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ON p-RADICAL GROUPS G AND THE NILPOTENCY INDICES OF $J(kG)$

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1. Introduction

Let kG be the group algebra of a finite group G over an algebraically closed field k of characteristic $p>0$, and let P be a Sylow p-subgroup of G.

Following Motose and Ninomiya [9] we call G *p-radical* if the induced module $(k_P)^G$ of the trivial kP-module k_P is completely reducible as a right kG-module.

In [10], Okuyama has proved that p-radical groups are p-solvable. And Tsushima has characterlized p-radical groups which are p-nilpotent by group theoretical properties (see Lemma 2.2). So it seems to be interesting to investigate the structure of p-radical groups of p-length 2 and in this paper we shall treat such a group with some additional properties.

Before describing our result we need to define some notations. Let $F=GF(q^n)$ be a finite field of q^n elements for prime q. Let V be the additive group of F. Let $T(q^n)$ be the set of semilinear transformations of the form $v\rightarrow av^{\sigma}$ with $v\in V$, $0\neq a\in F$, and σ a fied automorphism (see [11, p229]). Then we can consider the semidirect product $VT(q^n)$ of V by $T(q^n)$. Let λ be a generater of the multiplicative group of F and $\nu=\lambda^{q^{n/r}-1}$ for some integer r with $r|n$. Let $T_0=\{v\rightarrow av^{\sigma}|a\in <\nu>, \sigma\in \mathrm{Gal}(F/GF(q^{n/r}))\}$. Then we define $A_{q,n,r}= VT_0\subseteq VT(q^n)$.

Theorem 1. *Let G be a finite group with the following conditions.*

(1) $|G: O_{p',p,p'}(G)|=p$, $O_p(G)=1$ *and* $O^{p'}(G)=G$.

(2) *A Sylow p-subgroup P_0 of $O_{p',p}(G)$ is abelian.*

(3) $V=[O_{p'}(G),P_0]$ *is a minimal normal subgroup of G.*

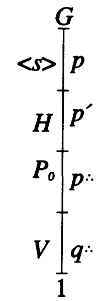
Then G is p-radical if and only if the following conditions (*A*), (*B*) *and* (*C*) *hold.*

(A) $\bar{G}=G/VP_0$ *is a Frobenius group with kernel* $O_{p'}(\bar{G})$.

(B) *V is an elementary abelian q-group for some prime* $q(\neq p)$.

(C) *One of the following* (1) *and* (2) *holds.*

(1) *The following* (i)-(vii) *hold.*



(i) $G=VN_G(P_0)$ *and* $V\cap N_G(P_0)=1$.

(ii) $P_0\lhd P_0H\lhd P_0H<s>$, *where* $|s|=p$ *and* H *is a* p'*-group.*

(iii) *By conjugation, we can regard* V *as an irreducible* $N_G(P_0)$*-module. Then* $V=V_1\times\cdots\times V_p$, *where* V_i, $1\leq i\leq p$, *are the homogeneous components of* V_{P_0}.

(iv) *Set* $P_i=C_{P_0}(V_1\times\cdots\times V_{i-1}\times V_{i+1}\times\cdots\times V_p)$, $1\leq i\leq p$.
Then $P_0=P_1\times\cdots\times P_p$.

(v) $V_1^{s^i}=V_{i+1}$, $P_1^{s^i}=P_{i+1}$, $0\leq i\leq p-1$.

(vi) V_i *and* P_i *are* H*-invariant,* $1\leq i\leq p$, *and* $VP_0=(V_1P_1)\times\cdots\times(V_pP_p)$.

(vii) *Set* $r=|H/C_H(V_1)|$ *and* $p^m=|P_1|$, $q^n=|V_1|$.
Then $r|n$ *and* $\dfrac{q^n-1}{q^{n/r}-1}=p^m$ *and* $V_iP_iH/C_H(V_i)\simeq A_{q,n,r}$, $1\leq i\leq p$.

(2) $C_G(v)\subseteq O_{p',p,p'}(G)$ *for any element* v *of* $V^{\#}$.

Next, let $t(G)$ be the nilpotency index of the radical $J(kG)$ of kG and let p^a be the order of Sylow p-subgroups of G.
Wallace [14] proved that if G is p-solvable, then $a(p-1)+1\leq t(G)\leq p^a$. If G has p-length 1, then by Motose and Ninomiya [8] $t(G)=a(p-1)+1$ if and only if P is elementary abelian.

All known examples of p-solvable group G with $t(G)=a(p-1)+1$ have p-length at most 2. Using Theorem 1, we can prove the following theorem.

Theorem 2. *If* G *is a* p*-radical group with* $t(G)=a(p-1)+1$, *then* $G=O_{p,p',p,p'}(G)$. *In particular, the* p*-length of* G *is at most* 2.

2. Preliminaries

In this section, we shall give some lemmas which will be used to prove the theorems.

