \frac{\eta}{2})$, and $d_{\widetilde{V}}(x,0) \ge d_V(x,0)$ in $\mathbf{R}^2$. We choose $\chi_\eta \in C_0^\infty(\mathbf{R}^2)$ such that

$$\begin{aligned} \mathrm{supp}\chi_\eta \subset B_V\left(0, s_0 - \frac{3}{4}\eta\right), \quad 0 \le \chi_\eta \le 1 \quad \text{in} \quad B_V\left(0, s_0 - \frac{3}{4}\eta\right), \\ \chi_\eta = 1 \quad \text{on} \quad B_V(0, s_0 - \eta). \end{aligned}$$

Let $\widetilde{\psi}(\lambda)(x) = \chi_\eta(x)\widetilde{\phi}(\lambda)(x)$. For $\theta \in E^*$ we set

$$\widetilde{\psi}_\theta(x) = \sum_{\gamma\in\Gamma} e^{i\gamma\cdot\theta}(\mathbf{T}_\gamma^B\widetilde{\psi})(x)(\in \mathcal{H}_{B,\theta} \cap C^\infty(\mathbf{R}^2)). \tag{3.21}$$

Then, by a direct computation we get

$$H(\lambda;\theta)\widetilde{\psi}_\theta(\lambda) = \widetilde{\mathcal{E}}(\lambda)\widetilde{\psi}_\theta(\lambda) + \widetilde{r}_\theta(\lambda) \tag{3.22}$$

where

$$\begin{aligned} \widetilde{r}_\theta(\lambda)(x) &= \textstyle\sum_{\gamma\in\Gamma} e^{i\gamma\cdot\theta}(\mathbf{T}_\gamma^B\widetilde{r}(\lambda))(x), \\ \widetilde{r}(\lambda)(x) &= -(\triangle\chi_\eta)\widetilde{\phi} - 2\nabla\chi_\eta\cdot\nabla\widetilde{\phi} - 2bi((x_2\partial_{x_1} - x_1\partial_{x_2})\chi_\eta)\widetilde{\phi}. \end{aligned}$$

We estimate $\|\widetilde{\psi}_\theta(\lambda)\|_{\mathcal{H}_{B,\theta}}$ and $\|\widetilde{r}_\theta(\lambda)\|_{\mathcal{H}_{B,\theta}}$. Because $\|\widetilde{\phi}\|_{L^2(\mathbf{R}^2)} = 1$ and $0 \leq \chi_\eta \leq 1$ in $\mathbf{R}^2$, we have

$$1 - \|(1-\chi_\eta)\widetilde{\phi}\|_{L^2(\mathbf{R}^2)} \leq \|\widetilde{\psi}\|_{L^2(\mathbf{R}^2)} \leq 1.$$

Using Lemma 3.1, we have

$$\begin{aligned}
\|(1-\chi_\eta)\widetilde{\phi}\|_{L^2(\mathbf{R}^2)} &\leq \|\widetilde{\phi}\|_{L^2(\mathbf{R}^2 \setminus B_V(0, s_0 - \eta))} \\
&= \|e^{-\lambda(1-\epsilon)\varphi} e^{\lambda(1-\epsilon)\varphi} \widetilde{\phi}\|_{L^2(\mathbf{R}^2 \setminus B_V(0, s_0 - \eta))} \\
&\leq C_\epsilon e^{-\lambda(1-\epsilon)(s_0 - \eta)} e^{\epsilon\lambda} \\
&= C_\epsilon e^{-\lambda(s_0-\eta) + \lambda\epsilon(s_0 - \eta + 1)}.
\end{aligned}$$

We choose $\epsilon > 0$ such that $\epsilon(s_0 - \eta + 1) < \eta$. Then we have

$$\|(1-\chi_\eta)\widetilde{\phi}\|_{L^2(\mathbf{R}^2)} = O(e^{-\lambda(s_0 - 2\eta)}),$$

and

$$\|\widetilde{\psi}\|_{L^2(\mathbf{R}^2)} = 1 + O(e^{-\lambda(s_0-2\eta)}). \tag{3.23}$$

Using (21), we have

$$\|\widetilde{\psi_\theta}\|^2_{\mathcal{H}_{B,\theta}} = \|\widetilde{\psi}\|^2_{L^2(\mathbf{R}^2)} + \sum_{\substack{\gamma,\gamma' \in \Gamma \\ \gamma \neq \gamma'}} e^{i(\gamma - \gamma')\cdot\theta} (\mathbf{T}^B_\gamma \widetilde{\psi}, \mathbf{T}^B_{\gamma'} \widetilde{\psi})_{L^2(E_0)}, \tag{3.24}$$

where $E_0 = [-\pi, \pi) \times [-\pi, \pi)$. We note that the summation of the right-hand side of (24) ranges over a finite set of indices because $\widetilde{\psi}$ is compactly supported. Let γ, $\gamma' \in \Gamma$, $\gamma \neq \gamma'$. Lemma 3.2 implies

$$\begin{aligned}
|(\mathbf{T}^B_\gamma \widetilde{\psi}, \mathbf{T}^B_{\gamma'} \widetilde{\psi})_{L^2(E_0)}| &\leq \int_{E_0} |\widetilde{\phi}(\lambda)(x-\gamma)\widetilde{\phi}(\lambda)(x-\gamma')| dx \\
&\leq C_\epsilon \int_{E_0} e^{-\lambda d_V(x-\gamma,0)+\lambda\epsilon} e^{-\lambda d_V(x-\gamma',0)+\lambda\epsilon} dx.
\end{aligned}$$

Because

$$d_V(x-\gamma, 0) + d_V(x - \gamma', 0) \geq d_V(\gamma, \gamma') \geq s_0,$$

we have

$$|(\mathbf{T}^B_\gamma \widetilde{\psi}, \mathbf{T}^B_{\gamma'} \widetilde{\psi})_{L^2(E_0)}| = O(e^{-\lambda(s_0 - 2\eta)}). \tag{3.25}$$

Combining (24) and (23), (25), we have

$$\|\widetilde{\psi_\theta}\|_{\mathcal{H}_{B,\theta}} = 1 + O(e^{-\lambda(s_0-2\eta)}), \tag{3.26}$$

where the error term is uniform with respect to $\theta \in E^*$.

Next we estimate $\|\widetilde{r}_\theta\|_{\mathcal{H}_{B,\theta}}$. We have

$$\|\widetilde{r}_\theta\|^2_{\mathcal{H}_{B,\theta}} = \sum_{\gamma,\gamma' \in \Gamma} e^{i(\gamma-\gamma')\cdot\theta}(\mathbf{T}^B_\gamma \widetilde{r}, \mathbf{T}^B_{\gamma'}\widetilde{r})_{L^2(E_0)}. \tag{3.27}$$

We again note that the summation of the right-hand side of (27) ranges over a finite set of indices. Let us recall

$$\widetilde{r}(\lambda)(x) = -(\triangle\chi_\eta)\widetilde{\phi} - 2\nabla\chi_\eta \cdot \nabla\widetilde{\phi} - 2bi((x_2\partial_{x_1} - x_1\partial_{x_2})\chi_\eta)\widetilde{\phi}.$$

We note that

$$\triangle\chi_\eta = 0, \nabla\chi_\eta = 0, (x_2\partial_{x_1} - x_1\partial_{x_2})\chi_\eta = 0 \quad \text{on} \quad B_V(0, s_0 - \eta).$$

So, Lemma 3.2 implies

$$|\widetilde{r}(\lambda)(x)| \le C_\eta e^{-\lambda(s_0 - 2\eta)} \quad \text{in} \quad \mathbf{R}^2.$$

Using (27) and the above inequality, we have

$$\|\widetilde{r}_\theta\|_{\mathcal{H}_{B,\theta}} = O(e^{-\lambda(s_0-2\eta)}), \tag{3.28}$$

where the error term is uniform with respect to $\theta \in E^*$.

Using (22), (26), and (28), we get

$$\operatorname{dis}(\widetilde{\mathcal{E}}(\lambda), \sigma(H(\lambda;\theta))) \le \frac{\|(H(\lambda;\theta) - \widetilde{\mathcal{E}}(\lambda))\widetilde{\psi}_\theta\|_{\mathcal{H}_{B,\theta}}}{\|\widetilde{\psi}_\theta\|_{\mathcal{H}_{B,\theta}}} \le \frac{\|\widetilde{r}_\theta\|_{\mathcal{H}_{B,\theta}}}{\|\widetilde{\psi}_\theta\|_{\mathcal{H}_{B,\theta}}} = O(e^{-\lambda(s_0-2\eta)}).$$

On the other hand,

$$\widetilde{\mathcal{E}}(\lambda) = v_1\lambda + o(\lambda), \mathcal{E}_1(\lambda;\theta) = v_1\lambda + o(\lambda), \mathcal{E}_2(\lambda;\theta) = v_2\lambda + o(\lambda),$$

where $v_1 = \sqrt{\mu_1} + \sqrt{\mu_2} < v_2$ and each error term is uniform with respect to $\theta \in E^*$. These two facts imply Theorem A. □

4. Proof of Theorem B

In this section, we describe the proof of Theorem B. For this purpose, we shall get the θ-dependence of the asymptotic behavior of $\mathcal{E}_1(\lambda;\theta)$. In this proof, the W.K.B. type analysis plays an important role.

