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HAUSDORFF MEASURE OF THE SAMPLE PATHS OF GAUSSIAN RANDOM FIELDS

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1. Introduction

Let $Y(t)$ $(t \in \boldsymbol{R}^N)$ be a real-valued, centered Gaussian random field with $Y(0)=0$. We assume that $Y(t)$ $(t \in \boldsymbol{R}^N)$ has stationary increments and continuous covariance function $R(t,s)=EY(t)Y(s)$ given by

$$R(t,s)=\int_{\boldsymbol{R}^N}(e^{i\langle t,\lambda\rangle}-1)(e^{-i\langle s,\lambda\rangle}-1)\Delta(d\lambda), \tag{1.1}$$

where $\langle x,y\rangle$ is the ordinary scalar product in $\boldsymbol{R}^N$ and $\Delta(d\lambda)$ is a nonnegative symmetric measure on $\boldsymbol{R}^N\backslash\{0\}$ satisfying

$$\int_{\boldsymbol{R}^N}\frac{|\lambda|^2}{1+|\lambda|^2}\Delta(d\lambda)<\infty. \tag{1.2}$$

Then there exists a centered complex-valued Gaussian random measure $W(d\lambda)$ such that

$$Y(t)=\int_{\boldsymbol{R}^N}(e^{i\langle t,\lambda\rangle}-1)W(d\lambda) \tag{1.3}$$

and for any Borel sets A, $B \subseteq \boldsymbol{R}^N$

$$E(W(A)\overline{W(B)})=\Delta(A\cap B) \quad \text{and} \quad W(-A)=\overline{W(A)}.$$

It follows from (1.3) that

$$E[(Y(t+h)-Y(t))^2]=2\int_{\boldsymbol{R}^N}(1-\cos\langle h,\lambda\rangle)\Delta(d\lambda). \tag{1.4}$$

We assume that there exist constants $\delta_0>0$, $0<c_1\le c_2<\infty$ and a non-decreasing, continuous function $\sigma:[0,\delta_0)\to[0,\infty)$ which is regularly varying at the origin with index α $(0<\alpha<1)$ such that for any $t\in\boldsymbol{R}^N$ and $h\in\boldsymbol{R}^N$ with $|h|\le\delta_0$

$$E[(Y(t+h)-Y(t))^2]\le c_1\sigma^2(|h|). \tag{1.5}$$

and for all $t \in \boldsymbol{R}^N$ and any $0 < r \leq \min\{|t|, \delta_0\}$

$$Var(Y(t) \mid Y(s) : r \leq |s-t| \leq \delta_0) \geq c_2 \sigma^2(r). \tag{1.6}$$

If (1.5) and (1.6) hold, we shall say that $Y(t)$ $(t \in \boldsymbol{R}^N)$ is strongly locally σ-nondeterministic. We refer to Monrad and Pitt [14], Berman [4] [5] and Cuzick and Du Peez [6] for more information on (strongly) locally nondeterminism.

We associate with $Y(t)$ $(t \in \boldsymbol{R}^N)$ a Gaussian random field $X(t)$ $(t \in \boldsymbol{R}^N)$ in $\boldsymbol{R}^d$ by

$$X(t) = (X_1(t), \cdots, X_d(t)), \tag{1.7}$$

where $X_1, \cdots, X_d$ are independent copies of Y. The most important example of such Gaussian random fields is the fractional Brownian motion of index α (see Example 4.1 below).

It is well known (see [1], Chapter 8) that with probability 1

$$\dim X([0,1]^N) = \min\left(d, \frac{N}{\alpha}\right).$$

The objective of this paper is to consider the exact Hausdorff measure of the image set $X([0,1]^N)$. The main result is the following theorem, which generalizes a theorem of Talagrand [22].

Theorem 1.1. *If $N < \alpha d$, then with probability* 1

$$0 < \phi\text{-}m(X([0,1]^N)) < \infty, \tag{1.8}$$

where $\phi(s) = \psi(s)^N \log\log \frac{1}{s}$, ψ is the inverse function of σ and ϕ-$m(X([0,1]^N))$ is the ϕ-Hausdorff measure of $X([0,1]^N)$.

If $N > \alpha d$, then by a result of Pitt [17], $X([0,1]^N)$ a.s. has interior points and hence has positive d-dimensional Lebesgue measure. In the case of $N = \alpha d$, the problem of finding ϕ-$m(X([0,1]^N))$ is still open even in the fractional Brownian motion case.

The paper is organized as follows. In Section 2 we recall the definition and some basic facts of Hausdorff measure, Gaussian processes and regularly varying functions. In Section 3 we prove the upper bound and in Section 4, we prove the lower bound for ϕ-$m(X([0,1]^N))$. We also give some examples showing that the hypotheses in Theorem 1.1 are satisfied by a large class of Gaussian random fields including fractional Brownian motion.

Another important example of Gaussian random fields is the Brownian sheet or N-parameter Wiener process $W(t)$ $(t \in \boldsymbol{R}^N_+)$, see Orey and Pruitt [16]. Since $W(t)$ $(t \in \boldsymbol{R}^N_+)$ is not locally nondeterministic, Theorem 1.1 does not apply. The problem of finding exact Hausdorff measure of $W([0,1]^N)$ was solved by Ehm [7].

We will use K to denote an unspecified positive constant which may be different in each appearance.