Lemma 2.1. ([1, Theorem 6.5]). *Suppose that* $N\lhd G$. *Then the following* (1)–(3) *hold.*

(1) *If* G *is* p*-radical, so are* N *and* G/N.
(2) *If* N *is a* p*-group, then* G *is* p*-radical if and only if* G/N *is* p*-radical.*
(3) *If* G/N *is a* p'*-group, then* G *is* p*-radical if and only if* N *is* p*-radical.*

Lemma 2.2. ([13, Theorem 2]). *Let $G=PN$ be a p-nilpotent group with $N=O_{p'}(G)$. Then G is p-radical if and only if $[N,D]\cap C_N(D)=1$ for any p-subgroup D of G. In particular, if N is abelian, then G is p-radical.*

Lemma 2.3. *If G is p-radical, then $O^{p'}(G)$ is solvable.*

Proof. Suppose it is false and let G be a minimal couterexample. Then we have $G=O^{p'}(G)$. By Theorem 1 of [10], G is p-solvable. If $O_p(G)\neq 1$, then $G/O_p(G)$ is solvable since $G/O_p(G)$ is p-radical. Hence G is solvable, a contradiction. Hence $O_p(G)=1$, and so $O_{p'}(G)\neq 1$. Let P be a Sylow p-subgroup of $O_{p',p}(G)$, and set $W=<[O_{p'}(G),x]| x\in\Omega_1(Z(P))>$. If $W=1$, then $1\neq\Omega_1(Z(P))\subseteq C_G(O_{p'}(G))\subseteq O_{p'}(G)$, a contradiction. Since $G=O_{p'}(G)N_G(P)$, $1\neq W\lhd G$. Furthermore, for $x\in\Omega_1(Z(P))$ $[O_{p'}(G),x]$ is a normal subgroup of $O_{p'}(G)$ and is nilpotent by Thompson [12] as $C_{O_{p'}(G)}(x)\cap[O_{p'}(G),x]=1$ (see Lemma 2.2). Hence W is solvable. Since G/W is p-radical, G/W is solvable. This implies that G is solvable, contrary to our choice of G.

Let B be a block of kG. We call B a *p-radical block* if $k\otimes_P B$ is semisimple. Let $N\lhd G$ and b_0 a block of kN that is covered by B. Let T be the inertia group of b_0. Then there exists a unique block b of kT with $b^G=B$. We call b the *Fong correspondent of B w.r.t.(K,T)*. Then the next lemma holds.

Lemma 2.4. *The following* (1) *and* (2) *hold.*

(1) *B is p-radical if and only if $k\otimes_{P^y\cap T}b$ is a semisimple kT-module for any $y\in G$.*

(2) ([13] Tsushima) *If $|G:T|$ is a power of p, then B is p-radical if and only if b is p-radical.*

Proof. (1) Various facts are known about the relationship between B and b. $B=(kG)b(kG)$ and $J(B)=(kG)J(b)(kG)$. Furthermore, $(kG)b$ is a direct summand of B as a $k(G\times T)$-module. Hence $k\otimes_P(kG)b$ is a direct summand of $k\otimes_P B$ as a right kT-module. On the other hand, $k\otimes_P kG\simeq\bigoplus_{y\in P|G/T} k\otimes_{P^y\cap T}kT$ by Mackey decomposition, and so $k\otimes_P(kG)b\simeq$ $\bigoplus_{y\in P|G/T}\otimes_{P^y\cap T}b$ as right kT-modules.

Assume that B is p-radical. Since $(k\otimes_P B)J(b)\subseteq(k\otimes_P B)J(B)=0$, $(k\otimes_{P^y\cap T}b)J(b)=0$, and so $k\otimes_{P^y\cap T}b$ is semisimple.

Conversely, assume that $k\otimes_{P^y\cap T}b$ is a semisimple kT-module for any $y\in G$. Then $k\otimes_P(kG)b$ is a semisimple kT-module. Therefore $U=(k\otimes_P(kG)b)\otimes_T kG$ is semisimple by Fong's theory. Since $k\otimes_P B$ is a natural homomorphic image of U, it is also semisimple.

(2) Since $G=PT$ by assumption, $P^y\cap T$ is a Sylow p-subgroup of T for any $y\in G$, and hence $\mathrm{k}\otimes_{P^y\cap T}b$ is semisimple if and only if b is p-radical. Therefore (2) follows from (1).

Let $G\triangleright V$. We let $\mathrm{Irr}(V)$ be the set of ordinary irreducible characters of V and let $I_G(\varphi)$ be the inertia group of $\varphi\in\mathrm{Irr}(V)$.
Furthermore, for a block B of kG, let $\mathrm{Irr}(B)$ be the set of irreducible characters of G belonging to B.

Lemma 2.5. *Let $G=LV\triangleright V$, where V is an abelian p'-group with $L\cap V=1$. Let $\varphi\in\mathrm{Irr}(V)$ with $I_G(\varphi)=G$. Then L is p-radical if and only if all blocks which cover φ are p-radical.*

Proof. Let b be a block of kG which covers φ. Let $\chi\in\mathrm{Irr}(b)$. Then each irreducible constituent of χ_V is φ. Set $U=\mathrm{Ker}\,\varphi$. Then we have $\mathrm{Ker}\,\chi\supseteq U$, and so $\mathrm{Ker}\,b\supseteq U$. Set $\bar{G}=G/U$, then $\bar{G}=\bar{L}\times\bar{V}$. Let e be a centrally primitive idempotent corresponding to φ. Then the sum of all blocks which cover φ is isomorphic to $k\bar{L}\otimes_k ke\simeq kL$. Then the lemma follows immediately.