First, we define a distance between the subspaces of a Hilbert space H. Let E, F be closed subspaces of H, and let Π_F be the orthogonal projection onto F. We define

$$\overrightarrow{d}(E,F) = \sup_{x\in E, \|x\|=1} \operatorname{dis}(x, F) = \|(1-\Pi_F)|_E\|_H.$$

Proposition 4.1 (cf. [2, Proposition 2.5]). *Let A be a selfadjoint operator in H. Let $I \subset \mathbf{R}$ be a compact interval. Let $\psi_1, \psi_2, \cdots, \psi_N \in \mathcal{D}(A)$ be linearly independent, and $\mu_1, \mu_2, \cdots, \mu_N \in I = [\alpha, \beta]$ be such that*

$$A\psi_j = \mu_j \psi_j + r_j, \quad \|r_j\| \leq \epsilon \quad (j = 1, 2, \cdots, N).$$

Suppose that there exists a constant $a > 0$ such that

$$\sigma(A) \cap [\alpha - 2a, \alpha] = \emptyset, \quad \sigma(A) \cap [\beta, \beta + 2a] = \emptyset.$$

Let E be the subspace of H spanned by $\psi_1, \psi_2, \cdots, \psi_N$ and let F be the range of $E_A(I)$, $E_A(\cdot)$ being the spectral projection associated with A.

Then, we have

$$\overrightarrow{d}(E, F) \leq \frac{N^{1/2}\epsilon}{a\sqrt{\lambda_s^{\min}}},$$

where $\lambda_s^{\min}$ is the smallest eigenvalue of the matrix $S = ((\psi_j, \psi_k)_H)_{1 \leq i,j \leq N}$.

For $\theta \in E^*$ and $\widetilde{\psi}_\theta(\lambda)$ defined in (21), let

$$E_\theta(\lambda) = \{k\widetilde{\psi}_\theta(\lambda); k \in \mathbf{C}\},$$

and let $F_\theta(\lambda)$ be the eigenspace of $H(\lambda; \theta)$ associated with $\mathcal{E}_1(\lambda; \theta)$. Using the decay estimates of eigenfunctions in §3 and this proposition, we have the following.

Lemma 4.2.

$$\overrightarrow{d}(E_\theta(\lambda), F_\theta(\lambda)) = O(e^{-(s_0 - 2\eta)\lambda}) \quad (\lambda \to \infty) \tag{4.1}$$

where the error term is uniform with respect to $\theta \in E^*$.

Proof. First we recall the following estimates. (See §3 (26), (22), and (28).)

$$\|\widetilde{\psi}_\theta\|_{\mathcal{H}_{B,\theta}} = 1 + O(e^{-\lambda(s_0 - 2\eta)}), \tag{4.2}$$

$$H(\lambda; \theta)\widetilde{\psi}_\theta(\lambda) = \widetilde{\mathcal{E}}(\lambda)\widetilde{\psi}_\theta(\lambda) + \widetilde{r}_\theta(\lambda), \tag{4.3}$$

$$\|\widetilde{r}_\theta\|_{\mathcal{H}_{B,\theta}} = O(e^{-\lambda(s_0 - 2\eta)}), \tag{4.4}$$

where the error terms in (2) and (4) are uniform with respect to $\theta \in E^*$.

In §2, we have shown that

$$\widetilde{\mathcal{E}}(\lambda) = v_1\lambda + o(\lambda), \mathcal{E}_1(\lambda; \theta) = v_1\lambda + o(\lambda), \mathcal{E}_2(\lambda; \theta) = v_2\lambda + o(\lambda), v_1 < v_2, \tag{4.5}$$

where each error term is uniform with respect to $\theta \in E^*$.

We set $k = (v_2 - v_1)/4$ (> 0). Then (5) implies

$$\begin{aligned} \mathcal{E}_1(\lambda;\theta) &\in [(v_1 - k)\lambda, (v_1 + k)\lambda], \\ \sigma(H(\lambda;\theta)) \cap [(v_1 - 2k)\lambda, (v_1 - k)\lambda] &= \emptyset, \\ \sigma(H(\lambda;\theta)) \cap [(v_1 + k)\lambda, (v_1 + 2k)\lambda] &= \emptyset \end{aligned} \tag{4.6}$$

for sufficiently large λ.

Applying Proposition 4.1, we have

$$\begin{aligned} \overrightarrow{d}(E_\theta(\lambda), F_\theta(\lambda)) &\leq \frac{\|\widetilde{r}_\theta\|_{\mathcal{H}_{B,\theta}}}{\frac{k}{2}\lambda\|\widetilde{\psi}_\theta\|^2_{\mathcal{H}_{B,\theta}}} \\ &= O(e^{-\lambda(s_0 - 2\eta)}), \end{aligned}$$

where we used (2) and (4) in the last equality, and the last term is uniform with respect to $\theta \in E^*$. □

Lemma 4.3.

$$\mathcal{E}_1(\lambda;\theta) = \widetilde{\mathcal{E}}(\lambda) + \sum_{\gamma \in \Gamma \setminus \{0\}} e^{i\gamma\cdot\theta}(\mathbf{T}^B_\gamma \widetilde{r}, \widetilde{\psi})_{L^2(\mathbf{R}^2)} + O(e^{-(2s_0 - 5\eta)\lambda}) \quad (\lambda \to \infty)$$

where the error term is uniform with respect to $\theta \in E^*$.

Proof. Let Π_{F_θ} be the orthogonal projection onto F_θ. We set

$$v_\theta = \Pi_{F_\theta}\widetilde{\psi}_\theta. \tag{4.7}$$

Lemma 4.2 implies

$$\begin{aligned} \|v_\theta - \widetilde{\psi}_\theta\|_{\mathcal{H}_{B,\theta}} &= \|(\Pi_{F_\theta} - 1)\widetilde{\psi}_\theta\|_{\mathcal{H}_{B,\theta}} \\ &\leq \|\widetilde{\psi}_\theta\|_{\mathcal{H}_{B,\theta}} \overrightarrow{d}(E_\theta(\lambda), F_\theta(\lambda)) \\ &= O(e^{-(s_0 - 2\eta)}), \end{aligned} \tag{4.8}$$

$$\begin{aligned} \|\widetilde{\psi}_\theta\|^2_{\mathcal{H}_{B,\theta}} &= \|v_\theta\|^2_{\mathcal{H}_{B,\theta}} + \|v_\theta - \widetilde{\psi}_\theta\|^2_{\mathcal{H}_{B,\theta}} \\ &= \|v_\theta\|^2_{\mathcal{H}_{B,\theta}} + O(e^{-2(s_0 - 2\eta)\lambda}). \end{aligned} \tag{4.9}$$

Recalling again the relations (2)∼(4), we have

$$\begin{aligned} H(\lambda;\theta)(v_\theta - \widetilde{\psi}_\theta) &= H(\lambda;\theta)(\Pi_{F_\theta} - 1)\widetilde{\psi}_\theta \\ &= (\Pi_{F_\theta} - 1)H(\lambda;\theta)\widetilde{\psi}_\theta \end{aligned} \tag{4.10}$$

$$
\begin{aligned}
&= (\Pi_{F_\theta} - 1)(\widetilde{\mathcal{E}}(\lambda)\widetilde{\psi}_\theta + \widetilde{r}_\theta) \\
&= \widetilde{\mathcal{E}}(\lambda)(v_\theta - \widetilde{\psi}_\theta) + (\Pi_{F_\theta} - 1)\widetilde{r}_\theta \\
&= O(e^{-(s_0-3\eta)\lambda}) \quad \text{in} \quad \mathcal{H}_{B,\theta},
\end{aligned}
$$

where we used (3) in the third equality and (8), (4) in the fifth equality. So we get

$$
\begin{aligned}
(4.11)\qquad &(H(\lambda;\theta)\widetilde{\psi}_\theta, \widetilde{\psi}_\theta)_{\mathcal{H}_{B,\theta}} \\
&= (H(\lambda;\theta)v_\theta, v_\theta)_{\mathcal{H}_{B,\theta}} + (H(\lambda;\theta)(\widetilde{\psi}_\theta - v_\theta), \widetilde{\psi}_\theta - v_\theta)_{\mathcal{H}_{B,\theta}} \\
&= (H(\lambda;\theta)v_\theta, v_\theta)_{\mathcal{H}_{B,\theta}} + O(e^{-(2s_0-5\eta)\lambda}).
\end{aligned}
$$

Using (3) and $H(\lambda;\theta)v_\theta = \mathcal{E}_1(\lambda;\theta)v_\theta$, we get

$$
\begin{aligned}
(4.12)\qquad &\mathcal{E}_1(\lambda;\theta)\|v_\theta\|^2_{\mathcal{H}_{B,\theta}} \\
&= \widetilde{\mathcal{E}}(\lambda)\|\widetilde{\psi}_\theta\|^2_{\mathcal{H}_{B,\theta}} + (\widetilde{r}_\theta, \widetilde{\psi}_\theta)_{\mathcal{H}_{B,\theta}} + O(e^{-(2s_0-5\eta)\lambda}) \\
&= \widetilde{\mathcal{E}}(\lambda)\|v_\theta\|^2_{\mathcal{H}_{B,\theta}} + (\widetilde{r}_\theta, \widetilde{\psi}_\theta)_{\mathcal{H}_{B,\theta}} + \widetilde{\mathcal{E}}(\lambda)(\|\widetilde{\psi}_\theta\|^2_{\mathcal{H}_{B,\theta}} - \|v_\theta\|^2_{\mathcal{H}_{B,\theta}}) \\
&\quad + O(e^{-(2s_0-5\eta)\lambda}).
\end{aligned}
$$

Using (9), (2), (4), we have

$$
(4.13)\qquad \mathcal{E}_1(\lambda;\theta) = \widetilde{\mathcal{E}}(\lambda) + (\widetilde{r}_\theta, \widetilde{\psi}_\theta)_{\mathcal{H}_{B,\theta}} + O(e^{-(2s_0-5\eta)\lambda}).
$$