2. Preliminaries

Let Φ be the class of functions $\phi:(0,\delta)\rightarrow(0,1)$ which are right continuous, monotone increasing with $\phi(0+)=0$ and such that there exists a finite constant $K>0$ for which

$$\frac{\phi(2s)}{\phi(s)}\leq K, \quad \text{for} \quad 0<s<\frac{1}{2}\delta.$$

For $\phi\in\Phi$, the ϕ-Hausdorff measure of $E\subseteq \boldsymbol{R}^N$ is defined by

$$\phi\text{-}m(E)=\liminf_{\varepsilon\rightarrow 0}\left\{\sum_i\phi(2r_i): E\subseteq \cup_{i=1}^{\infty}B(x_i,r_i),\ r_i<\varepsilon\right\},$$

where $B(x,r)$ denotes the open ball of radius r centered at x. It is known that ϕ-m is a metric outer measure and every Borel set in $\boldsymbol{R}^N$ is ϕ-m measurable. The Hausdorff dimension of E is defined by

$$\begin{aligned}\dim E&=\inf\{\alpha>0:\ s^{\alpha}\text{-}m(E)=0\}\\ &=\sup\{\alpha>0:\ s^{\alpha}\text{-}m(E)=\infty\}.\end{aligned}$$

We refer to [F] for more properties of Hausdorff measure and Hausdorff dimension.

The following lemma can be easily derived from the results in [18] (see [23]), which gives a way to get a lower bound for ϕ-$m(E)$. For any Borel measure μ on $\boldsymbol{R}^N$ and $\phi\in\Phi$, the upper ϕ-density of μ at $x\in\boldsymbol{R}^N$ is defined by

$$\bar{D}_{\mu}^{\phi}(x)=\limsup_{r\rightarrow 0}\frac{\mu(B(x,r))}{\phi(2r)}.$$

Lemma 2.1. *For a given $\phi\in\Phi$ there exists a positive constant K such that for any Borel measure μ on $\boldsymbol{R}^N$ and every Borel set $E\subseteq\boldsymbol{R}^N$, we have*

$$\phi\text{-}m(E)\geq K\mu(E)\inf_{x\in E}\{\bar{D}_{\mu}^{\phi}(x)\}^{-1}.$$

Now we summarize some basic facts about Gaussian processes. Let $Z(t)$ $(t\in S)$ be a Gaussian process. We provide S with the following metric

$$d(s,t)=\|Z(s)-Z(t)\|_2,$$

where $\|Z\|_2=(E(Z^2))^{\frac{1}{2}}$. We denote by $N_d(S,\varepsilon)$ the smallest number of open d-balls of radius ε needed to cover S and write $D=\sup\{d(s,t): s,\ t\in S\}$.

The following lemma is well known. It is a consequence of the Gaussian isopermetric inequality and Dudley's entropy bound ([11], see also [22]).

Lemma 2.2. *There exists an absolute constant $K>0$ such that for any $u>0$, we have*

$$P\left\{\sup_{s,\ t\in S}|Z(s)-Z(t)|\geq K(u+\int_0^D\sqrt{\log N_d(S,\varepsilon)}d\varepsilon)\right\}\leq\exp\left(-\frac{u^2}{D^2}\right).$$

Lemma 2.3. *Consider a function Ψ such that $N_d(S,\varepsilon)\leq\Psi(\varepsilon)$ for all $\varepsilon>0$. Assume that for some constant $C>0$ and all $\varepsilon>0$ we have*

$$\Psi(\varepsilon)/C\leq\Psi\left(\frac{\varepsilon}{2}\right)\leq C\Psi(\varepsilon).$$

Then

$$P\{\sup_{s,\ t\in S}|Z(s)-Z(t)|\leq u\}\geq\exp(-K\Psi(u)),$$

where $K>0$ is a constant depending only on C.

This is proved in [21]. It gives an estimate for the lower bound of the small ball probability of Gaussian processes. Similar problems have also been considered be Monrad and Rootzén [15] and by Shao [20].

We end this section with some lemmas about regularly varying functions. Let $\sigma(s)$ be a regularly varying function with index α $(0<\alpha<1)$. Then σ can be written as

$$\sigma(s)=s^\alpha L(s),$$

where $L(s):[0,\delta_0)\to[0,\infty)$ is slowly varying at the origin in the sense of Karamata and hence can be represented by

$$L(s)=\exp\left(\eta(s)+\int_s^A\frac{\varepsilon(t)}{t}dt\right), \tag{2.1}$$

where $\eta(s):[0,\delta_0)\to\boldsymbol{R}$, $\varepsilon(s):(0,A]\to\boldsymbol{R}$ are bounded measurable functions and

$$\lim_{s\to0}\eta(s)=c,\quad |c|<\infty;\quad \lim_{s\to0}\varepsilon(s)=0.$$

In the following, Lemma 2.4 is an easy consequence of (2.1) and Lemma 2.5 can be deduced from Theorem 2.6 and 2.7 in Seneta [19] derectly.

Lemma 2.4. *Let $L(s)$ be a slowly varying function at the origin and let $U=U(s):[0,\infty)\to[0,\infty)$ satisfying*

$$\lim_{s\to 0} U(s)=\infty \quad \textit{and} \quad \lim_{s\to 0} sU(s)=0.$$

Then for any $\varepsilon>0$, as s small enough we have

$$U(s)^{-\varepsilon}L(s)\leq L(sU(s))\leq U(s)^{\varepsilon}L(s)$$

and

$$U(s)^{-\varepsilon}L(s)\leq L(sU(s)^{-1})\leq U(s)^{\varepsilon}L(s).$$

Lemma 2.5. *Let σ be a regularly varying function at the origin with index $\alpha>0$. Then there is a constant $K>0$ such that for $r>0$ small enough, we have*

$$\int_1^{\infty}\sigma(re^{-u^2})du\leq K\sigma(r), \tag{2.2}$$

$$\int_0^1\sigma(rs)ds\leq K\sigma(r), \tag{2.3}$$

$$\int_0^1\sigma(rs)s^{N-1}ds\leq K\sigma(r). \tag{2.4}$$

Let $\sigma:[0,\delta_0)\to[0,\infty)$ be non-decreasing and let ψ be the inverse function of σ, that is

$$\psi(s)=\inf\{t\geq 0:\sigma(t)\geq s\}.$$

then $\psi(s)=s^{1/\alpha}f(s)$, where $f(s)$ is also a slowly varying function and

$$\sigma(\psi(s))\sim s \quad \text{and} \quad \psi(\sigma(s))\sim s \quad \text{as} \quad s\to 0. \tag{2.5}$$

