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EXISTENCE AND UNIQUENESS OF SOLUTIONS FOR INTEGRODIFFERENTIAL EQUATIONS WITH TIME DELAY IN HILBERT SPACE

Dedicated to Professor Hiroki Tanabe on his sixtieth birthday

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0. Introduction

In this paper we consider the following integrodifferential equations with time delay in a real Hilbert space H

$$(0.1)\qquad \begin{cases} \dfrac{d}{dt}u(t)+Au(t)+A_1u(t-h)+\displaystyle\int_{-h}^{0} a(-s)A_2u(t+s)ds=f(t), \\ u(0)=x,\ u(s)=y(s)\quad -h\le s<0. \end{cases}$$

Here, A is a positive definite self-adjoint operator and A_1, A_2 are closed linear operators with domains containing that of A. The notation h denotes a fixed positive number and a $(\cdot)$ is a real valued function belonging to $C^3([0,h])$.

The equations of the type (0.1) were investigated by G. Di Balasio, K. Kunisch and E. Sinestrari [3], S. Nakagiri [5], H. Tanabe [9] and D.G. Park and S.Y. Kim [6], etc. Particularly, G. Di Balasio, K. Kunisch and E. Sinestrari [3] showed the existence and uniqueness of a solution for $f\in L^2(0,T;\ H)$, $Ay\in L^2(-h,0;\ H)$ and $x\in(D(A),H)_{1/2,2}$. In [6] D.G. Park and S.Y. Kim also got a similar result to [3] under the following conditions:

$$(0.2)\qquad \begin{cases} f\in L^2(0,T;\ H;\ tdt) \text{ and } Ay\in L^2(-h,0;\ H\text{: } tdt), \\ A^{-1/2}f\in L^2(0,T;\ H) \text{ and } A^{1/2}y\in L^2(-h,0;\ H), \\ f(t) \text{ and } Ay(t-h) \text{ are improperly integrable at } t=0, \\ x \text{ is an arbitray element of } H, \end{cases}$$

where $f \in L^2(0,T;\ H;\ tdt)$ means that f is a measurable function with values in H on $[0,T]$ and $\int_0^T |f(t)|_H^2 tdt < \infty$, and A^α is a fractional power of A.

Since the equation (0.1) is of parabolic type, we want x to be an arbitrary element of H. Then the integral in (0.1) exists only in the improper sense no matter what nice functions f and Ay may be. Hence, it would be considered natural to investigate our problem under the hypothesis such as (0.2).

In this paper our object is to study the existence and uniqueness of the solution of the equation (0.1) under assumptions weaker than (0.2) for the initial function y and the inhomogeneous term f, which are, roughly speaking, stated as follows:

$$(0.3)\qquad \begin{cases} f \in \cap_{\delta>0} L^2(\delta,\ T;\ H) \text{ and } Ay \in \cap_{\delta>0} L^2(-h+\delta,\ 0;\ H), \\ f(t) \text{ and } Ay(t-h) \text{ are improperly integrable at } t=0. \end{cases}$$

First we show the existence and uniqueness of a weak solution for which $A^{-\alpha}u$ is continuous in $[0,T]$ for an arbitrary positive number α. Next, under some conditions which are weaker than the assumption (0.2) but stronger than (0.3) we show the existence of the unique solution of (0.1) in $C([0,T];\ H)$.

We enumerate the contents of this paper as follows.
In section 1 we give some notations, assumptions and theorems.
In section 2 we introduce some fundamental lemmas for later use.
In section 3 we investigate approximate solutions of some parabolic equation associated with the equation (0.1) and prove the existence and uniqueness of a solution to the equation.
In section 4 we study the relation between this parabolic equation and the equation (0.1). In section 5 we give the proof of Theorem 1 concerned with a weak solution and in section 6 we show Theorem 2 for a continuous solution. In the final section we state some remarks for our assumptions.

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1. Notations, Assumptions and Theorems

Let H be a real Hilbert space with the inner product $(\cdot,\cdot)$ and norm $|\cdot|_H$. The operator A is positive definte self-adjoint in H. The fractional power A^α of A is defined for all real numbers using the spectral resolution, and A^α is bounded if $\alpha<0$. In this paper by the graph norm of A^α we mean the norm $|A^{\alpha}\cdot|_H$ for any real number α. If $\alpha\geq 0$, $D(A^{\alpha})$ is a Hilbert space with this graph norm. If $\alpha>0$, we denote by $D(A^{-\alpha})$ the pre-Hiblert space H endowed with the graph norm of $A^{-\alpha}$. We denote $\bigcap_{n=1}^{\infty} D(A^n)$ by $D(A^{\infty})$. Let e^{-tA} be the analytic semigroup generated by $-A$. We denote the convolution of funcitons f and g by $f*g$:

$$(f*g)(t)=\int_0^t f(t-s)g(s)ds,$$

and the product of operators $P_1P_2\cdots P_n$ by $\prod_{i=1}^{n} P_i$. We use the usual notations $L^2(0,T;\ S)$, $W^{1,2}(0,T;\ S)$ etc., to denote variable spaces of functions with values in a Banach space S. In particular $L^2(\delta,T;\ D(A))$ is the space of all measurable functions from $[\delta,T]$ to the domain of A such that $|Au|_H^2$ is integrable on $[\delta,T]$. For the sake of simplicity we put

$$L^2_{loc}((a,b];\ H)=\cap_{\delta>0}L^2(a+\delta,b;\ H).$$

When we are concerned with convergence, we mean the strong convergence in H unless otherwise stated.

Throughout this paper we denote

$$\lim_{\varepsilon\to 0}\int_{a+\varepsilon}^{t} f(s)ds \quad\text{and}\quad \lim_{\varepsilon\to 0} A^{-\alpha}\int_{a+\varepsilon}^{t} f(s)ds \text{ by}$$

$$\int_{a+}^{t} f(s)ds \quad\text{and}\quad A^{-\alpha}\int_{a+}^{t} f(s)ds \quad\text{respectively.}$$

We denote by Q the set of functions satisfying the following two conditions:

1) $f\in L^2_{loc}((0,h];\ H)$,

2) for any $\alpha>0$, $A^{-\alpha}\int_{0+}^{t} f(s)ds$ exists and there is a function $q\in C((0,h];\ H)$ such that, for any $t\in(0,h]$, $A^{-\alpha}\int_{a+}^{t} f(s)ds=A^{-\alpha}q(t)$ and there exists $\int_{a+}^{t} q(s)ds$.

The second condition means that the improper integral $q(t)=\int_{a+}^{t} f(s)ds$ exists in the graph norm of $A^{-\alpha}$ for any $\alpha>0$ and is a continuous function with values in H in $(0,h]$ such that the improper integral $\int_{a_+}^{t} q(s)ds$ exists.

Suppose A_1 and A_2 are closed linear mappings with domains containing that of A. For the sake of simplicity we assume $T=Nh$ for some natural number N.

With regard to the equation (0,1) we employ the terminology of a weak solution on $[0,T]$ defined as follows.

DEFINITION 1.1. We say that a function u defined on $[-h,T]$ is a weak solution of the equation (0.1) if the following four conditions are satisfied:

1) $u\in L^2_{loc}((nh,(n+1)h];\ D(A))\cap W^{1,2}_{loc}((nh,(n+1)h];\ H)$ for $n=0,1,\cdot\cdot,N-1$ and $u\in C([0,T];\ D(A^{-\alpha}))$ for any $\alpha>0$,
2) $\lim_{t\to 0} A^{-\alpha}u(t)=A^{-\alpha}x$ for any $\alpha>0$ and $u(s)=y(s)$ for $-h\leq s<0$,
3) for each $n{:}n=0,1,2,\cdots,N-1$, $Au(\cdot+nh)$ belongs to Q,
4) the function u satisfies the equation (0.1) for *a.e.* $t\in 0,T]$

REMARK 1. Let u be a weak solution of (0.1). Since $u\in W^{1,2}_{loc}((nh,(n+1)h];H)$, u is continuous in $(nh,(n+1)h]$ for $n=0,1,\cdots,N-1$ where the continuity at $t=(n+1)h$ means the left continuity. From 3) we know that there exists

$$A^{-\alpha}\int_{nh+}^{(n+1)h} Au(s)ds \text{ for } n=0,1\cdots N-1 \text{ and any } \alpha>0.$$

Then the integral in the left side of the equation (0.1) should be understood in the following sense

$$A^{\alpha}\lim_{\epsilon\to 0}A^{-\alpha}\{(\int_{-h}^{-t+nh}+\int_{-t+nh+\epsilon}^{0})a(-s)A_2u(s+t)ds\}$$

for $nh<t<(n+1)h$ and any $\alpha>0$ where $n=0,1,\cdots N-1$(i.e. the integral exists in the improper sense in the graph norm of $A^{-\alpha}$ for any $\alpha>0$ and its value belongs to H for *a.e* t).

