Kawakami, T.
Osaka J. Math.
31 (1994), 831–835

NONISOMORPHIC ALGEBRAIC MODELS OF NASH MANIFOLDS AND COMPACTIFIABLE C^∞ MANIFOLDS

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(Received March 5, 1993)

1. Introduction

A well-known theorem of A. Tognoli [9] asserts that every compact C^∞ submanifold M of $\boldsymbol{R}^n$ with 2 dim $M+1\leq n$, one can find a C^∞ imbedding $e: M\rightarrow\boldsymbol{R}^n$, arbitrarily close in the C^∞ topology to the inclusion map $M\rightarrow\boldsymbol{R}$, such that $e(M)$ is a nonsingular algebraic subset of $\boldsymbol{R}^n$. In particular, M admits an algebraic model. Here an algebraic model of M means a nonsingular algebraic subset of some Euclidean space diffeomorphic to M.J. Bochnak and W. Kucharz showed in [4] that M has a continuous family of birationally nonequivalent algebraic models when M is a connected closed manifold with dim $M\geq 1$. In this paper, we consider algebraic models of a given affine Nash manifold and a given compactifiable C^∞ manifold. Here we say that a C^∞ manifold M is *compactifiable* if there exists a compact C^∞ manifold Y with boundary such that M is C^∞ diffeomorphic to the interior of Y. M. Shiota proved in [8, Remark 6.2.11] that any affine Nash manifold admits an algebraic model. We prove that either any Nash manifold or any compactifiable C^∞ manifold have an infinite family of birationally nonequivalent algebraic models. More precisely, we prove the following.

Theorem 1. *Each affine Nash manifold M with dim* $M\geq 1$ *has an infinite family of nonsingular algebraic subsets* $\{X_n\}_{n\in N}$ *of some Euclibean space such that each* X_n *is Nash diffeomorphic to M and that* X_n *is not birationally equivalent to* X_m *for* $n\neq m$.

Theorem 2. *Every compactifiable* C^∞ *manifold M with dim* $M\geq 1$ *has an infinite family of nonsingular algebraic subsets* $\{X_n\}_{n\in N}$ *of some Euclidean space such that each* X_n *is* C^∞ *diffeomorphic to M and that* X_n *is not birationally equivalent to* X_m *for* $n\neq m$.

Theorem 2 is a refinement of [3, Corollary 3.3]. We have next corollary because a nonsingular algebraic subset is an affine Nash submanifold.

Corollary 3. *Any nonsingular algebraic subset with dim* $M \geq 1$ *has infinitely many birationally nonequivalent algebraic models in some Euclidean space.*

2. Proof of Results

Recall the notation of [3]. For any two positive integers n and d, let $P(n,d)$ denote the projective space associated with the vector space of all hamogeneous polynomials in $\boldsymbol{R}[T_0,\cdots,T_n]$ of degree d. For H in $P(n,d)$ and its representative K, we set

$$V(H)=\{[x_0:,\cdots,:x_n]\in \boldsymbol{R}P^n | K(x_0,\cdots,x_n)=0\},$$
$$V(H,\boldsymbol{C})=\{[x_0:,\cdots,:x_n]\in \boldsymbol{C}P^n | K(x_0,\cdots,x_n)=0\}.$$

We will identify H with K.

A subset Σ of $P(n,d)$ is said to be *thin* if it is contained in the union of countably many algebraic subsets of $P(n,d)$ different from $P(n,d)$. We prepare next lemma to prove Theorem 1.

Lemma. *Let* X *be a compact irreducible nonsingular algebraic set. Let* Y *and* Z_i $(1\leq i\leq s)$ *be nonsingular algebraic hypersurfaces of* X *satisfying the following seven conditions*:

(1) *The dimension of* Y *is greater than or equal to* 2.
(2) *Each* $W_i := Y\cap Z_i$ *is nonsingular.*
(3) *Every* Z_i *intersects* Y *transversally at any point of* W_i.
(4) *Any* W_i *intersects transversally in* Y *one another.*
(5) *There exist two* C^∞ *submanifolds* Y_1 *and* Y_2 *of* Y *with same boundary such that* Y *is the attaching space of* Y_1 *and* Y_2 *by their boundaries.*
(6) *The union* $\bigcup_{i=1}^s W_i$ *is contained in* Y_2.
(7) *A* C^∞ *smoothing of the* C^∞ *manifold with cornered boundary* $Y_3=\{x\in Y_2 | dist(x,\bigcup_{i=1}^s W_i)\geq \varepsilon\}$ *for small* $\varepsilon>0$ *is* C^∞ *diffeomorphic to the cartesian product of the boundary of* Y_2 *and* $[0,1]$.