Throught this paper, we let $\mathscr{F}$ be the family of all finite group G such that $t(G)=a(p-1)+1$, where p^a is the order of a Sylow p-subgroup of G.

Lemma 2.6. *Let G be a p-solvable group and $N\triangleleft G$. If $G\in\mathscr{F}$, then $G/N\in\mathscr{F}$.*

Proof. Let p^a, p^b be the orders of Sylow p-subgroups of G and N, respectively.

At first, assume N is a p'-group. By Theorem 2.2, 3.3 of [14], $a(p-1)+1\le t(G/N)\le t(G)=a(p-1)+1$. Hence $t(G/N)=a(p-1)+1$, and so $G/N\in\mathscr{F}$.

Next, assume N is a p-group. By Theorem 2.4, 3.3 of [14], $b(p-1)+(a-\mathrm{b})(\mathrm{p}-1)+1\le t(N)+t(G/N)-1\le t(G)=a(p-1)+1$. Hence $t(N)=b(p-1)+1$ and $t(G/N)=(a-b)(p-1)+1$, and so $G/N\in\mathscr{F}$.

Now, we shall consider the general case. If $N\neq 1$, $0_{p'}(N)\neq 1$ or $0_p(N)\neq 1$ since N is p-solvable. By induction on $|N|$, the lemma follows.

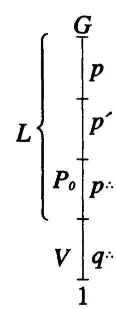
Lemma 2.7. *Let $G=LV\trianglerighteq V$, where V is a p-group and L is a p-radical p-nilpotent group with $L\cap V=1$. If $G\in\mathscr{F}$, then $\langle VO_{p'}(L),s\rangle\in\mathscr{F}$ for any p-element $s\in L$.*

Proof. Set $H=O_{p'}(L)$. By Theorem 3.1 of [14], $|s|=p$ and V is elementary abelian. Since L is p-radical, $[H,s]\langle s\rangle$ is a Frobenius group by Lemma 2.2. By Theorem 2.7 of [5], $gr\ kV$ is semisimple as a kG-module. Let $N=V[H,s]\langle s\rangle\lhd G$. *Then* $gr\ kV$ is semisimple as a kN-module, and so $N\in\mathscr{F}$ by Theorem 2.7 of [5]. Set $M=\langle VO_{p'}(L),s\rangle$. Since $M\trianglerighteq N$ and M/N is a p'-group, $M\in\mathscr{F}$.

3. Proof of Theorem 1

First we shall prove the if -part of Theorem 1. Let G be a finite group with conditions (1)–(3) in Theorem 1 and we assume that G is p-radical.

Lemma 3.1. *$V=[O^{p'}(G),P_0]$ is an elementary abelian q-group for some prime $q(\neq p)$ and $N_G(P_0)$ is a complement of V in G.*



Proof. Since G is p-radical with $O^{p'}(G)=G$, G is solvable by Lemma 2.3. In particular, V is solvable. Then V is an elementary abelian q-group for some prime $q(\neq p)$ by the condition (3) in Theorem 1.

Next, $G=VN_G(P_0)$ by the Frattini argument. Since $[N_V(P_0),P_0]\subseteq V\cap P_0=1$, $N_V(P_0)\subseteq C_V(P_0)=1$. Thus $N_G(P_0)$ is a complement of V in G.

Now, set $L=N_G(P_0)$ and let P be a Sylow p-subgroup of L.

Lemma 3.2. *$\bar{L}=L/P_0$ is a Frobenius group with kernel $O_{p'}(\bar{L})$ and complement $\bar{P}$.*

Proof. By the condition (1) in Theorem 1, $O^{p'}(\bar{L})=\bar{L}$. Then $\bar{L}=[O_{p'}(\bar{L}),\bar{P}]\bar{P}$ since $\bar{L}=O_{p',p}(\bar{L})$. Furthermore, since $\bar{L}\simeq G/VP_0$, $\bar{L}$ is p-radical by Lemma 2.1 (1), and so $[O_{p'}(\bar{L}),\bar{P}]\cap C_{O_{p'}(\bar{L})}(\bar{P})=1$. Thus $\bar{L}$ is a Frobenius group.

By Lemma 3.2, G satisfies the condition (A) in Theorem 1.

Lemma 3.3. *Let $\varphi\in\mathrm{Irr}(V)$ with $\varphi\neq 1_V$. Then one of the following*

(1) and (2) holds.

(1) $I_G(\varphi) \subseteq O_{p',p,p'}(G)$.

(2) *$L = I_L(\varphi)P_0$ and $I_L(\varphi) \cap P_0 = 1$.*

Proof. Set $T = I_G(\varphi)$ and assume that $T \nsubseteq O_{p',p,p'}(G)$. Let b_0 be a block of kV with $\varphi \in b_0$. Let B be a block of kG which covers b_0 and let b be the Fong correspondent of B *w.r.t.*(V,T). Let D be a defect group of b and let P^* be a Sylow p-subgroup of G which contains D. Since B is p-radical, $k \otimes_{P^{*y} \cap T} b$ is semisimple for any $y \in G$ by Lemma 2.4. On the other hand, there exists an irreducible kT-module in b with vertex D (see [1, Lemma 4.6]). Hence $P^{*yt} \cap T \supseteq D$ for some $t \in T$, in particular $P^{*yt} \cap T = D$. Set $\bar{G} = G/VP_0$. Since $D \nsubseteq O_{p',p,p'}(G)$, $\bar{D}^{\bar{y}\bar{t}} = \bar{P}^{*\bar{y}\bar{t}} = \bar{D} \neq 1$. Since $\bar{G}$ is a Frobenius group, $\overline{yt} \in \bar{P}^*$. Hence $y \in TP^* = TP_0$. This implies that $G = TP_0$, and so $L = (L \cap T)P_0$.