ASSUMPTION.

A-1) $f \in L^2_{loc}((0,T]: H)$ and $y \in L^2_{loc}((-h,0]; D(A))$.

A-2) The restriction of f to $[0,h]$ and $Ay(\cdot -h)$ belong to Q.

A-3) Let α be any nonnegative number. Then we have the following estimate

$$\text{Max}\ \{|A^{-\alpha}A_1A^{-1+\alpha}x|_H,\ |A^{-\alpha}A_2A^{-1+\alpha}x|_H\} \le C|x|_H$$

for any $x \in D(A^\alpha)$, *where* C is a constant depending only on α.

A-4) The function $a(\cdot)$ belongs to $C^3([0,h])$.

We now state our theorem.

Theorem 1. *Under the assumptions* A-1), A-2), A-3) *and* A-4) *a weak solution* $u(t)$ *of the equation* (0.1) *exists and is unique.*

Next we state the definition of a continuous solution of the equation (0.1). We first define the function space F_{-1} by

$$F_{-1} = \{g \in L^2_{loc}((0,h];\ H);\ \int_{0+}^{t} g(s)ds \text{ exists for any } t \in (0,h]\}.$$

DEFINITION 1.2. Substitute 0 for α in 1), 2) of Definitition 1.1 and F_{-1} for Q in 3) of the same definition. Then the weak solution $u(t)$ of the equation (0.1) is called a continuous solution of (0.1).

We put $(Kf)(t) = \int_{t/2}^{t} e^{-(t-s)A} f(s)ds$ for any $f \in L^2_{loc}((0,h];\ H)$. If $g \in L^2(0,h;\ H)$, $\int_0^t e^{-(t-s)A} g(s)ds$ belongs to $L^2(0,h;\ D(A)) \cap W^{1,2}(0,h;\ H)$ (see Vol. 2 Theorem 3.2 in Lions-Magenes [4]). Let δ be any small positive number. Putting $g(t)=f(t)$ if $t \ge \delta/2$, $=0$ if $0 \le t \le \delta/2$ and using the above result, we see the operator K is continuous from $L^2_{loc}((0,h];\ H)$ to $L^2_{loc}((0,h];\ D(A)) \cap C((0,h];\ H)$. Then, from $A_1K = A_1A^{-1}AK$, A_1K is a continuous operator from $L^2_{loc}((0,h];\ H)$ to $L^2_{loc}((0,h];\ H)$. Thus, for any natural number i, $(KA_1)^iK$ is a continuous operator from $L^2_{loc}((0,h];\ H)$ to $L^2_{loc}((0,h];\ D(A)) \cap C((0,h];\ H)$.

We define function spaces F_n by

$$F_n = \{g \in F_{n-1};\ \lim_{t \to 0}((KA_1)^n Kg)(t) = 0\} \text{ for } n = 0,1,\cdots,N-1$$

inductively.

ASSUMPTION

A-5) $f(\cdot)-A_1y(\cdot-h)\in F_{N-1}$ and $A_1y(\cdot-h)\in F_{-1}$

A-6) Let α be any real number and $x\in D(A^{\alpha})\cap H$. Then the operator A_1 holds the same estimate as that in A-3).

Theorem 2. *Let the assumptions* A-1), A-4), A-5) *and* A-6) *be satisfied. Then a continuous solution* $u(t)$ *of the equation* (0.1) *exists and is unique.*

REMARK 2. The assumption A-1) plus A-2) is Weaker than (0.3), and A-5) is stronger than (0.3). But A-5) is weaker than the assumption (0.2) and we have $L^1(0,h;\ H)\cap L^2_{loc}((0,h];\ H)\subset F_{N-1}$ (see Appendix). We see the following relations of inclusion: $F_{N-1}\subset F_{N-2}\subset\cdots\subset F_0\subset F_{-1}\subset Q$. If A is a bounded operator it follows that $F_{N-1}=Q$.

REMARK 3. 1) If $f\in Q$, then for $t>0$, the improper integral $\int_{0+}^{t} e^{-(t-s)A}f(s)ds$ exists and belongs to $L^2_{loc}((0,h];\ D(A))\cap C((0,h];\ H)\cap C([0,h];\ D(A^{-\alpha}))$ for any $\alpha>0$. Indeed, for $f\, L^2_{loc}((0,h];H)$, we have $(Kf)(t)\in L^2_{loc}((0,h];\ D(A))\cap C((0,h];\ H)$. Using an integration by parts and the analytic semigroup properties we see

$$(1.1)\quad \int_{0+}^{t/2} e^{-(t-s)A}f(s)ds=e^{-(t/2)A}A^{\alpha}\cdot A^{-\alpha}\int_{0+}^{t/2} f(s)ds$$
$$-\int_{0+}^{t/2} Ae^{-(t-s)A}A^{\alpha/2}\cdot A^{-\alpha/2}\int_{0+}^{s} f(\tau)d\tau ds \text{ in } D(A^{-\alpha}).$$

Since $\int_{0+}^{t/2} |Ae^{-(t-s)A}A^{\alpha/2}|_{H\to H}ds\le c\cdot t^{-\alpha/2}$ both terms of the right side of (1.1) are well defined in H. So $\int_{0+}^{t} e^{-(t-s)A}f(s)ds$ exists in H. Moerover $\int_{0+}^{t/2} e^{-(t-s)A}f(s)ds\ \in C((0,h];\ D(A^{\infty}))$. Hence $\int_{0+}^{t} e^{-(t-s)A}f(s)ds\in L^2_{loc}((0,h];\ D(A))\cap C((0,h];\ H)$. Since both functions of the right side of

$$\int_{0+}^{t} e^{-(t-s)A}f(s)ds=\int_{0+}^{t} f(s)ds-\int_{0}^{t} Ae^{-(t-s)A}\int_{0+}^{s} f(\tau)d\tau ds$$

are continuous in $D(A^{-\alpha})$ on $[0,h]$ it follows $\int_{0+}^{t} e^{-(t-s)A}f(s)ds$ belongs to

$C([0,h];\ D(A^{-\alpha}))$.

2) If $f \in F_0$ $\int_{0+}^{t} e^{-(t-s)A} f(s)ds$ is continuous in $[0,h]$. Indeed, from our assumption for f it follows that $\int_{t/2}^{t} e^{-(t-s)A} f(s)ds$ is continuous on $[0,h]$ and (1.1) yields $\int_{0+}^{t/2} e^{-(t-s)A} f(s)ds \in C([0,h];\ H)$.

3) If $f \in Q$, $\int_{0+}^{t} e^{-(t-s)A} f(s)ds$ belongs to F_{-1}. Let ε and δ be sufficiently small positive numbers. Then

$$\int_{\delta}^{\epsilon} \int_{0+}^{t} e^{-(t-s)A} f(s)dsdt =$$

$$\int_{\delta}^{\epsilon} \int_{0+}^{t} e^{-(t-s)A} f(s)dsdt + \int_{\delta}^{\epsilon} \int_{0+}^{t} e^{-(t-s)A} f(s)dsdt = I_4 + I_5.$$

Changing the order of integration and integrating by parts we see

$$I_4 = \int_{\delta}^{\epsilon} A^{\alpha} e^{-(\epsilon-s)A} \cdot A^{-\alpha} \int_{\delta}^{s} f(\tau)d\tau ds.$$

Then from the analytic semigroup property and $f \in Q$ we get $\lim_{\epsilon,\delta\to 0} I_4 = 0$. Using the integration by parts we see

$$I_5 = \int_{\delta}^{\epsilon} A^{\alpha} e^{-(t-\delta)A} \cdot A^{-\alpha} \int_{0+}^{\delta} f(\tau)d\tau dt$$

$$- \int_{\delta}^{\epsilon} \int_{0+}^{\delta} A^{1+\alpha} e^{-(t-s)A} \cdot A^{-\alpha} \int_{0+}^{s} f(\tau)d\tau ds dt.$$

From the analytic semigroup property and $f \in Q$ we have $\lim_{\epsilon,\delta\to 0} I_5 = 0$. Then our assertion is proved.