Then, we can find an infinite family of nonsingular algebraic subsets $\{S_m\}_{m\in N}$ *of some Euclidean space such that each* S_m *is Nash diffeomorphic to* $Y-\bigcup_{i=1}^s W_i$ *and that* S_m *is not birationally equivalent to* $S_{m'}$ *for* $m\neq m'$.

Proof. By [3, Proposition 2.5], there exists an algebraic imbedding

$f: X \to \boldsymbol{R}P^n$ (for some n), such that the Zariski closure U of $f(X)$ in $\boldsymbol{C}P^n$ is nonsingular. Then we identify X, Y and Z_i $(1 \leq i \leq s)$ with $f(X)$, $f(Y)$ and $f(Z_i)$ $(1 \leq i \leq s)$, respectively. Let h be a homogeneous polynomial of $\boldsymbol{R}[T_0, \cdots, T_n]$ so that $Y = X \cap V(h)$. For each positive integer l, set

$$h_l = (T_0^2 + \cdots + T_n^2)^l h.$$

Observe that $X \cap V(h_l) = X \cap V(h) = Y$.

By [6, Theorem 6.5 and Theorem 7.5], there exists a positive integer l_0 such that for every integer $l > l_0$, one can find a thin subset Σ_l of $P(n, \deg h_l)$ with the property that for every $H_l \in P(n, \deg h_l) - \Sigma_l$, the complex hypersurface $V(H_l, \boldsymbol{C})$ is nonsingular and transverse to U, and that the variety $U(l) = U \cap V(H_l, \boldsymbol{C})$ is irreducible. If H_l is sufficiently close to h_l then $X(l)$ defined by

$$X(l) = X \cap V(H_l) = U(l) \cap \boldsymbol{R}P^n$$

is a nonsingular irreducible real algebraic hypersurface of X, and the Zariski closure of $X(l)$ in $\boldsymbol{C}P^n$ is equal to $U(l)$. Since the sequence of the Hodge numbers $h^{d-1,0}(U(l))$, where $d = \dim X$, diverges infinity as l increases [cf 6, Proof of Lemma 1. p240], one can find an increasing sequence $\{l_m\}_{m \in N}$ of positive integers so that the sequence $h^{d-1,0}(U(l_m))$ is also increasing. In particular, the varieties $U(l_m)$ and $U(l_{m'})$ are not birationally equivalent for $m \neq m'$, because the Hodge numbers $h^{p,0}$ are birational invariants [5, p190 Exercise 8.8]. It follows that $X(l_m)$ and $X(l_{m'})$ are not birationally equivalent for $m \neq m'$. If the approximation is sufficiently close, one can easily check that the triple $(X, X(l_m), \{X(l_m) \cap Z_i\}_i)$ satisfies the conditions in the lemma. Put

$$S_m = X(l_m) - \bigcup_{i=1}^{s} (X(l_m) \cap Z_i)).$$

By the proof of [8, Remark 6.2.11], $Y - \bigcup_{i=1}^{s} W_i$ is Nash diffeomorphic to S_m. On the other hand, $X(l_m)$ and $X(l_{m'})$ are birationally equivalent if and only if S_m and $S_{m'}$ are birationally equivalent. Therefore $\{S_m\}_{m \in N}$ is the required one. □

We are in a position to prove Theorem 1.

Proof of Theorem 1.