3. Upper bound for ϕ-$m(X([0,1]^N))$

Let $Y(t)$ $(t\in \boldsymbol{R}^N)$ be a real-valued, centered Gaussian random field with stationary increments and a continuous covariance function $R(t,s)$ given by (1.1). We assume that $Y(0)=0$ and (1.5) holds. Let $X(t)$ $(t\in \boldsymbol{R}^N)$ be the (N,d) Gaussian random field defined by (1.7).

We start with the following lemma.

Lemma 3.1. *Let $Y(t)$ $(t\in \boldsymbol{R}^N)$ be a Gaussian process with $Y(0)=0$ satisfying* (1.5). *Then*

(i) *For any* $r>0$ *small enough and* $u \geq K\sigma(r)$, *we have*

$$P\left\{\sup_{|t|\leq r}|Y(t)|\geq u\right\}\leq \exp\left(-\frac{u^2}{K\sigma^2(r)}\right). \tag{3.1}$$

(ii) *Let*

$$\omega_Y(h)=\sup_{t,\ t+s\in[0,1]^N,\ |s|\leq h}|Y(t+s)-Y(t)|$$

be the uniform modulus of continuity of $Y(t)$ *on* $[0,1]^N$. *Then*

$$\limsup_{h\to 0}\frac{\omega_Y(h)}{\sigma(h)\sqrt{2c_1\log\frac{1}{h}}}\leq 1, \quad a.s. \tag{3.2}$$

Proof. Let $r<\delta_0$ and $S=\{t\colon |t|\leq r\}$. Since $d(s,t)\leq c_1\sigma(|t-s|)$, we have

$$N_d(S,\varepsilon)\leq K\left(\frac{r}{\psi(\varepsilon)}\right)^N$$

and

$$D=\sup\{d(s,t)\,;s,\ t\in S\}\leq K\sigma(r).$$

By simple calculations

$$\begin{aligned}\int_0^D\sqrt{\log N_d(S,\varepsilon)}\,d\varepsilon &\leq K\int_0^{K\sigma(r)}\sqrt{\log(Kr)/\psi(\varepsilon)}\,d\varepsilon\\ &\leq K\int_0^{Kr}\sqrt{\log(Kr)/t}\,d\sigma(t)\\ &\leq K\left(\sigma(r)+\int_0^K\frac{1}{u\sqrt{\log K/u}}\sigma(ur)du\right)\\ &\leq K\left(\sigma(r)+\int_K^\infty\sigma(re^{-u^2})du\right)\\ &\leq K\sigma(r),\end{aligned}$$

where the last inequality follows from (2.2). If $u\geq K\sigma(r)$, then by Lemma 2.2 we have

$$\begin{aligned}&P\left\{\sup_{|t|\leq r}|Y(t)|\geq 2Ku\right\}\\ &\leq P\left\{\sup_{|t|\leq r}|Y(t)|\geq K\left(u+\int_0^D\sqrt{\log N_d(S,\varepsilon)}\,d\varepsilon\right)\right\}\end{aligned}$$

$$\leq \exp\left(-\frac{u^2}{K\sigma^2(r)}\right).$$

This proves (3.1). The inequality (3.2) can be derived from Lemma 2.2 directly in a standard way (see also [13]).

In order to get the necessary independence, we will make use of the spectral representation (1.3). Given $0<a<b<\infty$, we consider the process

$$Y(a,b,t)=\int_{a\leq|t|\leq b}(e^{i\langle t,\lambda\rangle}-1)W(d\lambda).$$

Then for any $0<a<b<a'<b'<\infty$, the processes $Y(a,b,t)$ and $Y(a',b',t)$ are independent. The next lemma expresses how well $Y(a,b,t)$ apporximates $Y(t)$.

Lemma 3.2. *Let $Y(t)$ ($t\in \boldsymbol{R}^N$) be defined by* (1.3). *If* (1.5) *holds, then there exists a constant $B>0$ such that for any $B<a<b$ we have*

$$\|Y(a,b,t)-Y(t)\|_2\leq K[|t|^2a^2\sigma^2(a^{-1})+\sigma^2(b^{-1})]^{\frac{1}{2}}. \tag{3.3}$$

Proof. First we claim that for any $u>0$ and any $h\in \boldsymbol{R}^N$ with $|h|=1/u$ we have

$$\int_{|\lambda|<u}\langle h,\lambda\rangle^2\Delta(d\lambda)\leq K\int_{\boldsymbol{R}^N}(1-\cos\langle h,\lambda\rangle)\Delta(d\lambda) \tag{3.4}$$

$$\int_{|\lambda|\geq u}\Delta(d\lambda)\leq K\left(\frac{u}{2}\right)^N\int_{[-1/u,\,1/u]^N}dv\int_{\boldsymbol{R}^N}(1-\cos\langle v,\lambda\rangle)\Delta(d\lambda). \tag{3.5}$$

For $N=1$, (3.4) and (3.5) are the truncation inequalities in [12] p209. For $N>1$ a similar proof yields (3.4) and (3.5).