2. Preliminaries

In this secton we list some lemmas which will be used throughout this paper. In what follows we suppose the assumptions A-3) and A-4) are satisfied.

Let $R(\cdot)$ be the operator valued function satisfying the following integral equation

(2.1) $R+aA_2A^{-1}+R^*aA_2{}^{-1}=0.$

We denote the derivative $\frac{d}{dt}R(t)$ by $\dot{R}(t)$ and also $\frac{d^2}{dt^2}R(t)$ by $\ddot{R}(t)$.

Lemma 2.1 1) *For each* $t\in[0,T]$ $R(t)$, $\dot{R}(t)$ *and* $\ddot{R}(t)$ *are continuous linear operators from H to H and their operator norms are uniformly bounded in* $[0,T]$.

2) *Let* α *be any positive number. Then we have the following estimate*:

$$\text{Max}\quad \{|A^{-\alpha}R(t)A^{\alpha}x|_H,\ |A^{-\alpha}\dot{R}(t)A^{\alpha}x|_H,\ |A^{-\alpha}\ddot{R}(t)A^{\ \alpha}x|_H\}\leq C|x|_H$$

for any $x\in D(A^{\alpha})$ *where C is a constant independent of t and x.*

3) $R*aA_2A^{-1}=aA_2A^{-1}*R.$

Proof. The assumptions A-3), A-4) and the equality (2.1) readily yield the conclusions of the lemma.

Using the semigroup property we get the following lemma.

Lemma 2.2. *There exists a constant C that is nonnegative number. Moreover it follows that*

$|e^{-tA}A_1x|_H\leq C\cdot t^{-1}|x|_H$ *for any* $x\in D(A)$ *and*

$|A^{\alpha}e^{-tA}A_1A^{\beta}e^{-sA}x|_H\leq C\cdot t^{\ -\alpha+\epsilon}s^{-\beta-1}|x|_H$ *for any* $x\in H$.

For any $f\in L^2_{loc}$ $((0,h];\ H)$ we consider the following approximate functions $f_n\in L^2(0,h;\ H)$ of f:

$$f_n(t)=\begin{cases} f(t) & \text{if}\quad 1/\text{n}\leq t\leq h\\ 0 & \text{if}\quad 0\leq t\leq 1/n\quad \textit{for}\ n=1,2,\cdots.\end{cases}$$

Lemma 2.3. *Let f(t) be a function satisfying assumptions* A-1) *and* A-2). *Then we have the following.*

1) f_n $L^2(0,h;\ H)$ *for each* $n=1,2,\ \cdots$.

2) *For any* $\alpha>0$ $A^{-\alpha}\int_0^t f_n(s)ds$ *uniformly concerges to* $A^{-\alpha}\int_{+0}^t f(s)ds$

in $[0,h]$ *as* $n\to\infty$.

3) *For any* $\delta>0$ *it follows that*

$$\lim_{n\to\infty}\int_\delta^h |f_n(s-f(s)|_H^2 ds=0.$$

Proof. With the aid of

$$f_n(t)-f(\mathrm{t})=\begin{cases} 0 & \text{if } 1/n\le t \\ -f(t) & \text{if } 0\le t\le 1/n \end{cases}$$

the assumptions A-1) and A-2) imply the assertions of the lemma.

Lemma 2.4. *If u belongs to Q there exist*

$$A^\alpha \lim_{\delta\to 0} A^{-\alpha}\int_\delta^t R(t-s)u(s)ds \text{ and } A^\alpha \lim_{\delta\to 0} A^{-\alpha}\int_\delta^t \dot{R}(t-s)u(s)ds$$

where α *is any positive number. Moreover the values of these limites are independent of* α *and belong to* F_{-1}.

Proof. Using the integration by parts we have

$$(2.2)\quad \lim_{\delta\to 0} A^{-\alpha}\int_\delta^t R(t-s)u(s)ds$$

$$=A^{-\alpha}R(0)A^\alpha\{A^{-\alpha}\int_{0+}^t u(\tau)d\tau\}-\lim_{\delta\to 0}A^{-\alpha}R(t-\delta)A^\alpha\cdot A^{-\alpha}\int_{0+}^\delta u(\tau)d\tau$$

$$+\lim_{\delta\to 0}A^{-\alpha}\int_\delta^t \dot{R}(t-s)A^\alpha A^{-\alpha}\int_{0+}^s u(\tau)d\tau ds=I_1-I_2+I_3.$$

Let q be the function in the definition of Q associated with u. Then

$$I_1=A^{-\alpha}R(0)q(t),\ I_2=\lim_{\delta\to 0}A^{-\alpha}R(t-\delta)A^\alpha\cdot A^{-\alpha}q(\delta)$$

$$\text{and } I_3=\lim_{\delta\to 0}A^{-\alpha}\int_\delta^t \dot{R}(t-s)q(s)ds.$$

Noting that q is improperly integrable at $t=0$ and using the integration by parts we see

$$I_3 = A^{-\alpha}\dot{R}(0)\int_{0+}^{t} q(\tau)d\tau + A^{-\alpha}\int_0^t \ddot{R}(t-s)\int_{0+}^{s} q(\tau)d\tau ds.$$

Then $A^{\alpha}(I_1+I_3)$ is a function independent of α and belonging to F_{-1}. Noting $A^{-\alpha}\, q(\delta)\in D(A^{\alpha})$ and using Lemma 2.1 we get $\lim_{\delta\to 0} I_2=0$. Replacing R by $\dot{R}$, using a similar method to the above and noting R is three times differentiable we know that the latter limit function in the lemma has also the same properties.

For any $u\in Q$ we set

$$(2.3)\quad \begin{cases} R*u(t) = A^{\alpha}\lim_{\delta\to 0} A^{-\alpha}\int_{\delta}^{t} R(t-s)u(s)ds, \\ \dot{R}*u(t) = A^{\alpha}\lim_{\delta\to 0} A^{-\alpha}\int_{\delta}^{t} \dot{R}(t-s)u(s)ds, \\ G(u)(t) = R(0)u(t) + (\dot{R}*u)(t). \end{cases}$$

Lemma 2.5. *Let α be an arbitrary positive number. Then we get the following.*

1) *If $u\in Q$, R^*u and $\dot{R}^*u$ belong to $F_{-1}\cap C([0,h];\ D(A^{-\alpha}))$.*

2) *$G(\cdot)$ is an operator from Q to Q.*

3) *The operator $G(\cdot)$ has the following inequatily*

$$\int_0^t |A^{-\alpha}(G(u)(s)-G(v)(s))|_H ds \le Const\int_0^t |A^{-\alpha}(u(s)-v(s))|_H ds$$

for any $u,\ v\in Q\cap\ L^1(0,h;\ D(A^{-\alpha}))$.

4) *If $A^{-\alpha}\ u\ \in([0,h];\ H)$ it follows*

$$\int_{0+}^{t} e^{-(t-s)A}G(u)(s)ds\ \in C([0,h];\ H) \text{ and}$$

$$|\int_{0+}^{t} e^{-(t-s)A}G(u)(s)ds|_H\ \le Const\int_0^t |A^{-\alpha}u(s)|\ {}_H ds.$$

Proof. Combining Lemma 2.2, Lemma 2.4 and (2.3) and following the proof of Lemma 2.4 we complete the proof of the lemma.

REMARK 4. If $u\in F_{-1}$, we know that

$$R*u(t)=\int_{0+}^{t} R(t-s)u(s)ds\in C([0,h];\ H),$$

and similarly for $\dot{R}*u(t)$. Hence, $G(u)(t)\in F_{-1}$.