A direct computation shows that

$$
(4.14)\qquad (\widetilde{r}_\theta, \widetilde{\psi}_\theta)_{\mathcal{H}_{B,\theta}} = (\widetilde{r}, \widetilde{\psi})_{L^2(\mathbf{R}^2)} + \sum_{\substack{\gamma\in\Gamma \\ \gamma\neq 0}} e^{i\gamma\cdot\theta}(\mathbf{T}^B_\gamma \widetilde{r}, \widetilde{\psi})_{L^2(\mathbf{R}^2)}.
$$

Because $\widetilde{r}$ and $\widetilde{\psi}$ are compactly supported and $\widetilde{r} = 0$ in $B_V(0, s_0 - \eta)$, Lemma 3.2 implies that

$$
(4.15)\qquad |(\widetilde{r}, \widetilde{\psi})_{L^2(\mathbf{R}^2)}| = O(e^{-(2s_0-4\eta)\lambda}).
$$

Combining (13), (14), and (15), we get the conclusion. □

Let $s_0' = \min_{\gamma\in\Gamma\setminus(\Lambda\cup\{0\})} d_V(\gamma, 0)$. Since V is periodic, it is easy to see that $s_0 < s_0' \leq 2s_0$. Then, Lemma 3.2 implies

$$
(4.16)\quad \mathcal{E}_1(\lambda;\theta) = \widetilde{\mathcal{E}}(\lambda) + \sum_{\gamma\in\Lambda} e^{i\gamma\cdot\theta}(\mathbf{T}^B_\gamma \widetilde{r}, \widetilde{\psi})_{L^2(\mathbf{R}^2)} + \widetilde{O}(e^{-s_0'\lambda}) \quad (\lambda \to \infty).
$$

where $\widetilde{O}(e^{-s_0'\lambda})$ means $O_\eta(e^{-(s_0'-\eta)\lambda})$ for any $\eta > 0$, and the error term is uniform with respect to $\theta \in E^*$.

$(H.2)$ implies that : $\gamma \in \Lambda \Rightarrow -\gamma \in \Lambda$. After a straightfoward calculation, we have

$$(\mathbf{T}^B_\gamma \widetilde{r}, \widetilde{\psi})_{L^2(\mathbf{R}^2)} = \overline{(\mathbf{T}^B_{-\gamma} \widetilde{r}, \widetilde{\psi})_{L^2(\mathbf{R}^2)}} \quad \text{for any} \quad \gamma \in \Lambda. \tag{4.17}$$

For $\gamma \in \Lambda$ and $a > 0$, let

$$E^{(a)}_\gamma = \{x \in \mathbf{R}^2; d_V(0,x) + d_V(\gamma,x) \le s_0 + a\}.$$

Then, for sufficiently small $a > 0$, we have

$$E^{(2a)}_\gamma \subset B_V\left(0, s_0 - \frac{3}{4}\eta\right) \cup B_V\left(\gamma, s_0 - \frac{3}{4}\eta\right).$$

We choose an open domain Ω with smooth boundary such that

$$0 \notin \overline{\Omega}, \gamma \in \Omega, E^{(2a)}_\gamma \cap \overline{\Omega} \subset B_V(\gamma, s_0 - \eta), E^{(2a)}_\gamma \cap \Omega^c \subset B_V(0, s_0 - \eta).$$

Let $\widetilde{\Gamma}_\gamma = \partial\Omega \cap E^{(2a)}$ and let $n = (n_1, n_2)$ be the outer unit normal of $\partial\Omega$. Using the decay estimates of eigenfunctions, we get

Lemma 4.4. *We have mod* $O(\lambda^{-\infty} e^{-s_0\lambda})$

$$\begin{aligned}(\mathbf{T}^B_\gamma \widetilde{r}, \widetilde{\psi})_{L^2(\mathbf{R}^2)} \equiv & \int_{\widetilde{\Gamma}_{-\gamma}} \left\{ \widetilde{\phi} \frac{\partial}{\partial n} \overline{(\mathbf{T}^B_{-\gamma} \widetilde{\phi})} - \overline{(\mathbf{T}^B_{-\gamma} \widetilde{\phi})} \frac{\partial}{\partial n} \widetilde{\phi} \right\} dS \\ & -2bi \int_{\widetilde{\Gamma}_{-\gamma}} \widetilde{\phi} \overline{\mathbf{T}^B_{-\gamma} \widetilde{\phi}} (x_2 n_1 - x_1 n_2) dS.\end{aligned} \tag{4.18}$$

Proof. Using the Green's formula, we have

$$\begin{aligned}
&(\mathbf{T}^B_\gamma \widetilde{r}, \widetilde{\psi})_{L^2(\mathbf{R}^2)} \\
&= (\widetilde{r}, \mathbf{T}^B_{-\gamma} \widetilde{\psi})_{L^2(\mathbf{R}^2)} \\
&= (-(\triangle \chi_\eta)\widetilde{\phi} - 2\nabla\chi_\eta \cdot \nabla\widetilde{\phi} - 2bi(L\chi_\eta)\widetilde{\phi}, \chi_\eta(x+\gamma)\mathbf{T}^B_{-\gamma}\widetilde{\phi})_{L^2(\mathbf{R}^2)} \\
&= \int_{\mathbf{R}^2} (\nabla\chi_\eta)(x)(\nabla\chi_\eta)(x+\gamma)\widetilde{\phi}(x)\overline{(\mathbf{T}^B_{-\gamma}\widetilde{\phi})(x)}dx \\
&\quad + \int_{\mathbf{R}^2} (\nabla\chi_\eta)(x)[\widetilde{\phi}\nabla\overline{(\mathbf{T}^B_{-\gamma}\widetilde{\phi})} - \overline{(\mathbf{T}^B_{-\gamma}\widetilde{\phi})}\nabla\widetilde{\phi}]\chi_\eta(x+\gamma)dx \\
&\quad - 2bi \int_{\mathbf{R}^2} (L\chi_\eta)(x)\widetilde{\phi}\chi_\eta(x+\gamma)\overline{\mathbf{T}^B_{-\gamma}\widetilde{\phi}}dx \\
&=: I_1 + I_2 + I_3,
\end{aligned} \tag{4.19}$$

where $L = x_2\partial_{x_1} - x_1\partial_{x_2}$. We choose $\chi_{E^{(a)}_{-\gamma}} \in C^\infty_0(\mathbf{R}^2)$ such that

$$\chi_{E^{(a)}_{-\gamma}} = 1 \quad \text{on} \quad E^{(a)}_{-\gamma}, \quad \text{supp}\ \chi_{E^{(a)}_{-\gamma}} \subset E^{(2a)}_{-\gamma}.$$

We compute these terms mod $O(\lambda^{-\infty}e^{-s_0\lambda})$ in the following way :

$$I_1 \equiv 0, \tag{4.20}$$

$$\begin{aligned} I_2 \equiv & \int_{\widetilde{\Gamma}_{-\gamma}} [\widetilde{\phi}\frac{\partial}{\partial n}\overline{(\mathbf{T}^B_{-\gamma}\widetilde{\phi})} - \overline{(\mathbf{T}^B_{-\gamma}\widetilde{\phi})}\frac{\partial}{\partial n}\widetilde{\phi}]dS \\ & - 2bi\int_{\Omega} \chi_\eta \chi_{E^{(a)}_{-\gamma}}[\widetilde{\phi}L\overline{\mathbf{T}^B_{-\gamma}\widetilde{\phi}} + \overline{\mathbf{T}^B_{-\gamma}\widetilde{\phi}}L\widetilde{\phi}]dx, \end{aligned} \tag{4.21}$$

$$\begin{aligned} I_3 \equiv & \, 2bi\int_{\Omega} \chi_\eta \chi_{E^{(a)}_{-\gamma}}[\widetilde{\phi}L\overline{\mathbf{T}^B_{-\gamma}\widetilde{\phi}} + \overline{\mathbf{T}^B_{-\gamma}\widetilde{\phi}}L\widetilde{\phi}]dx \\ & - 2bi\int_{\widetilde{\Gamma}_{-\gamma}} \widetilde{\phi}\overline{\mathbf{T}^B_{-\gamma}\widetilde{\phi}}(x_2 n_1 - x_1 n_2)dS. \end{aligned} \tag{4.22}$$

Lemma 4.4 is an immediate consequence of (19)~(22).

First we estimate I_1. We note that

$$\nabla\chi_\eta = 0 \quad \text{on} \quad B_V(s_0-\eta), \quad \text{supp}\ \chi_\eta \subset B_V\left(0, s_0 - \frac{3}{4}\eta\right).$$

So, Lemma 3.2 implies

$$|(\nabla\chi_\eta)(x)\widetilde{\phi}(x)| \le C_\eta e^{-(s_0-2\eta)\lambda} \quad \text{in} \quad \mathbf{R}^2.$$

Similary we have

$$|(\nabla\chi_\eta)(x+\gamma)(\mathbf{T}^B_{-\gamma}\widetilde{\phi})(x)| \le C_\eta e^{-(s_0-2\eta)\lambda} \quad \text{in} \quad \mathbf{R}^2.$$

So, (20) is proved.