Next, set $Q = (L \cap T)P_0$. Since P_0 is abelian, $Q \lhd L$. Since $[V,Q] \subseteq \ker\varphi \neq V$ and $[V,Q]$ is L-invariant, $[V,Q] = 1$ by the minimality of V, and so $Q \subseteq O_p(G) = 1$.

Now assume that $C_G(v) \nsubseteq O_{p',p,p'}(G)$ for some $v \in V^\#$. As V is a p'-group, this condition is equivalent to the condition that $I_G(\varphi) \nsubseteq O_{p',p,p'}(G)$ for some $\varphi \neq 1_V \in \mathrm{Irr}(V)$ (see for example [3]§13). Then for such a φ, $I_L(\varphi)P_0 = L$ and $I_L(\varphi)P_0 = 1$ by Lemma 3.3.

Let H be a Hall p'-subgroup of $I_L(\varphi)$ (which is also a Hall p'-subgroup of L). Then $H \lhd I_L(\varphi)$ and $I_L(\varphi) = H\langle s\rangle$ for some element s of order p in $P \backslash P_0$.

We continue our discussion by assuming that there exists an element s of order p in $N_P(H) \backslash P_0$ and shall prove that the condition (C)(1) in Theorem 1 holds.

Lemma 3.4. *If W is a subgroup of V with $|V\colon W| = q$ and $[s,V] \subseteq W$, then there exists a Hall p'-subgroup H_1 of L with $[H_1,V] \subseteq W$.*

Proof. Let φ be an irreducible character of V with kernel W. Then since $[s,V] \subseteq W$, $s \in I_G(\varphi)$. Now the result follows by Lemma 3.3.

Lemma 3.5. *Assume that $V = W \times W^s \times \cdots \times W^{s^{p-1}}$, where W is P_0H-invariant. Then the following* (1) *and* (2) *hold.*

(1) *If W_1 is a subgroup of W with $|W\colon W_1| = q$, then there exsts a Hall p'-subgroup H_1 of L with $[H_1,W] \subseteq W_1$. In particular, W_1 is H_1-invariant.*

(2) *Let W_0 be an irreducible P_0-module of W. Then W_0 is H-invariant.*

Proof. (1) We can easily see that $V=W\times[V,s]$ as V is an abelian p'-group. Put $U=W_1\times[V,s]$. By Lemma 3.4, there exists a Hall p'-subgroup H_1 of L with $[H_1,V]\subseteq U$. Since $H_1\subseteq P_0H$, W is H_1-invariant, and so $[H_1,W]\subseteq W\cap U=W_1$. In particular, W_1 is H_1-invariant.

(2) If W_0 is not H-invariant, there exists an $h\in H$ with $W_0\neq W_0^h$. Let $w\in W_0^\#$. Since W is elementary abelian, there exist subgroups W' and W'' of W with $W_0=<w>\times W'$ and $W=W_0\times W_0^h\times W''$. Set $W_1=W'\times W_0^h\times W''$, then $|W:W_1|=q$ and $W_0\nsubseteq W_1$. By (1), W_1 is H_1-invariant for some Hall p'-subgroup H_1 of L. Since $h\in H\subseteq P_0H_1$, $h=ah_1$ for some $a\in P_0$ and $h_1\in H_1$. Furthermore, since $W_1\supseteq W_0^h=W_0^{h_1}$, $W_0=(W_0^{h_1})^{h_1^{-1}}\subseteq W_1^{h_1^{-1}}=W_1$, contrary to our choice of W_1.

Since L acts on V by conjugation, we can regard V as an L-module. Furthermore, since V is a minimal normal subgroup of G, V is an irreducible L-module.

Lemma 3.6. *Let $V=V_1\times\cdots\times V_n$, where V_i, $1\leq i\leq n$, are the homogeneous components of V with respect to P_0. Then $n=p$ and we may take $V_1^{s^i}=V_{i+1}$, $0\leq i\leq p-1$. Furthermore, V_i is an irreducible P_0-module which is H-invariant, $1\leq i\leq p$.*

Proof. We divide the proof of Lemma 3.6 into three steps.