Lemma 2.6. *If $f\in Q$ then it follows that $\int_t^h a(t+h-\tau)f(\tau)d\tau$ belongs to $F_{-1}\cap C([0,h];\ D(A^{-\alpha}))$.*

If $f\in F_{-1}$, $\int_t^h a(t+h-\tau)f(\tau)d\tau$ belongs to $C([0,h];\ H)$.

Proof. From A-2) it follows that $\int_\tau^h f(\xi)d\xi=q(h)-q(\tau)$ and $q(\cdot)\in F_{-1}$. Then the integration by parts, A-4) and A-2) yield conclusions.

3. Approximate equations

We study the existence, uniqueness and properties of a solution of the following initial value problem

$$\text{(3.1)} \qquad \begin{cases} \dot{u}+Au=f-G(u) \\ u(0)=x\in H. \end{cases}$$

This type of equation (3.1) was deeply investigated by J. Pruss [7], E. Sinestrari [8] and G.F. Webb [11] in case where f holds some regularity.

In this section we study the equation (3.1) assuming that f belongs to Q. We consider the following approximate equations of (3.1)

$$\text{(3.2)} \qquad \begin{cases} \dot{u}_n+Au_n=f_n-G(u_n) \quad on \quad [0,h] \\ u_n(0)=x\in H, \end{cases}$$

where f_n are the approximate functions defined just before Lemma 2.3.

Lemma 3.1. *For any natural number n, a slution $u_n(t)$ of (3.2) exists and is uunique. Moreover $u_n(t)$ belongs to $C([0,h];\ H)\cap W^{1,2}_{loc}((0,h];\ H)\cap L^2_{loc}((0,h];\ D(A))$ and satisfies*

$$\text{(3.3)} \qquad u_n(t)=e^{-tA}x+\int_0^t e^{-(t-s)A}f_n(s)ds-\int_0^t e^{-(t-s)A}G(u_n)(s)ds.$$

Proof. The integral equation (3.3) is easily solved by successive approximation. Since $f_n \in L^2(0,h;\ H)$, the maximal regularity result (Theorem 3.2 of [4], Vol. 2) yield the proof of the lemma.

Lemma 3.2. 1) *For each* $\alpha > 0$ *there exists a constant* C_α *such that for any* $n = 1,2,\cdots$,

$$\sup_{0 \le t \le h} |A^{-\alpha} u_n(t)|_H \le C_\alpha,$$

and there exists a constant C *such that for* $n = 1,2,\cdots$

$$\sup_{0 \le t \le h} |\int_0^t u_n(s)ds|_H \le C \ \textit{and} \ \sup_{0 \le t \le h} |\int_0^t Gu_n(s)ds|_H \le C.$$

2) *there exists a continuous decreasing function* $C(\cdot)$ *from* $(0,h]$ *to* $(0,\infty)$ *such that*

$$\sup_{0 \le s \le t \le h} |u_n(t)|_H \ \textit{and} \ \sup_{0 \le s \le t \le h} |Gu_n(t)|_H \le C(\delta).$$

Proof. Using the integration by parts we have

$$|A^{-\alpha} \int_0^t e^{-(t-s)A} f_n(s)ds|_H \le A^{-\alpha} \int_0^t f_n(\tau)d\tau|_H$$

$$+ |\int_0^t A^{-\alpha+1} e^{-(t-s)A} A^{\alpha/2} A^{-\alpha/2} \int_0^s f_n(\tau)d\tau ds|_H = I_1 + I_2.$$

From 2) of Lemma 2.3 we obtain $I_1 \le C$. Combining 2) of Lemma 2.3 and Lemma 2.2 we obtain $I_2 \le C$. Then we have $I_1 + I_2 \le C$. From 4) of Lemma 2.5, the equality (3.3) and the above results there exists a constant C such that

$$|A^{-\alpha} u_n(t)|_H \le C(1 + \int_0^t |A^{-\alpha} u_n(s)|_H ds).$$

By virtue of Gronwall's inequality $|A^{-\alpha} u_n(t)|_H$ is uniformly bounded on $[0,h]$. In view of 4) in Lemma 2.5 and the above result we know that $\int_0^t e^{-(t-s)A} G(u_n)(s)ds$ are uniformly continuous on $[0,h]$. Changing the order of integration and integrationg by parts we have

$$(3.4)\quad \int_0^t \int_0^s e^{-(s-\tau)A} f_n(\tau) d\tau ds = \int_0^t e^{-(t-\tau)A} A^{\alpha} \cdot A^{-\alpha} \int_0^{\tau} f_n(s) ds d\tau.$$

From 2) of Lemma 2.3 it follows that the right side of (3.4) is uniformly continuous on $[0,h]$. Thus integrating both sides of equation (3.3) over $[0,t]$ we conclude $\int_0^t u_n(s)ds$ are uniformly continiuous on $[0,h]$.
Using a similar method to 1) in Remark 3 and noting 2) and 3) of Lemma 2.3 we get $|\int_0^t e^{-(t-s)A} f_n(s)ds|_H \leq C$ for any $t \in [\delta,h]$ where C is independent of n but dependent on δ. From (3.3), 1) of this lemma and the above result we obtain that there exists a positive decreasing function $C(\delta)$ such that $|u_n(t)|_H \leq C(\delta)$ for any $t \in [\delta,h]$. Using the method of (2.2) and 1) of this lemma and noting the above result we know that $|G(u_n)(t)|_H$ are also smaller than Const$(C(\delta)+1)$ if $t \in [\delta,h]$, where we denote another positive decreasing function again by $C(\delta)$. Then the proof is complete.

For the sake of simplicity we denote a continuous decreasing function which will be used in subsequent estimates by the same notation $C(\delta)$ as that in Lemma 3.2.

Lemma 3.3. *For any $\delta \in (0,h)$ we have*

$$\int_{\delta}^t (|\dot{u}_n(s)|_H^2 + |Au_n(s)|_H^2) ds + \sup_{\delta \leq t \leq h} |A^{1/2} u_n(t)|_H \leq C(\delta).$$

Proof. Let γ be any positive number with $0<\gamma<h$. Noting 2) of Lemma 3.2 and 3) of Lemma 2.3 we have

$$|\int_{\gamma}^t (G(u_n)(s), \dot{u}_n(s)) ds|_H \leq C(\gamma) + 1/4 \int_{\gamma}^t |\dot{u}_n(s)|_H^2 ds$$

and

$$|\int_{\gamma}^t (f_n(s), \dot{u}_n(s)) ds|_H \leq C(\gamma) + 1/4 \int_{\gamma}^t |\dot{u}_n(s)|_H^2 ds.$$

Combining the following energy equality

$$\int_{\gamma}^t |\dot{u}_n(s)|_H^2 ds + 1/2 |A^{1/2} u_n(t)|_H^2 = 1/2 |A^{1/2} u_n(\gamma)|_H^2$$

$$+\int_\gamma^t (f_n(s),\dot{u}_n(s))ds-\int_\gamma^t (G(u_n)(s),\dot{u}_n(s))ds$$

and the above two inequalities we obtain

$$\int_\gamma^t |\dot{u}_n(s)|_H^2 ds+|A^{1/2}u_n(t)|_H^2\leq C(\gamma)+|A^{1/2}u_n(\gamma)|_H^2. \tag{3.5}$$

Multiplying both sides of (3.2) by $u_n(t)$ and integrating over $[\delta/2,t]$ we get

$$1/2|u_n(t)|_H^2-1/2|u_n(\delta/2)|_H^2+\int_{\delta/2}^t |A^{1/2}u_n(s)|_H^2 ds$$
$$=\int_{\delta/2}^t (f_n(s),u_n(s))ds-\int_{\delta/2}^t (G(u_n)(s),u_n(s))ds.$$

Setting $t=\delta$ in the above equality and using 2) of Lemma 3.2 and 3) of Lemma 2.3 we have

$$\int_{\delta/2}^\delta |A^{1/2}u_n(s)|_H^2 ds\leq C(\delta/2).$$

Hence there exists $s_{n,\delta}$, $\delta/2\leq s_{n,\delta}\leq\delta$, such that

$$|A^{1/2}u_n(s_{n,\delta})|_H^2\leq 4/\delta\cdot C(\delta/2).$$

Putting $\gamma=s_{n,\delta}$ in (3.5) and noting that $C(\cdot)$ is a decreasing function we get, for any $t\in[\delta,h]$,

$$\int_\delta^t |\dot{u}_n(s)|_H^2 ds+|A^{1/2}u_n(t)|_H^2\leq(1+4/\delta)C(\delta/2).$$

Combining the equations (3.2), 3) of Lemma 2.3 and the above results we get a decreasing function $C(\cdot)$ such that

$$\int_\delta^t |Au_n(s)|_H^2 ds\leq C(\delta).$$

This completes the proof.