Next we compute I_2. Using Lemma 3.2, we have

$$\begin{aligned} & |(\nabla\chi_\eta)(x)[\widetilde{\phi}\nabla\overline{(\mathbf{T}^B_{-\gamma}\widetilde{\phi})} - \overline{(\mathbf{T}^B_{-\gamma}\widetilde{\phi})}\nabla\widetilde{\phi}]\chi_\eta(x+\gamma)| \\ \le & \, C_\eta e^{-(d_V(x,0)+d_V(x+\gamma,0)-2\eta)\lambda} \\ \le & \, C_\eta e^{-(d_V(x,0)+d_V(x,-\gamma)-2\eta)\lambda}. \end{aligned}$$

So, we get

$$I_2 \equiv \int_{\mathbf{R}^2} \chi_{E^{(a)}_{-\gamma}}(\nabla\chi_\eta)(x)\left[\widetilde{\phi}\nabla\overline{(\mathbf{T}^B_{-\gamma}\widetilde{\phi})} - \overline{(\mathbf{T}^B_{-\gamma}\widetilde{\phi})}\nabla\widetilde{\phi}\right]\chi_\eta(x+\gamma)dx \qquad \text{mod} \quad O(\lambda^{-\infty}e^{-s_0\lambda}).$$

Because

$$\chi_\eta(x) = 1 \quad \text{on} \quad \text{supp}\chi_{E^{(a)}_{-\gamma}} \cap \Omega^c, \chi_\eta(x+\gamma) = 1 \quad \text{on} \quad \text{supp}\chi_{E^{(a)}_{-\gamma}} \cap \overline{\Omega},$$

we have

$$\begin{aligned}
I_2 &\equiv \int_\Omega \chi_{E^{(a)}_{-\gamma}} (\nabla \chi_\eta)[\tilde{\phi}\overline{\nabla(\mathbf{T}^B_{-\gamma}\tilde{\phi})} - \overline{(\mathbf{T}^B_{-\gamma}\tilde{\phi})}\nabla\tilde{\phi}]dx \quad \text{mod} \quad O(\lambda^{-\infty}e^{-s_0\lambda}) \\
&= -\int_\Omega \chi_\eta \nabla \chi_{E^{(a)}_{-\gamma}} [\tilde{\phi}\overline{\nabla(\mathbf{T}^B_{-\gamma}\tilde{\phi})} - \overline{(\mathbf{T}^B_{-\gamma}\tilde{\phi})}\nabla\tilde{\phi}]dx \\
&\quad - \int_\Omega \chi_\eta \chi_{E^{(a)}_{-\gamma}} [\tilde{\phi}\overline{\Delta(\mathbf{T}^B_{-\gamma}\tilde{\phi})} - \overline{(\mathbf{T}^B_{-\gamma}\tilde{\phi})}\Delta\tilde{\phi}]dx \\
&\quad + \int_{\partial\Omega} \chi_{E^{(a)}_{-\gamma}} [\tilde{\phi}\frac{\partial}{\partial n}\overline{(\mathbf{T}^{(a)}_{-\gamma}\tilde{\phi})} - \overline{(\mathbf{T}^B_{-\gamma}\tilde{\phi})}\frac{\partial}{\partial n}\tilde{\phi}]\chi_\eta dS.
\end{aligned}$$

Noting that

$$\tilde{\phi}\overline{\nabla(\mathbf{T}^B_{-\gamma}\tilde{\phi})} = O(e^{-(s_0+a-2\eta)\lambda}), \ \overline{(\mathbf{T}^B_{-\gamma}\tilde{\phi})}\nabla\tilde{\phi} = O(e^{-(s_0+a-2\eta)\lambda}) \text{ on } \mathrm{supp}\nabla\chi_{E^{(a)}_{-\gamma}},$$

we have

$$\begin{aligned}
I_2 \equiv &-\int_\Omega \chi_\eta \chi_{E^{(a)}_{-\gamma}} [\tilde{\phi}\overline{\Delta(\mathbf{T}^B_{-\gamma}\tilde{\phi})} - \overline{(\mathbf{T}^B_{-\gamma}\tilde{\phi})}\Delta\tilde{\phi}]dx \\
&+ \int_{\partial\Omega} \chi_{E^{(a)}_{-\gamma}} [\tilde{\phi}\frac{\partial}{\partial n}\overline{(\mathbf{T}^B_{-\gamma}\tilde{\phi})} - \overline{(\mathbf{T}^B_{-\gamma}\tilde{\phi})}\frac{\partial}{\partial n}\tilde{\phi}]\chi_\eta dS \quad \text{mod} \quad O(\lambda^{-\infty}e^{-s_0\lambda})
\end{aligned}$$

for sufficientry small η. We further compute

$$\begin{aligned}
&\tilde{\phi}\overline{\Delta(\mathbf{T}^B_{-\gamma}\tilde{\phi})} - \overline{(\mathbf{T}^B_{-\gamma}\tilde{\phi})}\Delta\tilde{\phi} \\
&= \tilde{\phi}\overline{H(\lambda)\mathbf{T}^B_{-\gamma}\tilde{\phi}} - \overline{(\mathbf{T}^B_{-\gamma}\tilde{\phi})}H(\lambda)\tilde{\phi} - 2bi\tilde{\phi}\overline{L\mathbf{T}^B_{-\gamma}\tilde{\phi}} - 2bi\overline{(\mathbf{T}^B_{-\gamma}\tilde{\phi})}L\tilde{\phi} \\
&= \tilde{\phi}\overline{\mathbf{T}^B_{-\gamma}H(\lambda)\tilde{\phi}} - \overline{(\mathbf{T}^B_{-\gamma}\tilde{\phi})}H(\lambda)\tilde{\phi} - 2bi\tilde{\phi}\overline{L\mathbf{T}^B_{-\gamma}\tilde{\phi}} - 2bi\overline{(\mathbf{T}^B_{-\gamma}\tilde{\phi})}L\tilde{\phi}.
\end{aligned}$$

Because

$$\begin{aligned}
H(\lambda)\tilde{\phi} &= \tilde{\mathcal{E}}(\lambda)\tilde{\phi} && \text{on} \quad \mathrm{supp}\chi_\eta, \\
\mathbf{T}^B_{-\gamma}H(\lambda)\tilde{\phi} &= \tilde{\mathcal{E}}(\lambda)\mathbf{T}^B_{-\gamma}\tilde{\phi} && \text{on} \quad \mathrm{supp}\chi_\eta(x+\gamma),
\end{aligned}$$

we have

$$\begin{aligned}
&\tilde{\phi}\overline{\Delta(\mathbf{T}^B_{-\gamma}\tilde{\phi})} - \overline{(\mathbf{T}^B_{-\gamma}\tilde{\phi})}\Delta\tilde{\phi} \\
&= \tilde{\phi}\tilde{\mathcal{E}}(\lambda)\overline{(\mathbf{T}^B_{-\gamma}\tilde{\phi})} - \overline{(\mathbf{T}^B_{-\gamma}\tilde{\phi})}\tilde{\mathcal{E}}(\lambda)\tilde{\phi} - 2bi\tilde{\phi}\overline{L\mathbf{T}^B_{-\gamma}\tilde{\phi}} - 2bi\overline{(\mathbf{T}^B_{-\gamma}\tilde{\phi})}L\tilde{\phi} \\
&= -2bi\tilde{\phi}\overline{L\mathbf{T}^B_{-\gamma}\tilde{\phi}} - 2bi\overline{(\mathbf{T}^B_{-\gamma}\tilde{\phi})}L\tilde{\phi}
\end{aligned}$$

on $\mathrm{supp}\chi_{E^{(a)}_{-\gamma}} \cap \mathrm{supp}\chi_\eta$. So, we get

$$\begin{aligned}
I_2 \equiv &-2bi\int_\Omega \chi_\eta \chi_{E^{(a)}_{-\gamma}} [\tilde{\phi}\overline{L\mathbf{T}^B_{-\gamma}\tilde{\phi}} + \overline{\mathbf{T}^B_{-\gamma}\tilde{\phi}}L\tilde{\phi}]dx \\
&+ \int_{\partial\Omega} \chi_{E^{(a)}_{-\gamma}} \left[\tilde{\phi}\frac{\partial}{\partial n}\overline{(\mathbf{T}^B_{-\gamma}\tilde{\phi})} - \overline{(\mathbf{T}^B_{-\gamma}\tilde{\phi})}\frac{\partial}{\partial n}\tilde{\phi}\right]\chi_\eta dS \quad \text{mod} \quad O(\lambda^{-\infty}e^{-s_0\lambda}).
\end{aligned}$$

From this formula one can easily derive (21) by noting that

(4.23) $$\chi_\eta(x) = 1 \quad \text{on} \quad \widetilde{\Gamma}_{-\gamma},$$

(4.24) $$d_V(x,0) + d_V(x,-\gamma) \geq s_0 + a \quad \text{if} \quad \chi_{E^{(a)}_{-\gamma}}(x) \neq 1.$$

Finally we compute I_3. A similar argument shows that

$$I_3 \equiv -2bi \int_{\mathbf{R}^2} \chi_{E^{(a)}_{-\gamma}} (L\chi_\eta)\widetilde{\phi}\overline{(\mathbf{T}^B_{-\gamma}\widetilde{\phi})}\chi_\eta(x+\gamma)dx \quad \text{mod} \quad O(\lambda^{-\infty}e^{-s_0\lambda}).$$