STEP 1. *s induces a regular permutation representation on the set $\{V_1,\cdots,V_n\}$.*

Proof. Suppose it is false, and let s fix V_1. Since P_0 is abelian, $\bar{P}_0=P_0/C_{P_0}(V_1)$ is cyclic. Since $\overline{<s>P_0}=<s>P_0/C_{P_0}(V_1)$ has an abelian subgroup of (p,p) type, there exists an $x\in<s>P_0$ with $C_{V_1}(x)\neq1$ and $\bar{x}\neq1$. If $x\in P_0$, then $C_{V_1}(x)=1$ since V_1 is homogeneous as a P_0-module. Hence $x\notin P_0$. Let $v\in C_{V_1}(x)^\#$, then $x\in C_P(v)$. Now by Lemma 3.3 and the remark before Lemma 3.4, $C_{P_0}(v)=1$. So $C_{P_0}(V_1)\subseteq C_{P_0}(v)=1$ and P_0 is cyclic. Since $L\rhd P_0$ and P_0 is abelian, $\bar{L}=L/P_0$ acts on P_0 by conjugation. Since $O^{p'}(\bar{L})=\bar{L}$ and $\mathrm{Aut}(P_0)$ is abelian, $O_{p'}(\bar{L})$ centralizes P_0. Hence $L=O_{p',p}(L)$, and so $G=O_{p',p}(G)$, contrary to the condition (1) of Theorem 1. Hence s acts regularly on the set $\{V_1,\cdots,V_n\}$.

STEP 2. *Let $\{V_1,\cdots,V_t\}$ be an H-*orbit *of $\{V_1,\cdots,V_n\}$, and set $W=V_1\times\cdots\times V_t$. Then $V=W\times W^s\times\cdots\times W^{s^{p-1}}$.*

Proof. Suppose s fixes an H-orbit $\{V_1,\cdots,V_t\}$. Since H is a p'-group, $(p,t)=1$. Therefore s fixes V_i for some i, contrary to Step 1. Hence s doesn't fix an H-orbit $\{V_1,\cdots,V_t\}$ and this implies that $W\times W^s\times\cdots\times W^{s^{p-1}}\subseteq V$. Furthermore, as $W\times W^s\times\cdots\times W^{s^{p-1}}$ is L-invariant, we have $V=W\times W^s\times\cdots\times W^{s^{p-1}}$.

STEP 3. Proof of Lemma 3.6.

Take an irreducible P_0-submodule W_0 in V_1. Then by Lemma 3.5(2), W_0 is P_0H-invariant. Therefore $W_0\times W_0^s\times\cdots\times W_0^{s^{p-1}}$ is L-invariant and coincides with V. Thus $W_0=V_1$ (and $t=1$ in Step 2) and the lemma follows.

From Theorem 15.16 of [3], we obtain the following lemma.

Lemma 3.7. *Let $P_0\supseteq Q_1\supseteq Q_2\supseteq\Phi(P_0)$, where Q_1 and Q_2 are $H<s>$-invariant subgroups of P_0. If H acts non-trivially on Q_1/Q_2, then $|Q_1/Q_2|\geq p^p$.*

Lemma 3.8. *Let ϕ be a homomorphism of P_0 into $P_0/C_{P_0}(V_1)\times\cdots\times P_0/C_{P_0}(V_p)$ which is defined by the rule $\phi(x)=(C_{P_0}(V_1)x,\cdots,C_{P_0}(V_p)x)$. Then ϕ is an isomorphism.*

Proof. $\mathrm{Ker}\,\phi=C_{P_0}(V)=1$ and P_0 is isomorphic to a subgroup of $P_0/C_{P_0}(V_1)\times\cdots\times P_0/C_{P_0}(V_p)$. On the other hand, $P_0/C_{P_0}(V_i)$ is cyclic, $1\leq i\leq p$. Hence the rank of P_0 is at most p and $|P_0/\Phi(P_0)|\leq p^p$, and so $|P_0/\Phi(P_0)|=p^p$ by Lemma 3.7.

Suppose next that ϕ is not any epimorphism. Let P_0 have exponent p^m. Then $p^m=|P_0/C_{P_0}(V_1)|$ as $P_0/C_{P_0}(V_1)$ is cyclic. Set $\Omega_{m-1}(P_0)=\{x\in P_0|x^{p^{m-1}}=1\}$. Then $\Omega_{m-1}(P_0)$ is $<s>H$-invariant and $P_0\supsetneqq\Omega_{m-1}(P_0)\supsetneqq\Phi(P_0)$. By Lemma 3.7, H acts trivially on both $P_0/\Omega_{m-1}(P_0)$ and $\Omega_{m-1}(P_0)/\Phi(P_0)$ which is a contradiction.

Set $P_i=C_P(V_1\times\cdots\times V_{i-1}\times V_{i+1}\times\cdots\times V_p)$, $1\leq i\leq p$. Then $P_0=P_1\times\cdots\times P_p$ by Lemma 3.8, and so $VP=(V_1P_1)\times\cdots\times(V_pP_p)$. P_i is cyclic as it acts on V_i irreducibly and faithfully.

Lemma 3.9. *$C_{P_1H}(v)$ contains a Hall p'-subgroup of P_1H for any $v\in V_1$.*

Proof. Set $K=P_1H$. Then K acts on $\mathrm{Irr}(V_1)$ and on the set of elements of V_1. We claim that P_1-orbits coincide with K-orbits on

V_1. Let $A_1, \cdots, A_m$ be P_1-orbits on $\mathrm{Irr}(V_1)$ and let $B_1, \cdots, B_n$ be P_1-orbits on the set of elements of V_1. By Corollary 6.3.3 of [3], we have $m=n$.