For the convergence of $u_n(t)$ we get the following lemma,

Lemma 3.4. *For each* $\alpha>0$, $\{A^{-\alpha}u_n(t)\}$ *is a Cauchy sequence in* $C([0,h];\ H)$, *Moreover* $\{u_n(t)\}$ *is also a Cauchy sequence in* $C([\delta,h];\ H)$ *where* δ *is any positive number.*

Proof. The first part follows from Gronwall's inequality satisfied by $|A^{-\alpha}(u_n(t)-u_m(t))|_H$ and the fact that $\int_0^t e^{-(t-s)A}A^{-\alpha}f_n(s)ds$ is a Cauchy sequence in $C([0,h];\ H)$, both of which are established following the proof of 1) of Lemma 3.2. Combining this with Lemma 3.3 and using the interpolation ineqality

$$|u|_H^2 \leq |A^{1/2}u|_H \cdot |A^{-1/2}u|_H$$

we know that $\{u_n(t)\}$ is a Cauchy sequence in $C([\delta,h];\ H)$.
Thus the proof of the lemma is complete.

We put

$$\lim_{n\to\infty} u_n(t)=u(t) \text{ in } (0,\mathrm{h}]. \tag{3.6}$$

Lemma 3.5. *The function u has the following properties*:

1) *u belongs to* F_{-1},
2) $u\in L^2_{loc}((0,h];\ D(A))\cap W^{1,2}_{loc}((0,h];\ H)$,
 $A^{-\alpha}u\in C([0,h];\ H)$ *and* $u(0)=x$,
3) $\lim_{n\to\infty}\int_0^t u_n(s)ds=\int_{0+}^t u(s)ds$ *for any* $t\in[0,h]$,
4) $\lim_{n\to\infty} G(u_n)(t)=G(u)(t)$ *for any* $t\in(0,h]$.

Proof. In the proof of 1) of Lemma 3.2 we showed that $\int_0^t u_n(s)ds$ are uniformly continuous in $[0,h]$. Especially $\lim_{t\to 0}\sup_n\left|\int_0^t u_n(s)ds\right|_H=0$. Using this and Lemma 3.4 we get that $\int_{0+}^t u(s)ds$ exists and 3) holds. Then $u\in F_{-1}$. 2) is a direct consequence of Lemma 3.3 and Lemma 3.4. From $u\in F_{-1}$ we have

$$G(u)(t)=R(0)u(t)+\dot{R}(0)\int_{0+}^t u(s)ds+\int_0^t \ddot{R}(t-s)\int_{0+}^s u(\tau)d\tau ds. \tag{3.7}$$

Then 3) yields 4).

DEFINITION 3.1. We call a function $u(t)$ defined on $[0,h]$ and satisfying the following conditions a weak solution of the initial value problem(3.1).

1) $u \in L^2_{loc}((0,h]; D(A)) \cap W^{1,2}_{loc}((0,h]; H) \subset C((0,h]; D(A^{1/2}))$.
2) $\lim_{t \to 0} A^{-\alpha}u(t) = A^{-\alpha}x$ for any $\alpha > 0$.
3) The improper integral $\int_{0+}^{t} u(s)ds$ exists.
4) The function u satisfies the equation (3.1) for *a.e* $t \in [0,h]$, where $G(u)$ is defined by (3.7).

Lemma 3.6. *A weak solution $u(t)$ of* (3.1) *exists.*

Proof. With the aid of the equations (3.2), Lemma 3.3, Lemma 3.4 and Lemma 3.5 and using the well known argument on approximate solutions we obtain that the function $u(t)$ of (3.6) satisfies the conditions of the weak solution of (3.1).

Lemma 3.7. *Let u be a weak solution of* (3.1). *Then $Au \in Q$ and $A^{-\alpha}u \in C([0,h]; H)$ for any $\alpha > 0$.*

Proof. From the definition we know that $A^{-\alpha}u$ belongs to $C([0,h]; H)$. From the equation (3.1) we have

$$A^{-\alpha}\int_{0+}^{t} Au(s)ds = A^{-\alpha}\{x - u(t) + \int_{0+}^{t} f(s)ds - \int_{0+}^{t} G(u)(s)ds\}.$$

Thus noting that $\int_{0+}^{t} u(s)ds$ exists we get $Au \in Q$.

Lemma 3.8. *A weak solution of* (3.1) *is unigue.*

Proof. Let u and v be two weak solutions of (3.1). Since u satisfies the equation (3.1) in H we have

$$A^{-\alpha}u(t) = A^{-\alpha}e^{-(t-\varepsilon)A}u(\varepsilon)$$
$$+ A^{-\alpha}\int_{\epsilon}^{t} e^{-(t-s)A}f(s)ds - A^{-\alpha}\int_{\epsilon}^{t} e^{-(t-s)A}G(u)(s)ds.$$

Letting $\varepsilon \to 0$ in the above and recalling 1) of Remark 3 we get

$$(3.8)\quad A^{-\alpha}u(t)=A^{-\alpha}e^{-tA}x+A^{-\alpha}\int_{0+}^{t}e^{-(t-s)A}f(s)ds$$
$$-A^{-\alpha}\int_{0}^{t}e^{-(t-s)A}G(u)(s)ds.$$

Combining (3.8) and 3) of Lemma 2.5 we get

$$|A^{-\alpha}(u(t)-v(t))|_H \le Const\int_0^t |A^{-\alpha}(u(s)-v(s))|_H ds.$$

The uniqueness follows with the aid of Gronwall's inequality.

4. The relation between the associated equation and the original equation

This section is devoted to the study of relations between solutious of equations of (0.1) and (3.1). To prove our statment we use a similar method to that of M.G. Crandall and J.A. Nohel [2]. Throughout this section let f belong to Q and $\{f_n\}$ be approximate functions of f in Lemma 2.3.

Lemma 4.1. *Let ε be any small positive number. Then there exists a constant δ_0 depending only on ε such that for any γ and τ: $0<\gamma<\tau<\delta_0$,*

$$(4.1)\qquad |A^{-\alpha}\int_{\gamma}^{\tau}a(t-s)A_2A^{-1}f_n(s)ds|_H<\varepsilon \text{ and}$$

$$(4.2)\qquad |A^{-\alpha}\int_{\gamma}^{\tau}R(t-s)A_2A^{-1}f_n(s)ds|_H<\varepsilon.$$

Proof. Using a similar method to the proof of Lemma 2.4 and noting 2) of Lemma 2.1 we complete the proof.

Lemma 4.2. *For any $t\in[0,h]$ it follows*

$$(4.3)\quad \int_{0+}^{t}A^{-\alpha}R(t-s)A^{\alpha}\cdot\lim_{\delta\to 0}A^{-\alpha}\int_{\delta}^{s}a(s-\tau)A_2\ A^{-1}f(\tau)d\tau ds$$
$$=-A^{-\alpha}\int_{0+}^{t}R(t-\tau)f(\tau)d\tau-A^{-\alpha}\int_{0+}^{t}a(t-\tau)A_2A^{-1}f(\tau)d\tau$$

and

$$(4.4) \quad \int_{0+}^{t} a(t-s)A^{-\alpha}A_2A^{-1}A^{\alpha}\cdot\lim_{\delta\to 0}A^{-\alpha}R(s-\tau)f(\tau)d\tau ds$$
$$= -A^{-\alpha}\int_{0+}^{t} R(t-\tau)f(\tau)d\tau - A^{-\alpha}\int_{0+}^{t} a(t-\tau)A_2A^{-1}f(\tau)d\tau.$$

Proof. From (2.1) we see $R*aA_2A^{-1}*f_n = -R*f_n - aA_2A^{-1}*f_n$. Noting (4.1) and letting $n\to\infty$ for the above equalityu we have (4.3). Using a similar method to the above and noting 3) of Lemma 2.1 and (4.2) we get (4.4).