Now for any $a>\delta_0^{-1}$ and any $t\in \boldsymbol{R}^N\backslash\{0\}$, by (1.4), (1.5) and (3.4) we have

$$\begin{aligned}\int_{|\lambda|<a}(1-\cos\langle t,\lambda\rangle)\Delta(d\lambda)&\leq\int_{|\lambda|<a}\langle t,\lambda\rangle^2\Delta(d\lambda)\\&=|t|^2a^2\int_{|\lambda|<a}\langle t/(a|t|),\lambda\rangle^2\Delta(d\lambda)\leq K|t|^2a^2\sigma^2(a^{-1}).\end{aligned} \tag{3.6}$$

For $b>0$ large enough, by (3.5), (1.4), (1.5) and (2.4) we have

$$\begin{aligned}\int_{|\lambda|\geq b}\Delta(d\lambda)&\leq K\left(\frac{b}{2}\right)^N\int_{[-1/b,\,1/b]^N}\sigma^2(|v|)dv\\&\leq Kb^N\int_0^{\sqrt{N}b^{-1}}\sigma^2(\rho)\rho^{N-1}d\rho\leq K\sigma^2(b^{-1}).\end{aligned} \tag{3.7}$$

Combining (3.6) and (3.7), we see that there exists a constant $B>0$ such that $B<a<b$ implies

$$E[(Y(a,b,t)-Y(t))^2]=2\int_{\{|\lambda|<a\}\cup\{|\lambda|>b\}}(1-\cos\langle t,\lambda\rangle)\Delta(d\lambda)$$
$$\le 2\int_{|\lambda|<a}(1-\cos\langle t,\lambda\rangle)\Delta(d\lambda)+2\int_{|\lambda|>b}\Delta(d\lambda)$$
$$\le K[|t|^2a^2\sigma^2(a^{-1})+\sigma^2(b^{-1})].$$

This proves (3.3).

Lemma 3.3. *There exists a constant $B>0$ such that for any $B<a<b$ and $0<r<B^{-1}$ the following holds: let $A=r^2a^2\sigma^2(a^{-1})+\sigma^2(b^{-1})$ such that $\psi(\sqrt{A})\le\frac{1}{2}r$, then for any*

$$u\ge K\left(A\log\frac{Kr}{\psi(\sqrt{A})}\right)^{\frac{1}{2}}$$

we have

$$P\left\{\sup_{|t|\le r}|Y(t)-Y(a,b,t)|\ge u\right\}\le\exp\left(-\frac{u^2}{KA}\right). \tag{3.8}$$

Proof. Let $S=\{t: |t|\le r\}$ and $Z(t)=Y(t)-Y(a,b,t)$. Then

$$d(s,t)=\|Z(t)-Z(s)\|_2\le c_1\sigma(|t-s|).$$

Hence

$$N_d(S,\varepsilon)\le K\left(\frac{r}{\psi(\varepsilon)}\right)^N.$$

By Lemma 3.2 we have $D\le K\sqrt{A}$. As in the proof of Lemma 3.1,

$$\int_0^D\sqrt{\log N_d(S,\varepsilon)}\,d\varepsilon\le K\int_0^{K\sqrt{A}}\sqrt{\log(Kr)/\psi(\varepsilon)}\,d\varepsilon$$
$$\le K\int_0^{K\psi(\sqrt{A})/r}\sqrt{\log K/t}\,d\sigma(rt)$$
$$\le K\left[\sqrt{\log K/t}\,\sigma(rt)|_0^{K\psi(\sqrt{A})/r}+\int_0^{K\psi(\sqrt{A})/r}\frac{1}{t\sqrt{\log K/t}}\sigma(rt)dt\right]$$

$$\leq K\sqrt{A\log Kr/\psi(\sqrt{A})}+K\int_{\sqrt{\log Kr/\psi(\sqrt{A})}}^{\infty}\sigma(Kre^{-u^2})du$$
$$\leq K\sqrt{A\log Kr/\psi(\sqrt{A})},$$

at least for $r>0$ small enough, where the last step follows from (2.2). Hence (3.8) follows immediately from Lemma 2.2.

Let $X_1(a,b,t),\cdots,X_d(a,b,t)$ be independent copies of $Y(a,b,t)$ and let

$$X(a,b,t)=(X_1(a,b,t),\cdots,X_d(a,b,t)) \qquad (t\in \boldsymbol{R}^N).$$

Then we have the following corollary of Lemma 3.3.

Corollary 3.1. *Consider* $B<a<b$ *and* $0<r<B^{-1}$. *Let* $A=r^2a^2\sigma^2(a^{-1})+\sigma^2(b^{-1})$ *with* $\psi(\sqrt{A})\leq\frac{1}{2}r$. *Then for any*

$$u\geq K\left(A\log\frac{Kr}{\psi(\sqrt{A})}\right)^{\frac{1}{2}}$$

we have

$$P\left\{\sup_{|t|\leq r}|X(t)-X(a,b,t)|\geq u\right\}\leq\exp\left(-\frac{u^2}{KA}\right). \tag{3.9}$$

Lemma 3.4. *Given* $0<r<\delta_0$ *and* $\varepsilon<\sigma(r)$. *Then for any* $0<a<b$ *we have*

$$P\left\{\sup_{|t|\leq r}|X(a,b,t)|\leq\varepsilon\right\}\geq\exp\left(-\frac{r^N}{K\psi(\varepsilon)^N}\right). \tag{3.10}$$

Proof. It is sufficient to prove (3.10) for $Y(a,b,t)$. Let $S=\{t\colon |t|\leq r\}$ and define a distance d on S by

$$d(s,t)=\|Y(a,b,t)-Y(a,b,s)\|_2.$$

Then $d(s,t)\leq c_1\sigma(|t-s|)$ and

$$N_d(S,\varepsilon)\leq K\left(\frac{r}{\psi(\varepsilon)}\right)^N.$$

By Lemma 2.3 we have

$$P\left\{\sup_{|t|\leq r}|Y(a,b,t)|\leq\varepsilon\right\}\geq\exp\left(-\frac{r^N}{K\psi(\varepsilon)^N}\right).$$

This proves lemma 3.4.