Because

$$(L\chi_\eta)(x) = 0 \quad \text{on} \quad \text{supp}\chi_{E^{(a)}_{-\gamma}} \cap \Omega^c,$$
$$\chi_\eta(x+\gamma) = 1 \quad \text{on} \quad \text{supp}\chi_{E^{(a)}_{-\gamma}} \cap \Omega,$$

we have

$$\begin{aligned} I_3 &\equiv -2bi \int_\Omega \chi_{E^{(a)}_{-\gamma}} (L\chi_\eta)\widetilde{\phi}\overline{(\mathbf{T}^B_{-\gamma}\widetilde{\phi})}dx \\ &= 2bi \int_\Omega (L\chi_{E^{(a)}_{-\gamma}})\chi_\eta\widetilde{\phi}\overline{(\mathbf{T}^B_{-\gamma}\widetilde{\phi})}dx \\ &\quad + 2bi \int_\Omega \chi_{E^{(a)}_{-\gamma}}\chi_\eta(L\widetilde{\phi})\overline{(\mathbf{T}^B_{-\gamma}\widetilde{\phi})}dx \\ &\quad + 2bi \int_\Omega \chi_{E^{(a)}_{-\gamma}}\chi_\eta\widetilde{\phi}\overline{L\mathbf{T}^B_{-\gamma}\widetilde{\phi}}dx \\ &\quad - 2bi \int_{\partial\Omega} \chi_{E^{(a)}_{-\gamma}}\chi_\eta\widetilde{\phi}\overline{\mathbf{T}^B_{-\gamma}\widetilde{\phi}}(x_2n_1 - x_1n_2)dS. \end{aligned}$$

Noting that

$$\widetilde{\phi}\mathbf{T}^B_{-\gamma}\widetilde{\phi} = O(e^{-(s_0+a-2\eta)\lambda}) \quad \text{on} \quad \text{supp}L\chi_{E^{(a)}_{-\gamma}},$$

we have

$$2bi \int_\Omega (L\chi_{E^{(a)}_{-\gamma}})\chi_\eta\widetilde{\phi}\overline{(\mathbf{T}^B_{-\gamma}\widetilde{\phi})}dx \equiv 0 \quad \text{mod} \quad O(\lambda^{-\infty}e^{-s_0\lambda})$$

for sufficiently large λ. (23), (24) imply that

$$\begin{aligned} &\int_{\partial\Omega} \chi_{E^{(a)}_{-\gamma}}\chi_\eta\widetilde{\phi}\overline{\mathbf{T}^B_{-\gamma}\widetilde{\phi}}(x_2n_1 - x_1n_2)dS \\ &\equiv \int_{\widetilde{\Gamma}_{-\gamma}} \widetilde{\phi}\overline{\mathbf{T}^B_{-\gamma}\widetilde{\phi}}(x_2n_1 - x_1n_2)dS \quad \text{mod} \quad O(\lambda^{-\infty}e^{-s_0\lambda}). \end{aligned}$$

Therefore we get (22). □

To approximate $(\mathbf{T}^B_\gamma \widetilde{r}, \widetilde{\psi})_{L^2(\mathbf{R}^2)}$ mod $O(\lambda^{-\infty} e^{-s_0\lambda})$, we construct an approximate eigenfunction of $\widetilde{H}(\lambda)$ by the W.K.B. method which we explain below.

For the differential operator

$$H(\lambda) = (D_{x_1} + bx_2)^2 + (D_{x_2} - bx_1)^2 + \lambda^2 V(x) \quad \text{in} \quad \mathbf{R}^2,$$

we construct an asymptotic solution of the following type

$$(a_0(x) + a_1(x)\lambda^{-1} + a_2(x)\lambda^{-2} + \cdots)e^{-\lambda\varphi(x)},$$

where $\varphi(x)$ is a real valued C^∞ function defined near 0 in $\mathbf{R}^2$, and $a_0(x)$, $a_1(x)$, $\cdots$ are complex valued C^∞ functions defined near 0 in $\mathbf{R}^2$. For e_1, e_2, $\cdots$, $e_{N+1} \in \mathbf{C}$, we set

$$a(x) = \sum_{j=0}^{N} a_j(x)\lambda^{-j}, \quad E(\lambda) = \sum_{k=1}^{N+1} e_k \lambda^{2-k}.$$

Then we get the following identity :

$$\begin{aligned}(4.25) \qquad & e^{\lambda\varphi}(H(\lambda) - E(\lambda))\left(\sum_{j=0}^{N} a_j(x)\lambda^{-j}e^{-\lambda\varphi}\right) \\ &= \lambda^2(V - |\nabla\varphi|^2) + \lambda(M_\varphi a_0 - e_1 a_0) \\ &\quad + \sum_{l=0}^{N-1}\left\{M_\varphi a_{l+1} - 2biLa_l + b^2|x|^2 a_l - \Delta a_l - \sum_{\substack{j+k=l+2\\ j\geq 0, k\geq 1}} e_k a_j\right\}\lambda^{-l} \\ &\quad + \lambda^{-N}(-2biLa_N + b^2|x|^2 a_N - \Delta a_N) - \sum_{l=N}^{2N-2}\lambda^{-l}\sum_{\substack{j+k=l+2\\ j\geq 0, k\geq 1}} e_k a_j,\end{aligned}$$

where $L = x_2\partial_{x_1} - x_1\partial_{x_2}$ and $M_\varphi = 2\nabla\varphi\cdot\nabla + \Delta\varphi + 2biL\varphi$.

So, we shall consider the following equations in the neighborhood of the origin :

$$(4.26) \qquad V - |\nabla\varphi|^2 = 0,$$

$$(4.27) \qquad M_\varphi a_0 = e_1 a_0,$$

$$(4.28)_l \qquad M_\varphi a_{l+1} = 2biLa_l - b^2|x|^2 a_l + \Delta a_l + \sum_{\substack{j+k=l+2\\ j\geq 0, k\geq 1}} e_k a_j.$$

If we solve these equations, the right-hand side of (25) is $O(\lambda^{-N})$ in the neighborhood of the origin. Since $V(0) = 0$, special attentions should be paid in solving these eikonal equation (26) and transport equations (27), $(28)_l$. We make use of the arguments of Helffer-Sjöstrand [2].

We first consider the eikonal equation. For $\epsilon \geq 0$ sufficiently small, let Ω_ϵ be the set consisting of $\{0\}$ and the union of the interiors of all minimal geodesics starting from $\{0\}$ of length strictly less than $s_0 - \epsilon$. Here by geodesic we mean the curve satisfying that

$$\begin{cases} \gamma : [0, a] \to \mathbf{R}^2; \text{smooth curve}, \\ \gamma(t) \notin \Gamma \text{ for any } t \in (0, a], \\ \gamma(t) \to 0 \text{ as } t \to +0, \\ \gamma|_{(0,a]} \text{ is a geodesic of } \mathbf{R}^2 \backslash \Gamma \text{ with metric } V dx^2. \end{cases}$$

Ω_0 is an open set. Let $d(x) = d_V(x, 0)$, then we have

$$d(x) \in C^\infty(\Omega_0), |\nabla d(x)|^2 = V(x) \quad \text{in} \quad \Omega_0.$$

Namely, $d(x)$ solves the eikonal equation (26) (see [1] §4.4). Moreover, $d(x)$ has the following property (see [1] §2.3 and 3.2).

$$d(x) = \frac{1}{2}\sqrt{\mu_1}x_1^2 + \frac{1}{2}\sqrt{\mu_2}x_2^2 + O(|x|^3) \quad (|x| \to 0). \tag{4.29}$$

Next, we consider the transport equations. Let

$$X = 2\nabla d \cdot \nabla \quad \text{in} \quad \Omega_0.$$

Since this vector field vanishes at the origin, we must impose compatibility conditions on transport equations to guarantee the solvability.

Lemma 4.5. *Let $a(x)$ and $b(x)$ be $\mathbf{C}$-valued C^∞ functions in Ω_0 with*

$$a(0) = b(0) = 0.$$

Then, for any $c \in \mathbf{C}$, the initial value problem

$$\begin{cases} Xu = au + b \quad in \quad \Omega_0 \\ u(0) = c \end{cases}$$

has a unique solution.

The proof of this Lemma is the same as those in [1] Propositions 2.3.7 and 4.4.2, where this fact is proved when $a(x)$ and $b(x)$ are real valued.

Now we determine e_1, e_2, $\cdots$ in such a way that the above compatibility conditions are satisfied. To solve the first transport equation (27) :

$$2\nabla d \cdot \nabla a_0 = -(\Delta d + 2biLd - e_1)a_0,$$

we set

$$e_1 = (\Delta d)(0) + 2bi(Ld)(0) = (\Delta d)(0).$$

Using (29), we have $e_1 = \sqrt{\mu_1} + \sqrt{\mu_2}$. Lemma 4.5 implies that (27) with initial condition $a_0(0) = 1$ has a unique solution defined in Ω_0.

Next, we consider $(28)_0$:

$$2\nabla d \cdot \nabla a_1 = -(\Delta d + 2biLd - e_1)a_1 + (2biLa_0 - b^2|x|^2 a_0 + \Delta a_0 + e_2 a_0).$$

By choosing e_2 in such a way that

$$e_2 = -\frac{1}{a_0(0)}(2bi(La_0)(0) + (\Delta a_0)(0)) = -(\Delta a_0)(0),$$

one can see by Lemma 4.5 that $(28)_0$ with initial condition $a_1(0) = 0$ has a unique solution defined in Ω_0.

Inductively, $(28)_l$ $(l = 1, 2, \cdots)$ with initial condition $a_{l+1}(0) = 0$ has a unique solution defined in Ω_0 if we set $e_{l+2} = -(\Delta a_l)(0)$.

Using the Borel procedure, we have the following.