We consider two equations

$$(4.5) \quad \begin{cases} \dot{u}+Au=f+(R*f)-G(u)+R(t)x \\ u(0)=x\in H, \end{cases}$$

$$(4.6) \quad \begin{cases} \dot{u}+Au+A^{\alpha}\lim_{\delta\to 0}A^{-\alpha}\int_{\delta}^{t}a(t-s)A_2u(s)ds = f \\ u(0)=x\in H. \end{cases}$$

If $f\in Q$, then $f+R*f+R(\cdot)x\in Q$. A function u is a weak solution of (4.5) if it is a weak solution of (3.1) with f replaced by $f+R*f+R(\cdot)x$.

DEFINITION 4.1. We call a function $u(t)$ defined on $[0,h]$ and satisgying the following conditions a weak solution of the initial value problem (4.6).

1) $u\in L^2_{loc}((0,h];\ D(A))\cap W^{1,2}_{loc}((0,h];\ H)\subset C((0,h];\ D(A^{1/2}))$.
2) $\lim_{t\to 0}A^{-\alpha}u(t)=A^{-\alpha}x$.
3) $Au\in Q$.
4) The function u satisfies the equation (4.6) for $a.e$ $t\in[0,h]$.

REMARK 5. Let u be a weak solution of (4.6). From 3) of the above definition it follows that the integral in(4.6) makes sense and $\dot{u}$ belongs to Q. If $f\in F_{-1}$ and $u\in C([0,h];\ H)$, Au belongs to F_{-1}.

Proposition 4.3. A function u is a weak solution of (4.5) if and only if is a weak solution of (4.6).

Proof. Let u be a weak solution of (4.6). From the integration by parts it follows

$$(4.7)\quad \int_{0+}^{t} A^{-\alpha}R(t-s)A^{\alpha}\cdot A^{-\alpha}\dot{u}(s)ds$$

$$=A^{-\alpha}R(0)u(t)-A^{-\alpha}R(t)x+\int_{0+}^{t} A^{-\alpha}\dot{R}(t-s)A^{\alpha}\cdot A^{-\alpha}u(s)ds.$$

$$=A^{-\alpha}\{G(u)(t)-R(t)x\}.$$

Multiplying the equation (4.6) by $A^{-\alpha}$, taking the convolution of both sides of the equation and $A^{-\alpha}R(\cdot)A^{\alpha}$ and using (4.3) with $f=Au$ and (4.7) we have

$$A^{-\alpha}\int_{0+}^{t} a(t+s)A_2u(s)ds=A^{-\alpha}\{G(u)(t)-R(t)x-(R*f)(t)\}.$$

With the aid of the argument used in the proof of (3.8) and noting 3) of Remark 3 and 4) of Lemma 2.5 we get $u\in F_{-1}$. Then we obtain that u is a weak solution of (4.5). Conversely, let u be a weak solution of (4.5). From Lemma 3.7 it follows $Au\in Q$. Then $(f-\dot{u})\in Q$. Thus using the method of (4.7) and noting (4.5) we see

$$(4.8)\quad A^{-\alpha}(f-\dot{u})(t)+A^{-\alpha}(R*(f-\dot{u}))(t)=A^{-\alpha}Au(t).$$

Taking the convolution of both sides of the equality (4.8) and $a(\cdot)A^{-\alpha}A_2A^{-1}A^{\alpha}$ and using (4.4) we have

$$A^{-\alpha}\int_{0+}^{t} a(t-s)A_2u(s)ds=-A^{-\alpha}(R*(f-\dot{u}))(t).$$

Combining (4.8) and the above equality we obtain that u is a weak solution of (4.6).

Corollary 4.4. *The weak solution* $u(t)$ *of* (4.6) *exists and is unique on* $[0,h]$.

Proof. The conclusion follows from Lemma 3.6, Lemma 3.8 and Proposition 4.3.

5. Proof of Theorem 1

In this section we assume the conditions in Theorem 1.

We first study the equation (0.1) on $[0,h]$. We transform the equation (0.1) to the following form:

$$(5.1)\qquad \begin{cases} \dot{u}+Au+\int_0^t a(t-s)A_2u(s)ds \\ \quad =f(t)-A_1y(t-h)-\int_{t-h}^0 a(t-s)A_2y(s)ds=f_0(t), \\ u(0)=x. \end{cases}$$

Nothing $Ay(\cdot-h)\in Q$ and using Lemma 2.6 we see

$$\int_{t-h}^0 a(t-s)A_2y(s)ds\in F_{-1}.$$

Then it follows $f_0\in Q$. Therefore from Corollary 4.4 we get the existence and uniqueness of a weak solution of (5.1). We denote the weak solution on $[0,h]$ by $u_1(t)$. Next we consider the equation (0.1) on $[h,2h]$. From Lemma 3.7 Au_1 belongs to Q. Then it follows

$$f(t)-A_1u_1(t-h)-\int_{t-h}^0 a(t-s)A_2u_1(s)ds\in Q_1$$

where $Q_1=\{g\in L^2_{loc}((h,2h];\ H);\ g(\cdot-h)\in Q\}$.
For t; $h\leq t\leq 2h$ we also transform the equation (0.1) to the following form

$$(5.2)\qquad \begin{cases} \dot{u}+Au+\int_h^{t+h} a(t-s)A_2u(s)ds \\ \quad =f(t)-A_1u_1(t-h)-\int_{t-h}^h a(t-s)A_2u_1(s)ds \\ u(0)=u_1(h). \end{cases}$$

With the aid of the change of the variables $t\rightarrow\tilde{t}=t-h$ in the above equation and using Corollary 4.4 we obtain that there exists the weak solution of (5.2) and is unique. Iterating this prosess, we get the global existence and uniqueness of the solution of the equation (0.1). Then the proof of Theorem 1 is complete.

6. Proof of Theorem 2

Throughout this section we suppose that the assumptions A-4) and A-6) hold. Moreover we assume $A_1y(\cdot-h)\in F_{-1}$, $f(\cdot)-A_1y(\cdot-h)F_1$ and $f\in L^2_{loc}((0,T];\ H)$

If the weak solution u of (0.1) is continuous on $[0,T]$, the proof of Theorem 2 is complete. Thus we shall show that the weak solution u of (0.1) is continuous on $[0,mh)$ if and only if $f(\cdot)-A_1y(-h)\in F_{m-1}$ where $0<m\leq N$.

We define the space C_0 by $C_0=\{g\in C([0,h];\ H);\ g(0)=0\}$ and recall the definitions of the operator K:

$$(Kf)(t)=\int_{0+}^{t/2} e^{-(t-s)A}f(s)ds.$$

We define two other integral operatros L and E by

$$(Lf)(t)=\int_{0+}^{t/2} e^{-(t-s)A}f(s)ds, \qquad (Ef)(t)=\int_{0+}^{t} e^{-(t-s)A}f(s)ds$$

for $f\in Q$. If $f\in Q$, then Ef is the unique weak solution of (3.1) with $G(\cdot)=0$ and $x=0$. In view of Lemma 3.7 $AEf\in Q$. Hence EA_1Ef is defined and $AEA_1Ef\in Q$. Continuing this process we see that for any integer $i\geq 0$ $(EA_1)^iEf$ is defined and belongs to $L^2_{loc}((0,h];\ D(A))\cap C((0,h];\ H)$. Since $(Lf)(t)=e^{-(t/2)A}(Ef)(t/2)$, ALf also belongs to Q, and so deos AKf. Hence, LA_1Kf is also defined as an element of $L^2_{loc}((0,h];(A))\cap C((0,h];\ H)$.