Proposition 3.1. *There exists a constant $\delta_1 > 0$ such that for any $0 < r_0 \le \delta_1$, we have*

$$P\left\{\exists r \in [r_0^2, r_0] \text{ such that } \sup_{|t| \le r} |X(t)| \le K\sigma(r(\log\log\frac{1}{r})^{-\frac{1}{N}})\right\} \tag{3.11}$$
$$\ge 1 - \exp\left(-(\log\frac{1}{r_0})^{\frac{1}{2}}\right).$$

Proof. We follow the line of Talagrand [22]. Let $U = U(r_0) \ge 1$, where $U(r)$ satisfying

$$U(r) \to \infty \quad \text{as} \quad r \to 0 \tag{3.12}$$

and for any $\varepsilon > 0$

$$r^{\varepsilon} U(r) \to 0 \quad \text{as} \quad r \to 0, \tag{3.13}$$

will be chosen later. For $k \ge 0$, let $r_k = r_0 U^{-2k}$. Let k_0 be the largest integer such that

$$k_0 \le \frac{\log \frac{1}{r_0}}{2 \log U},$$

then for any $0 \le k \le k_0$ we have $r_0^2 \le r_k \le r_0$. In order to prove (3.11), it suffices to show that

$$P\left\{\exists k \le k_0 \text{ such that } \sup_{|t| \le r_k} |X(t)| \le K\sigma(r_k(\log\log\frac{1}{r_k})^{-\frac{1}{N}})\right\} \tag{3.14}$$
$$\ge 1 - \exp\left(-(\log\frac{1}{r_0})^{\frac{1}{2}}\right).$$

Let $a_k = r_0^{-1} U^{2k-1}$ and we define for $k = 0, 1, \cdots$

$$X_k(t) = X(a_k, a_{k+1}, t),$$

then $X_0, X_1, \cdots$ are independent. By Lemma 3.4 we can take a constant K_1 such that for $r_0 > 0$ small enough

$$P\left\{\sup_{|t| \le r_k} |X_k(t)| \le K_1 \sigma(r_k(\log\log\frac{1}{r_k})^{-\frac{1}{N}})\right\} \tag{3.15}$$
$$\ge \exp\left(-\frac{1}{4}\log\log\frac{1}{r_k}\right)$$

$$=\frac{1}{(\log\frac{1}{r_k})^{\frac{1}{4}}}.$$

Thus, by independence we have

$$P\left\{\exists k\le k_0,\quad \sup_{|t|\le r_k}|X_k(t)|\le K_1\sigma(r_k(\log\log\frac{1}{r_k})^{-1/N})\right\} \tag{3.16}$$

$$\ge 1-\left(1-\frac{1}{(2\log 1/r_0)^{1/4}}\right)^{k_0}$$

$$\ge 1-\exp\left(-\frac{k_0}{(2\log 1/r_0)^{1/4}}\right).$$

Let

$$A_k=r_k^2a_k^2\sigma^2(a_k^{-1})+\sigma^2(a_{k+1}^{-1})$$

$$=U^{-2+2\alpha}r_k^{2\alpha}L^2(r_kU)+U^{-2\alpha}r_k^{2\alpha}L^2(r_k/U).$$

Let $\beta=2\min\{1-\alpha,\alpha\}$ and fix an $\varepsilon<\frac{1}{2}\beta$. Then by Lemma 2.4, we see that as r_0 small enough

$$U^{-\beta-\varepsilon}\sigma^2(r_k)\le A_k\le U^{-\beta+\varepsilon}\sigma^2(r_k).$$

Notice that r_0 for small enough we have

$$\psi(\sqrt{A_k})\ge\psi(U^{-(\beta+\varepsilon)/2}\sigma(r_k))$$

$$=(U^{-\beta/2}\sigma(r_k))^{1/\alpha}f(U)^{-\beta/2}\sigma(r_k))$$

$$=U^{-\beta/(2\alpha)}r_kL(r_k)^{1/\alpha}f(U^{-\beta/2}\sigma(r_k))$$

$$\ge KU^{-(\beta+\varepsilon)/2/(2\alpha)}r_k,$$

the last inequality follows from (2.5). It follows from Corollary 3.1 that for

$$u\ge K\sigma(r_k)U^{-\frac{\beta-\varepsilon}{2}}(\log U)^{1/2},$$

we have

$$P\left\{\sup_{|t|\le r_k}|X(t)-X_k(t)|\ge u\right\}\le\exp\left(-\frac{u^2U^{\beta-\varepsilon}}{K\sigma^2(r_k)}\right). \tag{3.17}$$

Hence, if we take

$$U=(\log 1/r_0)^{\frac{1}{\beta-\varepsilon}},$$

then as r_0 small enough

$$\sigma(r_k)U^{-\frac{\beta-\varepsilon}{2}}(\log U)^{1/2} \le \sigma(r_k(\log\log\frac{1}{r_0})^{-\frac{1}{N}}).$$