If M is compact then the theorem follows from [3](or [4]) and the fact that two compact Nash manifolds are Nash diffeomorphic if and only if they are C^∞ diffeomorphic. We now suppose that M is not compact. Using [7], we can assume that M is the interior of a compact

C^∞ manifold with boundary L_1. By [2], there exist a compact C^∞ manifold with boundary L_2 in some $\boldsymbol{R}^n$ and compact C^∞ submanifolds X_i $(1 \leq i \leq s)$ of Int L_2 such that the following three conditions are satisfied: (1)the boundary of L_2 is C^∞ diffeomorphic to the boundary of L_1, (2)each X_i intersects transversally one another, and (3)a C^∞ smoothing of the C^∞ manifold with cornered boundary $\{x \in L_2 | dist(x,X) \geq \varepsilon\}$ for small $\varepsilon > 0$ is C^∞ diffeomorphic to the cartesian product of the boundary of L_2 and [0,1]. Here $X = \bigcup_{i=1}^s X_i$.

Let L denote the attaching space of L_1 and L_2 by the above diffeomorphism between their boundaries. We regard M,L_1,L_2 and X_i as submanifolds of L.

On the other hand, if dim $M=1$ then, we can apply the method of the proof of [4] to (L,X) becasuse X consists of finite points. Therefore we have a continuous family of birationally nonequivalent algebraic models of M. But in general we do not know whether we are able to apply or not.

We return to the proof of Theorem 1. We may assume dim $M \geq 2$. According to a relative Nash theorem [1], one can imbed L in some $\boldsymbol{R}^t$ so that L is a nonsingular algebraic subset of $\boldsymbol{R}^t$ and each X_i is nonsingular algebraic subset of L. Moreover, by blowing up all X_i, we may suppose that every X_i is of codimension 1 in L. By the proof of [8, Remark 6.2.11], M is Nash diffeomorphic to $L-X$.

We now construct an irreducible algebraic model Y of L so that each $f(X_i)$ is a nonsingular algebraic subset of Y, where f is a diffeomorphism from L to Y. Fix an algebraic imbedding $L \rightarrow \boldsymbol{RP}^l$ and we identify L, X and X_i with its images, respectively. It follows from the proof of [3, Corollary 3.3] that there exists an irreducible nonsingular algebraic curve C in L satisfying the following three conditions: (1)C has a trivial normal bundle in L, (2)C is a disjoint s C^∞curves $C_1,\cdots,C_s$, each connected component of L containing precisely one C_i and (3)C does not intersect X.

Consider the triple $C \subset L \subset \boldsymbol{RP}^l$. Since $H_*^{alg}(\boldsymbol{RP}^l,\boldsymbol{Z}_2) = H_*(\boldsymbol{RP}^l,\boldsymbol{Z}_2)$ ($\boldsymbol{RP}^l$ has totally algebraic homology), it follows from the proof of [2, Theorem 3.1] that there exists a positive integer k and a Nash imbedding $e: L \times \{0\} \rightarrow \boldsymbol{RP}^l \times \boldsymbol{R}^k$ such that $Y := e(L)$ is a nonsingular algebraic subset of $\boldsymbol{RP}^l \times \boldsymbol{R}^k$, each $Z_i := e(X_i \times \{0\})$ is a nonsingular algebraic subset of $\boldsymbol{RP}^l \times \boldsymbol{R}^k$, and $C \times \{0\} \subset Y$. Since each connected component of Y contains a connected component of $C \times \{0\}$, Y is the required irreducible algebraic model of L.

Applying Lemma to the triple $(Y \times \boldsymbol{RP}^1, Y \times \{a\}, Z_i \times \boldsymbol{RP}^1)$(for some $a \in \boldsymbol{RP}^1$), we have the desired family. □

Proof of Theorem 2.

It is known that any compactifiable C^∞ manifold admits a nonaffine Nash manifold structure [7]. To apply Theorem 1, we show that each nonaffine Nash manifold M is C^∞ imbeddable into some Euclidean space as an affine Nash manifold. We may assume that M is connected. Using [7], we have a compact C^∞ manifold Y' with boundary K so that M is C^∞ diffeomorphic to the interior of Y'. Let Y be the double of Y'. By a relative Nash theorem [1], there exist nonsingular algebraic sets Z and Z' such that (Y,K) is pairwise C^∞ diffeomorphic to (Z,Z'). Since $Z-Z'$ is an affine Nash manifold, M is C^∞ diffeomorphic to a connected component of $Z'-Z$. An application of Theorem 1 completes the proof. □

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