Lemma 4.6. *One can construct*

$$\begin{aligned}
&e_1, e_2, \cdots \in \mathbf{R}\ (e_1 = \sqrt{\mu_1} + \sqrt{\mu_2}),\\
&\mathcal{E}(\lambda) \sim e_1\lambda + e_2 + e_3\lambda^{-1} + \cdots \quad (\lambda \to \infty),\\
&\mathbf{C}\text{-valued } C^\infty \text{ functions } a_0(x), a_1(x), \cdots \quad \text{in} \quad \Omega_0,\\
&\mathbf{C}\text{-valued } C^\infty \text{ function } a(x,\lambda) \quad \text{in} \quad \Omega_\epsilon,
\end{aligned}$$

satisfying that

$$\begin{cases}
a_0(x) \neq 0 \quad \text{in} \quad \Omega_0, a_0(0) = 1, a_j(0) = 0\ (j \geq 1),\\
a(x,\lambda) \sim \sum_{j=0}^{\infty} a_j(x)\lambda^{-j},\\
(H(\lambda) - \mathcal{E}(\lambda))\theta(\lambda) = O(\lambda^{-\infty})e^{-\lambda d(x)} \text{ in } \Omega_\epsilon \text{ where } \theta(\lambda) = \lambda^{1/2}a(x,\lambda)e^{-\lambda d(x)}.
\end{cases}$$

More precisely,

$$\begin{aligned}
&\max_{|\alpha|\leq 2} \sup_{x\in\Omega_\epsilon} |\partial_x^\alpha(a(x,\lambda) - \sum_{j=0}^{N} a_j\lambda^{-j})| = O(\lambda^{-(N+1)}) \quad \text{for any} \quad N \in \mathbf{N},\\
&\sup_{x\in\Omega_\epsilon} |e^{\lambda d(x)}(H(\lambda) - \mathcal{E}(\lambda))\theta(\lambda)| = O(\lambda^{-\infty}).
\end{aligned}$$

We fix $\epsilon > 0$. By deviding $\theta(\lambda)$ by $\|\theta(\lambda)\|_{L^2(\Omega_\epsilon)} \sim \sqrt{2\pi} + O(\lambda^{-1})$, one can normalize $\theta(\lambda)$ so that $\|\theta(\lambda)\|_{L^2(\Omega_\epsilon)} = 1$. Let K be a compact subset of Ω_ϵ. We can choose $\eta > 0$ sufficiently small such that $\Omega_\epsilon \subset B_V(0, s_0 - \eta)$. Let $\widehat{K}$ be the set composed of all minimal geodesics joining K to $\{0\}$. Then, $\widehat{K} \subset \Omega_\epsilon$. We choose $\widetilde{\Omega}$: an open neighborhood of $\widehat{K}$ such that $\widetilde{\Omega} \subset\subset \Omega_\epsilon$. We choose $\chi \in C_0^\infty(\mathbf{R}^2)$ such that $\chi = 1$ in a neighborhood of $\widehat{K}$ and $\mathrm{supp}\chi \subset \widetilde{\Omega}$. Recall that $\widetilde{\phi}(\lambda)$ is a normalized first eigenfunction of $\widetilde{H}(\lambda)$.

Let $E_1 = \{k(\chi\theta(\lambda)); k \in \mathbf{C}\}$ and $F_1 = \{k\widetilde{\phi}(\lambda); k \in \mathbf{C}\}$. Then the above lemma and Proposition 4.1 imply $\overrightarrow{d}(E_1, F_1) = O(\lambda^{-\infty})$. So we have

$$|(\chi\theta(\lambda), \widetilde{\phi}(\lambda))_{L^2(\mathbf{R}^2)}| = 1 + O(\lambda^{-\infty}).$$

So we can assume that $\widetilde{\phi}(\lambda)$ satisfies

$$(\chi\theta(\lambda), \widetilde{\phi}(\lambda))_{L^2(\mathbf{R}^2)} > 0$$

for sufficiently large λ.

Let $\omega(\lambda) = \chi(\widetilde{\phi}(\lambda) - \theta(\lambda))$. By the same argument as in [1] §4.4 of Helffer, we have the following lemma.

Lemma 4.7. *There exists $\widetilde{K}$: a neighborhood of $\widehat{K}$ with $\widetilde{K} \subset\subset \widetilde{\Omega}$ such that*

$$\omega = O(\lambda^{-\infty})e^{-\lambda d(x)} \quad in \quad H^2(\widetilde{K}).$$

This Lemma together with (18) implies that for any $\gamma \in \Lambda$ we have

$$\text{(4.30)}\quad (\mathbf{T}_\gamma^B \widetilde{r}, \widetilde{\psi})_{L^2(\mathbf{R}^2)} \equiv \int_{\widetilde{\Gamma}_{-\gamma}} \left\{ \theta \frac{\partial}{\partial n}(\overline{\mathbf{T}_{-\gamma}^B \theta}) - (\overline{\mathbf{T}_{-\gamma}^B \theta}) \frac{\partial}{\partial n}\theta \right\} dS$$
$$- 2bi \int_{\widetilde{\Gamma}_{-\gamma}} \theta \overline{\mathbf{T}_{-\gamma}^B \theta}(x_2 n_1 - x_1 n_2) dS \bmod O(\lambda^{-\infty} e^{-s_0\lambda}).$$

We have now arrived at the final step for proving Theorem B.

Lemma 4.8. *For $\gamma \in \Lambda$, there exists a constant $\widetilde{b_\gamma} \in \mathbf{C}\backslash\{0\}$ such that*

$$\text{(4.31)}\qquad (\mathbf{T}_\gamma^B \widetilde{r}, \widetilde{\psi})_{L^2(\mathbf{R}^2)} = (\widetilde{b_\gamma}\lambda^{3/2} + O(\lambda^{1/2}))e^{-s_0\lambda} \quad (\lambda \to \infty).$$

Proof. We insert $\theta(\lambda) = \lambda^{1/2}a(x, \lambda)e^{-\lambda d(x)}$ in (30) and use the definition of $\mathbf{T}_\gamma^B$: $(\mathbf{T}_\gamma^B u)(x) = e^{ib\gamma_1\gamma_2}e^{-ib(x_1\gamma_2 - x_2\gamma_1)}u(x - \gamma)$ to get

$$(\mathbf{T}_\gamma^B \widetilde{r}, \widetilde{\psi})_{L^2(\mathbf{R}^2)}$$

$$\begin{aligned}
&\equiv \lambda^2 e^{-ib\gamma_1\gamma_2}\int_{\widetilde{\Gamma}_{-\gamma}} a(x,\lambda)\overline{a(x+\gamma,\lambda)}e^{-ib(x_1\gamma_2-x_2\gamma_1)} \\
&\qquad \times\left(-\frac{\partial}{\partial n}d(x+\gamma)+\frac{\partial}{\partial n}d(x)\right)e^{-\lambda(d(x)+d(x+\gamma))}dS \\
&\quad + \lambda e^{-ib\gamma_1\gamma_2}\int_{\widetilde{\Gamma}_{-\gamma}}\left\{a(x,\lambda)\frac{\partial}{\partial n}(e^{-ib(x_1\gamma_2-x_2\gamma_1)}\overline{a(x+\gamma,\lambda)})-e^{-ib(x_1\gamma_2-x_2\gamma_1)}\right. \\
&\qquad \left.\times\overline{a(x+\gamma,\lambda)}\frac{\partial}{\partial n}a(x,\lambda)\right\}e^{-\lambda(d(x)+d(x+\gamma))}dS \\
&\quad - 2bi\lambda e^{-ib\gamma_1\gamma_2}\int_{\widetilde{\Gamma}_{-\gamma}} a(x,\lambda)\overline{a(x+\gamma,\lambda)}e^{-ib(x_1\gamma_2-x_2\gamma_1)}(x_2n_1-x_1n_2) \\
&\qquad \times e^{-\lambda(d(x)+d(x+\gamma))}dS \\
&=: I_{\gamma_1}+I_{\gamma_2}+I_{\gamma_3}.
\end{aligned}$$

We can assume that $\widetilde{\Gamma}_{-\gamma}$ intersects $\kappa_{-\gamma}$ transversally at $x_{-\gamma}$ where $x_{-\gamma}$ is the only point in $\widetilde{\Gamma}_{-\gamma}\cap\kappa_{-\gamma}$. Let η be the angle between $\overrightarrow{n}$ and ∇d at $x_{-\gamma}(\pi/2<\eta\leq\pi)$. Because $|\nabla d(x)|^2=V(x)$ in Ω_0, we have

$$\begin{aligned}
\frac{\partial}{\partial n}d(x_{-\gamma}) &= \overrightarrow{n}\cdot\nabla d(x_{-\gamma}) \\
&= |\nabla d(x_{-\gamma})|\cos\eta \\
&= \sqrt{V(x_{-\gamma})}\cos\eta,
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial}{\partial n}d(x+\gamma)|_{x=x_{-\gamma}} &= \frac{\partial}{\partial n}d_V(-\gamma,x)|_{x=x_{-\gamma}} \\
&= -\sqrt{V(x_{-\gamma})}\cos\eta.
\end{aligned}$$

So, decreasing a if necessary, we may assume that there exists a constant $C_0>0$ such that

$$-\frac{\partial}{\partial n}d(x+\gamma)+\frac{\partial}{\partial n}d(x)\leq -C_0 \quad \text{in} \quad \widetilde{\Gamma}_{-\gamma}. \tag{4.32}$$

$(H.8)$ implies

$$d(x)+d(x+\gamma)\geq s_0+Cd_V(x,x_{-\gamma})^2 \quad \text{in} \quad \widetilde{\Gamma}_{-\gamma}. \tag{4.33}$$

First we compute $I_{\gamma,1}$. Let

$$b_0(x)=a_0(x)\overline{a_0(x+\gamma)}e^{-ib(x_1\gamma_2-x_2\gamma_1)}\left(\frac{\partial}{\partial n}d(x)-\frac{\partial}{\partial n}d(x+\gamma)\right), \tag{4.34}$$

$$J(\lambda)=\int_{\widetilde{\Gamma}_{-\gamma}} b_0(x)e^{-\lambda(d(x)+d(x+\gamma))}dS. \tag{4.35}$$