Hence by taking

$$u = \frac{K_1}{2}\sigma(r_k(\log\log\frac{1}{r_0})^{-\frac{1}{N}})$$

in (3.17), we obtain

$$P\left\{\sup_{|t|\le r_k}|X(t)-X_k(t)| \ge \frac{K_1}{2}\sigma(r_k(\log\log\frac{1}{r_0})^{-\frac{1}{N}})\right\} \le \exp\left(-\frac{u^2U^{\beta-\varepsilon}}{K\sigma^2(r_k)}\right). \tag{3.18}$$

Combining (3.16) and (3.18) we have

$$\begin{aligned} &P\left\{\exists k \le k_0 \text{ such that } \sup_{|t|\le r_k}|X(t)| \le 2K_1\sigma(r_k(\log\log\frac{1}{r_k})^{-1/N})\right\} \\ &\ge 1-\exp\left(-\frac{k_0}{2(\log 1/r_0)^{1/4}}\right)-k_0\exp\left(-\frac{U^{\beta-\varepsilon}}{K(\log\log 1/r_0)^{(2\alpha)/N+\varepsilon}}\right). \end{aligned} \tag{3.19}$$

We recall that

$$\frac{\log\frac{1}{r_0}}{4\log U} \le k_0 \le \log\frac{1}{r_0}.$$

and hence for r_0 small enough, (3.11) follows from (3.19).

Now we are in a pposition to prove the uppper bound for ϕ-$m(X([0,1]^N))$.

Theorem 3.1. *Let* $\phi(s)=\psi(s)^N\log\log\frac{1}{s}$. *Then with probability* 1

$$\phi\text{-}m(X([0,1]^N))<\infty.$$

Proof. For $k\ge 1$, consider the set

$$R_k = \left\{t\in[0,1]^N : \exists r\in[2^{-2k},2^{-k}] \text{ such that } \sup_{|s-t|\le r}|X(s)-X(t)| \le K\sigma(r(\log\log\frac{1}{r})^{-1/N})\right\}.$$

By Proposition 3.1 we have

$$P\{t\in R_k\} \ge 1-\exp(-\sqrt{k/2}).$$

Denote the Lebesgue measure in $\boldsymbol{R}^N$ by L_N. It follows from Fubini's theorem that $P(\Omega_0)=1$, where

$$\Omega_0=\{\omega: L_N(R_k)\geq 1-\exp(-\sqrt{k/4}) \quad \text{infinitely often}\}.$$

On the other hand, by Lemma 3.1 ii), there exists an event Ω_1 such that $P(\Omega_1)=1$ and for all $\omega\in\Omega_1$, there exists $n_1=n_1(\omega)$ large enough such that for all $n\geq n_1$ and any dyadic cube C of order n in $\boldsymbol{R}^N$, we have

$$\sup_{s,t\in C}|X(t)-X(s)|\leq K\sigma(2^{-n})\sqrt{n}. \tag{3.20}$$

Now fix an $\omega\in\Omega_0\cap\Omega_1$, we show that ϕ-$m(X([0,1]^N))<\infty$. Consider $k\geq 1$ such that

$$L_N(R_k)\geq 1-\exp(-\sqrt{k/4}).$$

For any $x\in R_k$ we can find n with $k\leq n\leq 2k+k_0$ (where k_0 depends on N only) such that

$$\sup_{s,t\in C_n(x)}|X(t)-X(s)|\leq K\sigma(2^{-n}(\log\log 2^n)^{-1/N}), \tag{3.21}$$

where $C_n(x)$ is the unique dyadic cube of order n containing x. Thus we have

$$R_k\subseteq V=\cup_{n=k}^{2k+k_0}V_n$$

and each V_n is a union of dyadic cubes C_n of order n for which (3.21) holds. Clearly $X(C_n)$ can be covered by a ball of radius

$$\rho_n=K\sigma(2^{-n}(\log\log 2^n)^{-1/N}).$$

Since $\phi(2\rho_n)\leq K2^{-nN}=KL_N(C_n)$, we have

$$\begin{aligned}\sum_n\sum_{C\in V_n}\phi(2\rho_n)&\leq\sum_n\sum_{C\in V_n}KL_N(C_n)\\&=KL_N(V)<\infty.\end{aligned} \tag{3.22}$$

On the other hand, $[0,1]^N\backslash V$ is contained in a union of dyadic cubes of order $q=2k+k_0$, none of which meets R_k. There can be at most

$$2^{Nq}L_N([0,1]^N\backslash V)\leq K2^{Nq}\exp(-\sqrt{k}/4)$$

of such cubes. For each of these cubes, $X(C)$ is contained in a ball of radius $\rho=K\sigma(2^{-q})\sqrt{q}$. Thus for any $\varepsilon>0$

$$\sum\phi(2\rho)\leq K2^{Nq}\exp(-\sqrt{k}/4)2^{-Nq}q^{N/(2\alpha)+\varepsilon}\leq 1 \tag{3.23}$$

for k large enough. Since k can be arbitrarily large, Theorem 3.1 follows from (3.22) and (3.23).

4. Lower bound for ϕ-$m(X([0,1]^N))$

Let $Y(t)$ $(t\in \mathbf{R}^N)$ be a real-valued, centered Gaussian random field with stationary increments and a continuous covariance function $R(t,s)$ given by (1.1). We assume that $Y(0)=0$ and (1.6) holds. Let $X(t)$ $(t\in \mathbf{R}^N)$ be the (N,d) Gaussian random field defined by (1.7). In this section, we prove that if $N<\alpha d$, then

$$\phi\text{-}m(X([0,1]^N))>0 \quad \text{a.s.}$$

For simplicity we assume $\delta_0=1$ and let $I=[0,1]^N\cap B(0,1)$ (otherwise we consider a smaller cube). For any $0<r<1$ and $y\in \mathbf{R}^d$. Let

$$T_y(r)=\int_I 1_{B(y,r)}(X(t))dt$$

be the sojourn time of $X(t)$ $(t\in I)$ in the open ball $B(y,r)$. If $y=0$, we write $T(r)$ for $T_0(r)$.