$(EA_1)^iEf$ is formally expressed as follows:

$$\int_0^t\int_0^{s_i}\cdots\int_0^{s_1} e^{-(t-s_i)A}A_1\cdots e^{-(s_2-s_1)A}A_1e^{-(s_1-s_0)A}f(s_0)ds_0\cdots ds_i.$$

We divide the domain of integration of the above integral into the sum of the following sets

$$\{(s_0,s_1,\cdots s_i):\ 0\leq s_0\leq s_1/2 \text{ or } s_1/2\leq s_0\leq s_1,\ 0\leq s_1\leq s_2/2 \text{ or }$$
$$s_2/2\leq s_1\leq s_2,\cdots,\ 0\leq s_i\leq t/2 \text{ or } t/2\leq s_i\leq t\}$$

The number of these sets is 2^{i+1}. We denote them except the one with $s_1/2\leq s_0\leq s_1$, $s_2/2\leq s_1\leq s_2,\cdots$, $t/2\leq s_i\leq t$ by $\Delta^i_j, j=1,\cdots, 2^{i+1}-1$. We put

$$(\tilde{L}_{i,j}f)(t)=\int_{\Delta_j^i} e^{-(t-s_i)A}A_1\cdots e^{-(s_2-s_1)A}A_1e^{-(s_1-s_0)A}f(s_0)ds_0\cdot\cdot ds_i.$$

Lemma 6.1. The operators E_i have the following repesentations:

$$(EA_1)^iEf(t)=((KA_1)^iKf)(t)+(L_if)(t)$$

where $(L_if)(t)=\sum_{j=1}^{2^{i+1}-1}(\tilde{L}_{i,j}f)(t)$.

Proof. Inductively we show our statement. For $i=0$, it follows $(Ef)(t)=(Kf)(t)+\int_0^{t/2}e^{-(t-s)A}f(s)ds$. Then in this case the proof is complete.
Combining the definition of $(EA_1)^iEf$ and the assumption of the induction we get

$$\begin{aligned}((EA_1)^{i+1}Ef)(t)=((KA_1)^{i+1}Kf)(t)+(LA_1(KA_1)^iKf)(t)\\+(KA_1L_if)(t)+(LA_1L_if)(t).\end{aligned}$$

Putting $(L_{i+1}f)(t)=$ the sum of the last three terms of the right side of the above equality, we get the proof for $i+1$.

Lemma 6.2. *Let f belong to F_{-1} and p and q be any integers such that $0\leq p\leq q$. Then it follows*

1) $(KA_1)^pLA_1(KA_1)^{q-p}Kf\in L^2_{loc}((0,h];\ D(A^\infty))\cap C_0$,
2) $(KA_1)^pLf\in L^2_{loc}((0,h];\ D(A^\infty))\cap C_0$,
3) $|(KA_1)^pLA_1x|_H\leq Const\cdot|x|_H$ *for any* $x\in D(A)$.

Proof. We have the following equality

$$\begin{aligned}(6.1)\quad &(\prod_{k=0}^{q}e^{-(s_k-s_{k+1}A}A_1)\\&=(\prod_{i=0}^{p-1}A^{\alpha i}e^{-(s_i-s_{i+1})A}A^{1-\alpha(i+1)}\cdot A^{-1+\alpha(i+1)}A_1A^{-\alpha(i+1)})\\&\qquad\cdot(A^{p\alpha}e^{-(s_p-s_{p+1})A}A^{(q-p)\alpha+1})\cdot(A^{-1-(q-p)\alpha}A_1A^{(q-p)\alpha})\\&\cdot(\prod_{j=1+p}^{q}A^{-(p-j+1)\alpha}e^{-(s_j-s_{j+1})A}A^{1+(q-j)\alpha}\cdot A^{-1-(q-j)\alpha}A_1A^{(q-j)\alpha})=I_0\end{aligned}$$

where α be small positive number and $s_0 = t$. Then from the assumption A-4) and the semigroup property we get

$$(6.2) \qquad |I_0 x|_H \le C \prod_{i=0}^{p-1})(s_i - s_{i+1})^{-1+\alpha}(s_p - s_{p+1})^{-1-q\alpha} \prod_{j=p+1}^{q} (s_j - s_{j+1})^{-1+\alpha}|x|_H$$

for any $x \in D(A)$.

Let ε be any sufficiently small positive number and $r(t)$ be $\int_t^h |f(s)|_H ds$. We define a sequence $\{\xi_i\}_{i=0}^{\infty}$ satisfying the following conditions:

$$h = \xi_1 > \xi_2 > \cdots > \xi_i > \cdots > 0, \ \xi_{i+1} \ge \xi_i/2$$

$$\lim_{i \to \infty} \xi_i = 0 \text{ and } 0 \ \le r(\xi_{i+1}) - r(\xi_i) \le \varepsilon/2.$$

The existence of such a sequence is shown as follows. If $\xi_i, \cdots, \xi_i$ are already chosen, put

$$\xi_{i+1} = \min\{\xi;\ \xi \ge \xi_i/2,\ r(\xi) - r(\xi_i) \le \varepsilon/2\}.$$

Suppose that $\xi_i \to \xi_\infty > 0$. If i is so large that $\xi_i \le 2\xi_\infty$, then $\xi_{i+1} > \xi_\infty > \xi_i/2$. Hence, $r(\xi_{i+1}) - r(\xi_i) = \varepsilon/2$. This implies $r(\xi_\infty) = \infty$, which is obviously a contradiction. We put $s_\varepsilon(t) = \{(\xi_i - \xi_{i+1})/(\xi_{i-1} - \xi_i)\}(t - \xi_i) + \xi_{i+1}$ for $\xi_i < t \le \xi_{i-1}$. Then we have

i) $s_\varepsilon \in W_{loc}^{1,\infty}((0,h])$ and $t/2 < s_\varepsilon(t) < t$,

ii) $|\int_{s_\varepsilon(t)}^t e^{-(t-s)A} f(s) ds|_H < \varepsilon$ for any $t \in (0,h]$,

iii) $\int_{s_\varepsilon(t)}^t e^{-(t-s)A} f(s) ds \in C((0,h];\ H) \cap L_{loc}^2\ ((0,h];\ D(A))$.

For the sake of simplicity we put $t_\varepsilon = s_\varepsilon(t)$. Since $\int_{s_\varepsilon}^t e^{-(t-\xi)A} f(\xi) d\xi$ belongs to $D(A)$ from (6.1), (6,2), ii), iii) we get

$$(6.3) \qquad |(KA_1)^p LA_1 (KA_1)^{q-p} \int_{\cdot_\varepsilon}^{\cdot} e^{-(\cdot-\xi)A} f(\xi) d\xi|_H \le Const \cdot \varepsilon.$$

Using the integertion by parts we have

$$(Kf)(t)=\int_{t_\varepsilon}^{t} e^{-(t-s)A}f(s)ds-\int_{t/2}^{t_\varepsilon} Ae^{-(t-s)A}\int_{0+}^{s} f(\xi)d\xi ds+$$

$$\{e^{-(t-t_\varepsilon)A}\int_{0+}^{t_\varepsilon} f(\xi)d\xi-e^{-(t/2)A}\int_{0+}^{t/2} f(\xi)d\xi\}=I_1+I_2+I_3.$$

From i), ii) and iii) we get

$$I_1,\ I_2,\ I_3\in L^2_{loc}((0,h];\ D(A)) \quad \text{and} \quad \lim_{t\to 0} I_3=0.$$

Then using (6.2) we see

$$|(KA_1)^p LA_1(KA_1)^{q-p}I_3|_H\leq Const \sup_{0<s<t} |I_3|_H.$$

From (6.2) and (6.3) it follows that

$$|(KA_1)^p LA_1(KA_1)^{q-p}I_1|_H\leq Const\ \varepsilon.$$

Using a similar method to (6.1) and (6.2) we get

$$|(\prod_{k=0}^{q} e^{-(s_k-s_{k+1})A}A_1)Ae^{-(s_{q+1}-s_{q+2})A}x|_H\leq$$

$$C\prod_{i=0}^{p-1}(s_i-s_{i+1})^{-1+\alpha}(s_p-s_{p+1})^{-1-(q+1)\alpha}\prod_{j=p+1}^{q+1}(s_j-s_{j+1})^{-1+\alpha}|x|_H.$$

Then we see

$$|(KA_1)^p LA_1(KA_1)^{q-p}I_2|_H\leq Const \sup_{0<s<t} |\int_{0+}^{s} f(\xi)d\xi|_H H.$$

Therefore

$$\lim_{t\to 0}|((KA_1)^p LA_1(KA_1)^{q-p}Kf)(t)|_H=0.$$

Using a similar method to (6.1) and (6.2) we see

$$|A^j I_0 x|_H\leq c\prod_{i=0}^{p-1}(s_i-s_{i+1})^{-1+\alpha}(s_p-s_{p+1})^{-1+j-q\alpha}\prod_{j=p+1}^{q}(s_j-s_{j+1})^{-1+\alpha}|x|_H.$$

Then we get

$$|A^j((KA_1)^p LA_1(KA_1)^{q-p}Kf)(t)|_H \leq Const \cdot t^{-j}.$$

Then the proof of 1) is complete. We can prove 2) and 3) by an analogous method.