Then

$$b_0(x) \neq 0 \quad \text{in} \quad \widetilde{\Gamma}_{-\gamma}. \tag{4.36}$$

Let

$$x = c(t)(-\epsilon \leq t \leq \epsilon), c(0) = x_{-\gamma}, |c'(t)| = 1 \quad \text{on} \quad [-\epsilon, \epsilon] \tag{4.37}$$

be the curve representing $\widetilde{\Gamma}_{-\gamma}$ near $x_{-\gamma}$. Let

$$d_0(t) = d(c(t)) + d(c(t) + \gamma), \quad t \in [-\epsilon, \epsilon].$$

Because

$$c(t) = x_{-\gamma} + tp + O(t^2) \quad (t \to 0), \quad p \in \mathbf{R}^2, \ |p| = 1,$$

and there exists a constant $C' > 0$ such that

$$d_V(x, x_{-\gamma}) \geq C'|x - x_{-\gamma}| \quad \text{near} \quad x_{-\gamma},$$

(33) implies that there exists a constant $C'' > 0$ such that

$$d_0(t) \geq s_0 + C''t^2 \quad \text{in a neighborhood of } 0.$$

So, we have

$$d_0(t) = s_0 + \frac{1}{2} d_0''(0)t^2 + O(t^3) \quad (t \to 0), d_0''(0) > 0.$$

Then, we can apply the stationary phase method and get

$$\begin{aligned} J(\lambda) &\equiv \int_{-\epsilon}^{\epsilon} b_0(c(t)) e^{-\lambda d_0(t)} dt \\ &= e^{-s_0\lambda}(b_0(x_{-\gamma})\mu^{-1/2}\lambda^{-1/2}\sqrt{\pi} + O(\lambda^{-3/2})), \end{aligned}$$

where $\mu = (1/2)d_0''(0)$.

So, we have

$$I_{\gamma,1} \equiv e^{-s_0\lambda}(\widetilde{b_\gamma}\lambda^{3/2} + O(\lambda^{1/2})),$$

where

$$\widetilde{b_\gamma} = e^{-ib\gamma_1\gamma_2} b_0(x_{-\gamma})\sqrt{\pi}\mu^{-1/2} \in \mathbf{C}\backslash\{0\}.$$

A similar argument shows that

$$I_{\gamma,2} = e^{-s_0\lambda}O(\lambda^{1/2}), I_{\gamma,3} = e^{-s_0\lambda}O(\lambda^{1/2}).$$

So, we get the conclusion. □

We are now in a position of proving Theorem B.

Let $f(\theta) = \sum_{\gamma\in\Lambda} e^{i\gamma\cdot\theta}\widetilde{b_\gamma}$ for any $\theta \in E^*$. (17) and (31) imply that $\widetilde{b_\gamma} = \overline{\widetilde{b_{-\gamma}}}$ for any $\gamma \in \Lambda$. Let $b_0 = \max_{\theta\in E^*} f(\theta) - \min_{\theta\in E^*} f(\theta)$. Combining (16) and (31), we get

$$\text{(4.38)} \qquad \textit{length of } \mathcal{E}_1(\lambda; E^*) = (b_0\lambda^{3/2} + O(\lambda^{1/2}))e^{-s_0\lambda} \quad (\textit{as } \lambda \to \infty).$$

Since $\widetilde{b_\gamma} \neq 0$ for $\gamma \in \Lambda$, $f(\theta)$ is a non-constant real function. So we have $b_0 > 0$ and complete the proof of Theorem B. □

Appendix Eigenvalues and eigenfunctions of $H_0(\lambda)$

Let us first recall the following well-known fact on the Weyl operator $a^w(x, D_x)$ (cf. [7]). For a symplectic transformation χ on $\mathbf{R}^n_x \times \mathbf{R}^n_\xi$, there exists a unitary operator U on $L^2(\mathbf{R}^n)$ such that

$$U^{-1}a^w(x, D_x)U = (a \circ \chi)^w(x, D_x).$$

We use only the following two cases.

Case i) When χ is the map interchanging x_j, ξ_j by $\xi_j, -x_j$ respectively, leaving the other coodinates unchanged, U is the partial Fourier transformation with respect to x_j.

Case ii) If χ is the map $(x, \xi) \mapsto (Tx, {}^tT^{-1}\xi)$ where T is an $n \times n$ real matrix with $\det T \neq 0$, then $(Uf)(x) = |\det T|^{-1/2} f(T^{-1}x)$.

Next we describe the computation of eigenvalues and eigenfunctions of $H_0(\lambda)$. For $x = (x_1, x_2), \xi = (\xi_1, \xi_2) \in \mathbf{R}^2$, we set

$$p(x, \xi) = (\xi_1 + bx_2)^2 + (\xi_2 - bx_1)^2 + \lambda^2(\mu_1 x_1^2 + \mu_2 x_2^2).$$

Then, we have

$$(A.1) \qquad H_0(\lambda) = p^w(x, D_x).$$

Let U_1 be the Fourier transformation with respect to x_1. We set

$$\begin{aligned} p_1(x, \xi) &= p(\xi_1, x_2, -x_1, \xi_2) \\ &= \lambda^2\mu_1\xi_1^2 + (\xi_2 - b\xi_1)^2 + (-x_1 + bx_2)^2 + \lambda^2\mu_2 x_2^2. \end{aligned}$$

Then we have

$$p^w(x, D_x) = U_1 p_1^w(x, D_x) U_1^{-1}. \tag{A.2}$$

Let $T = \begin{pmatrix} \sqrt{\mu_1}\lambda & -b \\ 0 & 1 \end{pmatrix}$, and we set

$$\begin{aligned} p_2(x,\xi) &= p_1(Tx, {}^tT^{-1}\xi) \\ &= \xi_1^2 + \xi_2^2 + (x_1 x_2) \begin{pmatrix} \mu_1\lambda^2 & -2b\sqrt{\mu_1}\lambda \\ -2b\sqrt{\mu_1}\lambda & 4b^2 + \mu_2\lambda^2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \end{aligned}$$

For $f \in L^2(\mathbf{R}^2)$, we set

$$(U_2 f)(x) = \mu_1{}^{-1/4}\lambda^{-1/2} f(T^{-1}x).$$

U_2 is unitary on $L^2(\mathbf{R}^2)$, and we have

$$p_1^w(x, D_x) = U_2 p_2^w(x, D_x) U_2^{-1}. \tag{A.3}$$

Next, we diagonalize the matrix $\begin{pmatrix} \mu_1\lambda^2 & -2b\sqrt{\mu_1}\lambda \\ -2b\sqrt{\mu_1}\lambda & \mu_2\lambda^2 + 4b^2 \end{pmatrix}$ by an orthogonal matrix. Let $m_1(\lambda)$ and $m_2(\lambda)$ be the eigenvalues of this matrix such that $m_1(\lambda) < m_2(\lambda)$. Namely, we set

$$m_1(\lambda) = \frac{1}{2}(\mu_1 + \mu_2)\lambda^2 + 2b^2 - \left\{\frac{1}{4}(\mu_1 - \mu_2)^2\lambda^4 + 2b^2(\mu_1 + \mu_2)\lambda^2 + 4b^4\right\}^{1/2},$$

$$m_2(\lambda) = \frac{1}{2}(\mu_1 + \mu_2)\lambda^2 + 2b^2 + \left\{\frac{1}{4}(\mu_1 - \mu_2)^2\lambda^4 + 2b^2(\mu_1 + \mu_2)\lambda^2 + 4b^4\right\}^{1/2}.$$

Then, we get

$$m_1(\lambda) = \begin{cases} \mu_1\lambda^2 + O(1) & (\mu_2 > \mu_1) \\ \mu_1\lambda^2 - 2b\mu_1\lambda + O(1) & (\mu_2 = \mu_1) \\ \mu_2\lambda^2 + O(1) & (\mu_2 < \mu_1), \end{cases} \tag{A.4}$$

$$m_2(\lambda) = \begin{cases} \mu_2\lambda^2 + O(1) & (\mu_2 > \mu_1) \\ \mu_1\lambda^2 + 2b\mu_1\lambda + O(1) & (\mu_2 = \mu_1) \\ \mu_1\lambda^2 + O(1) & (\mu_2 < \mu_1). \end{cases} \tag{A.5}$$

Let $A(\lambda) = (a_1(\lambda), a_2(\lambda))$ where

(A.6)

$$(A.7)\quad \begin{aligned} a_1(\lambda) &= \begin{pmatrix} a_{11}(\lambda) \\ a_{21}(\lambda) \end{pmatrix} \\ &= \{(\lambda^2\mu_2 + 4b^2 - m_1(\lambda))^2 + 4b^2\mu_1\lambda^2\}^{-1/2} \times \begin{pmatrix} \lambda^2\mu_2 + 4b^2 - m_1(\lambda) \\ 2b\sqrt{\mu_1}\lambda \end{pmatrix} \\ a_2(\lambda) &= \begin{pmatrix} a_{12}(\lambda) \\ a_{22}(\lambda) \end{pmatrix} \\ &= \{(\mu_1\lambda^2 - m_2(\lambda))^2 + 4b^2\mu_1\lambda^2\}^{-1/2} \times \begin{pmatrix} 2b\sqrt{\mu_1}\lambda \\ \mu_1\lambda^2 - m_2(\lambda) \end{pmatrix} \end{aligned}$$