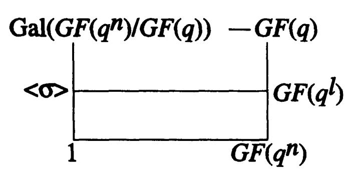
Let χ be an irreducible character in A_1, and set $W_1 = \mathrm{Ker}\,\chi$. Now $V = V_1 \times V_1^s \times \cdots \times V_1^{s^{p-1}}$, and V_1 is P_0H-invariant. Hence, by lemma 3.5 (1), there exists a Hall p'-subgroup H_1 of L with $[H_1, V_1] \subseteq W_1$. As $P_0 = P_1 C_{P_0}(V_1)$ by the remark before Lemma 3.9, we may take H_1 in K. Since $K = P_1H = P_1H_1$ and H_1 fixes χ, A_1 coincides with a K-orbit on $\mathrm{Irr}(V_1)$. Similarly, A_i is a K-orbit on $\mathrm{Irr}(V_1)$, $2 \le i \le m$. By Corollary 6.3.3 of [3] again, B_i is a K-orbit on the set of elements of V_1, $1 \le i \le m$, and our claim follows.

Let v be an element of V_1. For each element $g \in K$, we have $v^g = v^x$ for some $x \in P_1$ by the above claim. Hence $gx^{-1} \in C_K(v)$, and so $g \in C_K(v)P_1$. This implies that $K = C_K(v)P_1$. Hence $C_K(v)$ contains a Hall p'-subgroup of K.

Lemma 3.10. *Set $r = |H/C_H(V_1)|$ and $p^m = |P_1|$, $q^n = |V_1|$. Then $r|n$ and $\dfrac{q^n-1}{q^{n/r}-1} = p^m$ and $V_iP_iH/C_H(V_i) \simeq A_{q,n,r}$, $1 \le i \le p$.*

Proof. Set $N = V_1P_1H$ and $\bar N = V_1P_1H/C_H(V_1)$. By Lemma 3.9, $C_{\bar N}(\bar v)$ contains a Hall p'-subgroup of $\bar N$ for any $v \in V_1$. Hence every $\bar P_1$-orbit of V_1 contains an element which is centralized by $\bar H$. Since P_1 is cyclic, by Proposition 19.8 of [11] and Lemma 3.6, we can identify V_1 with $GF(q^n)$ in such a way that $\overline{P_1H} \subseteq T(q^n) = \{v \to av^\sigma | a \in GF(q^n)^\#, \sigma \in Gal(GF(q^n)/GF(q))\}$ and $\bar P_1 \subseteq \{v \to av | a \in GF(q^n)^\#\}$.

As $C_{\bar H}(\bar P_1)$ is contained in any Hall p'-subgroup of $\overline{P_1H}$, $C_{\bar H}(\bar P_1) = 1$. Since $\bar P_1$ is cyclic and $\bar H$ is a p'-group, $\bar H$ acts regularly on $\bar P_1$.



Let $\bar H = <\eta>$, where $\eta(v) = bv^\sigma$, for some $b \in GF(q^n)^\#$ and $\sigma \in \mathrm{Gal}(GF(q^n)/GF(q))$ for $v \in V_1$. By Proposition 19.8 of [11], $\bar H \simeq \overline{P_1H}/C_{\overline{P_1H}}(\bar P_1) \subseteq \mathrm{Gal}(GF(q^n)/GF(q))$, and so $r = |\bar H| = |\sigma|$ with $r|n$. Then $<\sigma> = \mathrm{Gal}(GF(q^n)/GF(q^l))$, where $n = lr$. Let $\bar P_1 = <x_a>$, where $x_a(v) = av$, $a \in GF(q^n)^\#$. If $a^i \in GF(q^l)$ for some i, then $\eta x_{a^i}(v) = \eta(a^iv) = b(a^iv)^\sigma = (a^i)^\sigma bv^\sigma = a^ibv^\sigma$. On the other hand, $x_{a^i}\eta(v) = x_{a^i}(bv^\sigma) = a^ibv^\sigma$. Hence $\eta x_{a^i} = x_{a^i}\eta$, and so $\eta^{-1}x_{a^i}\eta = x_{a^i}$. Since $x_{a^i} = (x_a)^i$, $\eta^{-1}(x_a)^i\eta = (x_a)^i$. Thus $a^i = 1$ as $\bar H$ acts regularly on $\bar P_1$. Hence $<a> \cap GF(q^l)^\# = 1$.

Since $<a> \subseteq GF(q^n)$ is a P_1-orbits, $\eta(a^i)=a^i$ for some i. Hence $\eta(a^i)=b(a^i)^\sigma=a^i$, and so $b=(a^i)(a^{-i})^\sigma \in <a>$. Thus $b=a^j$ for some j, and so $\eta=x_{a^j}\sigma$. Then $\bar{P}_1\bar{H}=<x_a><x_{a^j}\sigma>=<x_a><\sigma>$.

Let c be any element of $GF(q^n)^\#$. Since $\{c, ca, \cdots, ca^{p^m-1}\}$ is a $\bar{P}_1$-orbit, $(ca^i)^\sigma=ca^i$ for some i. Hence $ca^i \in GF(q^l)$. This implies that $GF(q^n)^\#=<a>GF(q^l)^\#$. Hence $q^n-1=(q^l-1)p^m=(q^{n/r}-1)p^m$. Furthermore, we have $\overline{V_1P_1H}=V_1<x_a><\sigma>\simeq A_{q,n,r}$.