Proposition 4.1. *There exist $\delta_2>0$ and $b>0$ such that for any $0<r<\delta_2$*

$$E(\exp(b\psi(r)^{-N}T(r)))\leq K<\infty. \tag{4.1}$$

Proof. We first prove that there exists a constant $0<K<\infty$ such that for any $n\geq 1$

$$E(T(r))^n\leq K^n n!\psi(r)^{Nn}. \tag{4.2}$$

For $n=1$, by (2.4) and (2.5) we have

$$\begin{aligned} ET(r)&=\int_I P\{X(t)\in B(0,r)\}dt \\ &\leq \int_I \min\{1,K(\frac{r}{\sigma(|t|)})^d\}dt \\ &\leq K\int_0^1 \min\{1,\frac{Kr^d}{\sigma(\rho)^d}\}\rho^{N-1}d\rho \\ &\leq K\int_0^{K\psi(r)} \rho^{N-1}d\rho+K\int_{K\psi(r)}^1 \frac{r^d\rho^{N-1}}{\sigma(\rho)^d}d\rho \\ &\leq K\psi(r)^N+Kr^d\psi(r)^{N-\alpha d}\int_1^\infty \frac{1}{t^{1+\alpha d-N}L(\psi(r)t)^d}dt \end{aligned} \tag{4.3}$$

$$\leq K\psi(r)^N + Kr^d\psi(r)^{N-\alpha d}/L(\psi(r))^d$$
$$\leq K\psi(r)^N.$$

For $n \geq 2$

$$E(T(r)^n) = \int_{I^n} P\{|X(t_1)| < r, \cdots, |X(t_n)| < r\} dt_1 \cdots dt_n. \tag{4.4}$$

Consider $t_1, \cdots, t_n \in I$ satisfying

$$t_j \neq 0 \quad \text{for} \quad j = 1, \cdots, n, \quad t_j \neq t_k \quad \text{for} \quad j \neq k.$$

Let $\eta = \min\{|t_n|,\ |t_n - t_i|,\ i = 1, \cdots, n-1\}$. Then by (1.6) we have

$$Var(X(t_n)|X(t_1), \cdots, X(t_{n-1})) \geq c_2\sigma^2(\eta). \tag{4.5}$$

Since conditional distributions in Gaussian processes are still Gaussian, it follows from (4.5) that

$$\begin{aligned} &P\{|X(t_n)| < r | X(t_1) = x_1, \cdots, X(t_{n-1}) = x_{n-1}\} \\ &\leq K \int_{|u| < r} \frac{1}{\sigma(\eta)^d} \exp\left(-\frac{|u|^2}{K\sigma^2(\eta)}\right) du. \end{aligned} \tag{4.6}$$

Similar to (4.3), we have

$$\begin{aligned} &\int_I dt_n \int_{|u| < r} \frac{1}{\sigma(\eta)^d} \exp\left(-\frac{|u|^2}{K\sigma^2(\eta)}\right) du \\ &\leq \int_I \min\{1,\ K(\frac{r}{\sigma(\eta)})^d\} dt_n \\ &\leq K \int_I \sum_{i=0}^{n-1} \min\{1,\ K(\frac{r}{\sigma(|t_n - t_i|)})^d\} dt_n \qquad (t_0 = 0) \\ &\leq Kn \int_0^1 \min\{1,\ \frac{Kr^d}{\sigma(\rho)^d}\} \rho^{N-1} d\rho \\ &\leq Kn\psi(r)^N. \end{aligned} \tag{4.7}$$

By (4.4), (4.6) and (4.7), we obtain

$$\begin{aligned} E(T(r))^n \leq K &\int_{I^{n-1}} P\{|X_1(t_1)| < r, \cdots, |X(t_{n-1})| < r\} dt_1 \cdots dt_{n-1} \\ &\cdot \int_I dt_n \int_{|u| < r} \frac{1}{\sigma(\eta)^d} \exp\left(-\frac{|u|^2}{K\sigma^2(\eta)}\right) du \end{aligned}$$

$$\leq Kn\psi(r)^N E(T(r))^{n-1}.$$

Hence, the inequality (4.2) follows from (4.3) and induction. Let $0<b<1/K$, then by (4.2) we have

$$E\exp(b\psi(r)^{-N}T(r))=\sum_{n=0}^{\infty}(Kb)^n<\infty.$$

This proves (4.1)

Proposition 4.2. *With probability* 1

$$\limsup_{r\to 0}\frac{T(r)}{\phi(r)}\leq\frac{1}{b}, \tag{4.8}$$

where $\phi(r)=\psi(r)^N\log\log 1/r$.