Then $A(\lambda)$ is an orthogonal matrix and the following equality holds :

$${}^tA(\lambda)\begin{pmatrix} \mu_1\lambda^2 & -2b\sqrt{\mu_1}\lambda \\ -2b\sqrt{\mu_1}\lambda & \lambda^2\mu_2 + 4b^2 \end{pmatrix} A(\lambda) = \begin{pmatrix} m_1(\lambda) & 0 \\ 0 & m_2(\lambda) \end{pmatrix}$$

Let

$$p_3(x,\xi) = p_2(A(\lambda)x, A(\lambda)\xi) = \xi_1^2 + \xi_2^2 + m_1(\lambda)x_1^2 + m_2(\lambda)x_2^2.$$

Then

$$p_3^w(x, D_x) = -\Delta + m_1(\lambda)x_1^2 + m_2(\lambda)x_2^2.$$

For $f \in L^2(\mathbf{R}^2)$, we set

$$(U_3 f)(x) = f({}^tA(\lambda)x).$$

U_3 is unitary on $L^2(\mathbf{R}^2)$, and we have

$$(A.8)\qquad p_2^w(x, D_x) = U_3 p_3^w(x, D_x) U_3^{-1}.$$

Let $U = U_1U_2U_3$. U is unitary on $L^2(\mathbf{R}^2)$. So, $(A.1)$, $(A.2)$, $(A.3)$, and $(A.8)$ imply

$$(A.9)\qquad H_0(\lambda) = U(-\Delta + m_1(\lambda)x_1^2 + m_2(\lambda)x_2^2)U^{-1}.$$

Namely, $H_0(\lambda)$ is unitarily equivalent to the Harmonic oscillator

$$-\Delta + m_1(\lambda)x_1^2 + m_2(\lambda)x_2^2.$$

The eigenvalues of $-\Delta + m_1(\lambda)x_1^2 + m_2(\lambda)x_2^2$ in $L^2(\mathbf{R}^2)$ are

$$(2j+1)\sqrt{m_1(\lambda)} + (2k+1)\sqrt{m_2(\lambda)} \quad (j, k \in \mathbf{N}),$$

and the corresponding eigenfunctions is

$$\begin{aligned} w_{j,k} &= m_1(\lambda)^{1/8} m_2(\lambda)^{1/8} Q_j(m_1(\lambda)^{1/4}x_1) Q_k(m_2(\lambda)^{1/4}x_2) \\ &\quad \times \exp\left(-\frac{1}{2}m_1(\lambda)^{1/2}x_1^2 - \frac{1}{2}m_2(\lambda)^{1/2}x_2^2\right), \end{aligned}$$

where Q_j is the Hermite polynomial of degree j. $\{w_{j,k}\}_{j,k\geq 0}$ is a complete orthonormal system in $L^2(\mathbf{R}^2)$. Therefore, the eigenvalues of $H_0(\lambda)$ are

$$\widetilde{\mathcal{E}}_{j,k}(\lambda) = (2j+1)\sqrt{m_1(\lambda)} + (2k+1)\sqrt{m_2(\lambda)} \quad (j,k \in \mathbf{N}),$$

and the corresponding eigenfunction is $(Uw_{j,k})(\lambda; x)$. So, $(A.4)$, and $(A.5)$ implies

$$\widetilde{\mathcal{E}}_{j,k}(\lambda) = v_{j,k}\lambda + O(1) \quad (\lambda \to \infty),$$

where $v_{j,k}(\lambda) = (2j+1)\sqrt{\min(\mu_1,\mu_2)} + (2k+1)\sqrt{\max(\mu_1,\mu_2)}$.

Next, we compute $(Uw_{j,k})(\lambda; x)$. We have

$$\begin{aligned} &(U_2U_3w_{j,k})(\lambda) \\ &= m_1(\lambda)^{1/8}m_2(\lambda)^{1/8}\mu_1^{-1/4}\lambda^{-1/2} \\ &\quad \times Q_j(m_1(\lambda)^{1/4}(a_{11}(\lambda)\lambda^{-1}\mu_1^{-1/2}(x_1+bx_2) + a_{21}(\lambda)x_2)) \\ &\quad \times Q_k(m_2(\lambda)^{1/4}(a_{12}(\lambda)\lambda^{-1}\mu_1^{-1/2}(x_1+bx_2) + a_{22}(\lambda)x_2)) \\ &\quad \times \exp\{(-m_1(\lambda)^{1/2}(a_{11}(\lambda)\lambda^{-1}\mu_1^{-1/2}(x_1+bx_2) + a_{21}(\lambda)x_2)^2 \\ &\quad - m_2(\lambda)^{1/2}(a_{12}(\lambda)\lambda^{-1}\mu_1^{-1/2}(x_1+bx_2) + a_{22}(\lambda)x_2)^2)/2\}. \end{aligned}$$

Let

$$\begin{aligned} c_{11}(\lambda) &= m_1(\lambda)^{1/4}\mu_1^{-1/2}\lambda^{-1}a_{11}(\lambda), \\ c_{12}(\lambda) &= m_2(\lambda)^{1/4}\mu_1^{-1/2}\lambda^{-1}a_{12}(\lambda), \\ c_{21}(\lambda) &= m_1(\lambda)^{1/4}(\mu_1^{-1/2}\lambda^{-1}ba_{11}(\lambda) + a_{21}(\lambda)), \\ c_{22}(\lambda) &= m_2(\lambda)^{1/4}(\mu_1^{-1/2}\lambda^{-1}ba_{12}(\lambda) + a_{22}(\lambda)). \end{aligned}$$

We have

$$\begin{aligned} &(U_2U_3w_{j,k})(\lambda; x) \\ &= m_1(\lambda)^{1/8}m_2(\lambda)^{1/8}\mu_1^{-1/4}\lambda^{-1/2}Q_j(c_{11}(\lambda)x_1 + c_{21}(\lambda)x_2)Q_k(c_{12}(\lambda)x_1 + c_{22}(\lambda)x_2) \\ &\quad \times \exp\left\{-\frac{1}{2}(c_{11}(\lambda)x_1 + c_{21}(\lambda)x_2)^2 - \frac{1}{2}(c_{12}(\lambda)x_1 + c_{22}(\lambda)x_2)^2\right\}. \end{aligned}$$

Let

$$f(x_1, x_2)$$

$$= (c_{11}(\lambda)x_1 + c_{21}(\lambda)x_2)^n(c_{12}(\lambda)x_1 + c_{22}(\lambda)x_2)^m$$
$$\times \exp\left\{-\frac{1}{2}(c_{11}(\lambda)x_1 + c_{21}(\lambda)x_2)^2 - \frac{1}{2}(c_{12}(\lambda)x_1 + c_{22}(\lambda)x_2)^2\right\} \quad (n, m \in \mathbf{N}).$$

A direct computation implies that

$$(A.10) \int e^{-ix_1\cdot\xi_1} f(\xi_1, x_2)d\xi_1$$
$$= (-c_{11}D_{x_1} + c_{21}x_2)^n(-c_{12}D_{x_1} + c_{21}x_2)^m \left\{\exp\left(-\frac{(c_{11}c_{22} - c_{12}c_{21})^2}{2(c_{11}^2 + c_{12}^2)}x_2^2\right)\right.$$
$$\left.\times \exp\left(i\frac{c_{11}c_{21} + c_{12}c_{22}}{c_{11}^2 + c_{12}^2}\right) \times \frac{\sqrt{2\pi}}{\sqrt{c_{11}^2 + c_{12}^2}} \exp\left(-\frac{1}{2(c_{11}^2 + c_{12}^2)}x_1^2\right)\right\}.$$

Because $(Uw_{j,k})(\lambda; x)$ is the Fourier transform with respect to x_1 of $(U_2U_3w_{j,k})(\lambda; x)$, $(Uw_{j,k})(\lambda; x)$ is a linear combination of $(A.10)$ whose coefficients are independent of λ.

Using $(A.4)$, $(A.5)$, $(A.6)$, and $(A.7)$, we have that there exist positive constants k_1, k_2, k_3, k_{11}, k_{12}, k_{21}, and k_{22} such that

$$(A.11) \qquad k_1\lambda \le \frac{1}{c_{11}(\lambda)^2 + c_{12}(\lambda)^2} \le k_2\lambda,$$
$$|c_{11}(\lambda)| \le k_{11}\lambda^{-1/2}, \quad |c_{12}(\lambda)| \le k_{12}\lambda^{-1/2},$$
$$|c_{21}(\lambda)| \le k_{21}\lambda^{1/2}, \quad |c_{22}(\lambda)| \le k_{22}\lambda^{1/2},$$
$$|c_{11}(\lambda)c_{21}(\lambda) + c_{12}(\lambda)c_{22}(\lambda)| \le k_3$$

for $\lambda \ge 1$.

Noting that

$$(c_{11}(\lambda)c_{22}(\lambda) - c_{12}(\lambda)c_{21}(\lambda))^2 = m_1(\lambda)^{1/2}m_2(\lambda)^{1/2}\mu_1^{-1}\lambda^{-2},$$

we can find positive constants k_4 and k_5 such that

$$(A.12) \qquad k_4 \le (c_{11}(\lambda)c_{22}(\lambda) - c_{12}(\lambda)c_{22}(\lambda))^2 \le k_5$$

for $\lambda \ge 1$. So we get

$$(A.13) \qquad |(Uw_{j,k})(\lambda; x)| \le C_{j,k}\lambda^{1/2}\exp(-c\lambda|x|^2) \quad \text{on} \quad \mathbf{R}^2,$$

where $C_{j,k}$ and c are positive constants independent of λ.

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