Now we completed the proof of the if-part of Theorem 1 and we shall prove the "only-if-part" of the theorem.

Lemma 3.11. *Let G satisfy the conditions* (1)–(3) *in Theorem* 1. *If G has the conditions* (*A*), (*B*) *and* (*C*) *of Theorem* 1, *then G is p-radical.*

Proof. Let $\varphi \in \mathrm{Irr}(V)$ and let B be a block of kG which covers φ. Then we shall prove that B is p-radical.

Case 1. $\varphi=1_V$.

Since L/P_0 is a Frobenius group, L is p-radical by Lemmas 2.1, 2.2. By lemma 2.5, B is p-radical.

Case 2. $\varphi\neq 1_V$.

Set $T=I_G(\varphi)$. Suppose that G satisfies the condition (C)(2) in Theorem 1. By the remark before Lemma 3.4, $T \subseteq O_{p',p,p'}(G)$. Set $N=VP_0$, then $D\subseteq N$, where D is a defect group of B. Then $J(B)=J(kN)B$ by Theorem 2.3 of [1]. By Lemma 2.2, N is p-radical, and so $(k\otimes_P B)J(B)=k\otimes_P J(kN)B \subseteq k\otimes_P J(kP_0)(kN)B=0$. Hence $k\otimes_P B$ is semisimple, and so B is p-radical.

Suppose next that G satisfies the condition (C)(1) in Theorem 1. Let v be an element of V. Then $v=v_1\cdots v_p$, where $v_i\in V_i$, $1\leq i\leq p$. Then, by (C)(2)(vii) in Theorem 1, there exist an $x_i\in P_i$ with $[v_i^{x_i}, H]=1$, $1\leq i\leq p$, where H is a Hall p'-subgroup of G. Set $x=x_1\cdots x_p$. Then $v^x=v_1^x\cdots v_p^x=v_1^{x_1}\cdots v_p^{x_p}$, and so $[v^x, H]=1$. This implies that $C_G(v)$ contains a Hall p'-subgroup of G for each $v\in V$. By a similar argument in the proof of Lemma 3.9, T contains a Hall p'-subgroup.

Since $V\subseteq T$ and $G=VL$, we have $T=V(L\cap T)$. $T\cap L$ is p-closed or $T\cap L/O_P(T\cap L)$ is a Frobenius group. In each case, $T\cap L$ is p-radical by Lemmas 2.1, 2.2. Let b be the Fong correspondent of B *w.r.t.*(V, T). Then b is p-radical by Lemma 2.5. Hence B is p-radical by Lemma 2.4(2).

4. Proof of Theorem 2

In this section, we shall prove the following theorem from which Theorem 2 follows by using Lemma 2.6.

Theorem 3. *If G is a p-radical group with $G/O_{p'}(G)\in\mathscr{F}$, then $G=O_{p,p',p,p'}(G)$.*

Proof. Suppose it is false and let G be a minimal counterexample of Theorem 3. Now we divide the proof into several steps. At first, we shall prove that G satisfies the conditions (1)–(3) in Theorem 1.

STEP 1. $O_p(G)=1$ *and* $O^{p'}(G)=G$.

Proof. Suppose that $O_p(G)\neq 1$. Set $\bar{G}=G/O_p(G)$. Then $\bar{G}$ is p-radical. Furthermore, since $\bar{G}/O_{p'}(\bar{G})$ is a homomorphic image of $G/O_{p'}(G)$, $\bar{G}/O_{p'}(\bar{G})\in\mathscr{F}$ by Lemma 2.6. By the minimality of G, $\bar{G}=O_{p,p',p,p'}(\bar{G})=O_{p',p,p'}(\bar{G})$, and so $G=O_{p,p',p,p'}(G)$, contrary to our choice of G. Next we assume that $O^{p'}(G)\subsetneq G$. Set $U=O^{p'}(G)$, then since U is p-radical and $U/O_{p'}(U)\in\mathscr{F}$, $U=O_{p,p',p,p'}(U)=O_{p,p',p}(U)$. Hence $G=O_{p,p',p,p'}(G)$, a contradiction.

Let V be a minimal normal subgroup of G. By Step 1 and Lemma 2.3, V is an abelian p'-group. Furthermore, let P_0 be a p-subgroup of G with $O_p(G/V)=P_0V/V$. Then we show the following Step 2.

STEP 2.

(1) $G/V=O_{p,p',p}(G/V)$, *in particular*, $G=O_{p',p,p',p}(G)$.
(2) $P_0\neq 1$ *is elementary abelian.*
(3) *Let M be a Hall p'-subgroup of G. Then there exists an* $s\in N_G(M)$ *with* $|s|=p$ *and* $G=O_{p',p,p'}(G)<s>$.
(4) $V=[O_{p'}(G),P_0]$.

Proof. Set $\bar{G}=G/V$. For (1), we note that $\bar{G}$ is p-radical and $\bar{G}\in\mathscr{F}$. Since $V\neq 1$, $\bar{G}=O_{p,p',p,p'}(\bar{G})=O_{p,p',p}(\bar{G})$, so that $G=O_{p',p,p',p}(G)$.