Proof. For any $\varepsilon>0$, it follows from (4.1) that

$$P\{T(r)\geq(1/b+\varepsilon)\psi(r)^N\log\log 1/r\}\leq\frac{K}{(\log 1/r)^{1+b\varepsilon}}. \tag{4.9}$$

Take $r_n=\exp(-n/\log n)$, then by (4.9) we have

$$P\{T(r_n)\geq(1/b+\varepsilon)\psi(r_n)^N\log\log 1/r_n\}\leq\frac{K}{(n/\log n)^{1+b\varepsilon}}.$$

Hence by Borel-Cantelli lemma we have

$$\limsup_{n\to\infty}\frac{T(r_n)}{\phi(r_n)}\leq\frac{1}{b}+\varepsilon. \tag{4.10}$$

It is easy to verify that

$$\lim_{n\to\infty}\frac{\phi(r_n)}{\phi(r_{n+1})}=1. \tag{4.11}$$

Hence by (4.10) and (4.11) we have

$$\limsup_{r\to 0}\frac{T(r)}{\phi(r)}\leq\frac{1}{b}+\varepsilon.$$

Since $\varepsilon>0$ is arbitrary, we obtain (4.8).

Since $X(t)$ $(t\in\boldsymbol{R}^N)$ has stationary increments, we derive the following

Corollary 4.1. *Fix $t_0 \in I$, then with probability* 1

$$\limsup_{r \to 0} \frac{T_{X(t_0)}(r)}{\phi(r)} \le \frac{1}{b}.$$

Theorem 4.1. *If $N < \alpha d$, then with probability* 1

$$\phi\text{-}m(X([0,1]^N)) > 0, \tag{4.12}$$

where $\phi(r) = \psi(r)^N \log\log 1/r$.

Proof. We define a random Borel measure μ on $X(I)$ as follows. For any Borel set $B \subseteq \boldsymbol{R}^d$, let

$$\mu(B) = L_N\{t \in I,\ X(t) \in B\}.$$

Then $\mu(\boldsymbol{R}^d) = \mu(X(I)) = L_N(I)$. By Corollary 4.1, for each fixed $t_0 \in I$, with probatility 1

$$\limsup_{r \to 0} \frac{\mu(B(X(t_0), r))}{\phi(r)} \tag{4.13}$$

$$\le \limsup_{r \to 0} \frac{T_{X(t_0)}(r)}{\phi(r)} \le \frac{1}{b}.$$

Let $E(\omega) = \{X(t_0) \colon t_0 \in I$ and (4.13) holds$\}$. Then $E(\omega) \subseteq X(I)$. A Fubini argument shows $\mu(E(\omega)) = 1$, a.s.. Hence by Lemma 2.1, we have

$$\phi\text{-}m(E(\omega)) \ge Kb.$$

This proves (4.12).

Proof of Theorem 1.1. It follows from Theorems 3.1 and 4.1 immediately.

EXAMPLE 4.1. Let $Y(t)$ $(t \in \boldsymbol{R}^N)$ be a real-valued fractional Brownian motion of index α $(0 < \alpha < 1)$ (see [10], Chapter 18). Its covariance function has the representation

$$R(s,t) = \frac{1}{2}(|s|^{2\alpha} + |t|^{2\alpha} - |t-s|^{2\alpha})$$

$$= c(\alpha) \int_{\boldsymbol{R}^N} (e^{i\langle t, \lambda \rangle} - 1)(e^{-i\langle s, \lambda \rangle} - 1) \frac{d\lambda}{|\lambda|^{N+2\alpha}},$$

where $c(\alpha)$ is a normalizing constant. Then (1.5) is verified and by a result of Pitt [17], (1.6) is also verified. In this case, Theorem 1.1 is proved by Goldman [9] for $\alpha=1/2$ and by Talagrand [22] for $0<\alpha<1$.

EXAMPLE 4.2 Let $Z(t)$ ($t\in\boldsymbol{R}^N$) be a real-valued mean zero stationary random field with covariance function

$$R(s,t)=\exp(-c|s-t|^{2\alpha}) \quad \text{with} \quad c>0 \quad \text{and} \quad 0<\alpha<1.$$

Then $Y(t)=Z(t)-Z(0)$ verifies the conditions (1.5) and (1.6). We can apply Theorem 1.1 to obtain the Hausdorff measure of $X([0,1]^N)$, where

$$X(t)=(X_1(t),\cdots,X_d(t))$$

and $X_1,\cdots,X_d$ are independent copies of Z. Other examples with absolutely continuous spectral measure can be found in Berman [2] p289, and Berman [4].

EXAMPLE 4.3. Now we give an example with discrete spectral measure. Let X_n ($n\geq 0$) and Y_n ($n\geq 0$) be independent standard normal random variables and a_n ($n\geq 0$) real numbers such that $\Sigma_n a_n^2<\infty$. Then for each t, the random series

$$Z(t)=\sum_{n=0}^{\infty} a_n(X_n\cos nt+Y_n\sin nt) \tag{4.14}$$

converges with probability 1 (see [10]), and $Z(t)$ ($t\in\boldsymbol{R}$) represents a stationary Gaussian process with mean 0 and covariance function

$$R(s,t)=\sum_{n=0}^{\infty} a_n^2\cos n(t-s).$$

By a result of Berman [4], there are many choices of a_n ($n\geq 0$) such that the process $Y(t)=Z(t)-Z(0)$ satisfies the hypotheses of Theorem 1.1 with

$$\sigma^2(s)=2\sum_{n=0}^{\infty} a_n^2(1-\cos ns).$$

Let $X(t)$ ($t\in\boldsymbol{R}$) be the Gaussian process in $\boldsymbol{R}^d$ associated with $Z(t)$ or $Y(t)$ ($t\in\boldsymbol{R}$) by (1.7). If $1<\alpha d$, then

$$0<\phi\text{-}m(X([0,1]^N))<\infty,$$

where $\phi(s)=\psi(s)\log\log\frac{1}{s}$ and ψ is the inverse function of σ. A special case of (4.14) is Example 3.5 in Monrad and Rootzén [15].

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