For (2) and (3), let P be a Sylow p-subgroup of G such that $N_P(M)$ is a Sylow p-subgroup of $N_G(M)$. If $P_0=1$, then $\bar{G}=O_{p',p}(\bar{G})$, and so $G=O_{p',p}(G)$, contrary to our choice of G. Hence $P_0\neq 1$. By Theorem 3.1 of [14], P_0 is elementary abelian. Hence $\bar{P}_0=[\bar{P}_0,\bar{M}]\times C_{\bar{P}_0}(\bar{M})$. Since $\bar{G}=\bar{P}_0N_{\bar{G}}(\bar{M})=\bar{P}_0N_{\bar{P}}(\bar{M})\bar{M}$, $\bar{G}=[\bar{P}_0,\bar{M}]\bar{M}N_{\bar{P}}(\bar{M})$. Furthermore, $[\bar{P}_0,\bar{M}]\bar{M}\lhd\bar{G}$ and $[\bar{P}_0,\bar{M}]\bar{M}\cap N_{\bar{P}}(\bar{M})=[\bar{P}_0,\bar{M}]\cap N_{\bar{P}}(\bar{M})=[\bar{P}_0,\bar{M}]\cap C_{\bar{P}_0}(\bar{M})=1$. Therefore $N_{\bar{P}}(\bar{M})$ is a complement of $[\bar{P}_0,\bar{M}]\bar{M}$ in $\bar{G}$. By Theorem

3.1 of [14], $N_{\bar{P}}(\bar{M})$ is elementary abelian. If $N_{\bar{P}}(\bar{M}) \subseteq \bar{P}_0$, then $\bar{G}=\bar{P}_0\bar{M}$. Hence $G=O_{p',p,p'}(G)$, contrary to our choice of G. Thus $N_{\bar{P}}(\bar{M}) \nsubseteq \bar{P}_0$. Therefore there exists an element $s \in N_P(M)$ with $|s|=p$ and $s \notin P_0$.

Now set $N=O_{p',p,p'}(G)<s>$. Since $N \lhd G$, N is p-radical by Lemma 2.1 (1). Furthermore, $\bar{N}=N/O_{p'}(G)=N/O_{p'}(N) \in \mathscr{F}$ by Lemma 2.7. If $N \subsetneq G$, then $N=O_{p,p',p,p'}(N)$. Since $N \lhd G$, $O_p(N) \subseteq O_p(G)=1$ and $O_{p',p}(N) \subseteq O_{p',p}(G)$. Hence $N=O_{p',p,p'}(N)$, and so $s \in O_{p',p}(N) \subseteq O_{p',p}(G)$, contrary to our choice of s. Thus $G=O_{p',p,p'}(G)<s>$.

For (4), we note $[O_{p'}(G),P_0] \subseteq O_{p'}(G) \cap P_0V=V$.

Step 3. Proof of Theorem 3.

By Step 2, there exists an s with $|s|=p$ and $G=O_{p',p,p'}(G)<s>$. By the remark before Lemma 3.4, the condition (C)(1) in Theorem 1 holds. We set $\bar{G}=G/V$, then $\bar{G} \in \mathscr{F}$ by Lemma 2.6. Since $\bar{P}_1$ is $\bar{H}$-invariant, $\bar{P}_1$ is $\bar{s}$-invariant (see the proof of Lemma 11(7) of [7]), and we have a contradiction.

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References

[1] W. Feit: "The representation theory of finite group," North-Holland, Amsterdam, 1982.
[2] D. Gorenstein: "Finite groups," Harper & Row, New York, 1968.
[3] I.M. Issacs: "Character theory of finite groups," Academic Press, New York, 1976.
[4] G. Karpilovsky: "The Jacobson radical of group algebra," North-Holland, Amsterdam, 1987.
[5] M. Lorenz: *On Loewy lengths of projective modules for p-solvable groups*, Comm. Algebra **13**(1985), 1193–1212.
[6] O. Manz: *On the modular version of Ito's theorem of character degrees for groups of odd order*, Nagoya Math J. **105**(1987), 121–128.
[7] K. Motose: *On the nilpotency index of the radical of a group algebra IV*, Math. J. Okayama Univ. **25**(1983), 35–42.
[8] K. Motose and Y. Ninomiya: *On the nilpotency index of the radical of a group algebra*, Hokkaido Math. J. **4**(1975), 261–264.
[9] K. Motose and Y. Ninomiya: *On the subgroup H of a group G such that $J(KH)KG \supset J(KG)$*, Math. J. Okayama Univ. **17**(1975), 171–176.
[10] T. Okuyama: *p-Radical groups are p-solvable*, Osaka J. Math. **23**(1986), 467–469.
[11] D.J. Passman: "Permutation groups", Benjamin, New York, 1968.
[12] J.G. Thompson: *Finite groups with fixed-point-free automorphisms of prime order*,

Proc. Nat. Acad. Sci. **45**(1959), 578–581.
[13] Y. Tsushima: *On p-radical groups*, J. Algebra **103**(1986), 80–86.
[14] D.A.R. Wallace: *Lower bounds for the radical of the group algebra of a finite p-souble group*, Proc. Edinburgh Math. Soc. **16**(1968/69), 127–134.

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