Lemma 6.3. *Let f belong to F_{-1}. Then, for any integer m, we see $L_m(t)f \in L^2_{loc}((0,h];\ D(A)) \cap C_0$.*

Proof. Lemma 6.1 and Lemma 6.2 yield the proof of the lemma.

Lemma 6.4. *We have the following five properties:*

1) $((EA_1)^k Eg)(\cdot) \in C_0 \cap L^2(0,h;\ D(A)) \cap C([0,h];\ D(A^{1/2}))$ *for any* $g \in L_2(0,h;\ H)$ *and* $k=0,1,2,\cdots$,

2) $\int_0^t A^\beta e^{-(t-s)A} g(s)ds \in C_0$ *for any g such that* $A^\alpha g \in C([0,h]; H)$, $Ag \in L^2_{loc}((0,h];\ H)$ *and* $0<\beta<\alpha$,

3) $((EA_1)^k e^{-\cdot A}y)(\cdot) \in C_0 \cap L^2(0,h;\ D(A)) \cap C([0,h];\ D(A^{1/2}))$ *for any* $y \in D(A^{1/2})$ *and* $k=1,2,\cdots$.

4) $\int_0^t A^\alpha e^{-(t-s)A} G(u)(s)ds \in C_0$ *where u is the weak solution of* (3.1) *and* $0<\alpha<1$,

5) $\int_0^t A^\beta e^{-(t-s)A} g(s)ds \in C_0$ *for any* $g \in C([0,h];\ D(A^{-\alpha}))$ *and* $0<\alpha+\beta<1$.

Proof. The well known semigroup property yields $Eg \in C_0 \cap L^2(0,h;\ D(A)) \cap C([0,h];\ D(A^{1/2}))$ for any $g \in L^2(0,h;\ H)$. Then $A_1 Eg \in L_2(0,h;\ H)$. Thus $(EA_1)Eg \in C_0 \cap L^2(0,h;\ D(A)) \cap C([0,h];\ D(A^{1/2}))$. Therefore 1) is proved. The ststements 2) and 5) follows from Lemma 2.2. Lemma 2.5 yields the proof of 4). Using a similar method to the proof of Lemma 6.2 we get

$$|A^{1/2}((EA_1)^k Ee^{-\cdot A}y)(s)|_H \leq C|A^{1/2}y|_H \qquad \text{for any} \quad y \in D(A^{1/2}).$$

On the other hand it follows

$$\lim_{s\to 0} |A^{1/2}((EA_1)^k Ee^{-\cdot A}y_m)(s)|_H = 0 \qquad \text{for any} \quad y_m \in D(A).$$

We choose $\{y_m\}$ such that $\lim_{m\to\infty} A^{1/2}y_m = A^{1/2}y$. Combining 1) of Remark 3 and the above results we get the proof of 3).

Let u be the weak solution of (0.1). Denote by u_n the restiction of u to $[(n-1)h, nh]$ for $n=1,2\cdots,N$, and $u_0(s)=y(s)$ for $-h\leq s<0$ and $u_1(0)=x$. Then, u_n is the solution of

$$(6.4)\qquad \begin{cases} \dot{u}+Au+\displaystyle\int_{(n-1)h}^{t} a(t-s)A_2u(s)ds \\ \qquad =f(t)-A_1u_{n-1}(t-h)-\displaystyle\int_{t-h}^{(n-1)h} a(t-s)A_2u_{n-1}(s)ds, \\ u((n-1)h)=u_{n-1}((n-1)h) \quad \text{for t: } (n-1)h\leq t\leq nh. \end{cases}$$

Put for $0\leq s\leq h$

$$w_n(s)=u_n(s+(n-1)h),\ w_0(s)=y(s-h),$$

$$g_n(s)=f(s+(n-1)h)-A_1w_{n-1}(s)-\int_{s-h}^{0} a(s-\tau)A_2\ w_{n-1}(\tau+h)d\tau,$$

$$g_1^n(s)=-\int_{s-h}^{0} a(s-\tau)A_2w_{n-1}(\tau+h)d\tau+(R*g_n)(s)+R(s)w_n(0).$$

Since $Aw_{n-1}\in Q$ we see $g_n\in Q$. Then, from Lemma 2.5, and Lemma 2.6, we get $g_n^1\in C([0,h];\ D(A^{-\alpha}))\cap F_{-1}$.
Then in the same manner as the proof of the equivalence of (4.5) and (4.6) we get from (6.4) that

$$(6.5)\qquad \begin{cases} \dot{w}_n(s)+Aw_n(s)=g_n^1(s)+f(s+(n-1)h) \\ \qquad -A_1w_{n-1}(s)-G(w_n)(s) \\ w_n(0)=u_n((n-1)h). \end{cases}$$

From 1) of Remark 3, 4) of Lemma 2.5 and (3.8) we get the representation of w_n as the following mild solution:

$$(6.6)\qquad \begin{aligned} w_n(s)&=e^{-sA}w_n(0)+(Eg_n^1)(s)+(Ef(\cdot+(n-1)h))(s) \\ &-(EA_1w_{n-1})(s)-(EG(w_n))(s)=I_1+I_2+I_3+I_4+I_5. \end{aligned}$$

From $g_n^1\in C([0,h];\ D(A^{-\alpha}))$ and 5) of Lemma 6.4 it follows $A^\beta I_2$ belongs to C_0 for $0<\beta<1$. Noting that w_n is a weak solution and using 4) in Lemma 6.4 we have $A^\alpha I_5 \in C_0$. Thus putting $\tilde{h}_n=(Eg_n^1)-(EG(w_n))$ and noting 1) of Remark 3 we get, for $n=1,2,\cdots$ and any β; $0<\beta<1$,

$$A^\beta \tilde{h}_n \in C_0 \text{ and } \tilde{h} \in L^2_{loc}((0,h];\ D(A)). \tag{6.7}$$

For any integer n, $1 \leq n \leq N-1$, we shall show there exists a function $h_n \in C_0 \cap L^2_{loc}((0,h];\ D(A))$ such that

$$\begin{aligned} w_1(s)-e^{-sA}x&=h_1(s)\\ &+K(f(\cdot)-A_1y(\cdot-h))(s)+L_1(f(\cdot)-A_1y(\cdot-h))(s), \end{aligned} \tag{6.8}$$

$$\begin{aligned} w_n(\mathrm{s})&=h_n(s)\\ &+\sum_{k=2}^{n}(-1)^{n-k}((EA_1)^{n-k}\{Ef(\cdot+(k-1)h)+e^{-\cdot A}w_k(0)\})(s)\\ &+(-1)^{n-1}((EA_1)^{n-1}e^{-\cdot A}x)(s)\\ &+(-1)^{n-1}((KA_1)^{n-1}K(f(\cdot)-A_1y(\cdot-h))(s)\\ &+L_n(f(\cdot)-A_1y(\cdot-h))(s) \end{aligned} \tag{6.9}$$

for $n=2,3,\cdots$ and

$$A^{\alpha_n}h_n \in C_0 \text{ and } h_n \in L^2_{loc}((0,h];\ D(A)) \text{ for } n= 1,2,\cdots \tag{6.10}$$

where α_n are some positive numbers smaller than 1. Here we note that $A_1e^{-\cdot A}x \in F_{-1}$ and so $(EA_1)^{n-1}e^{-\cdot A}x$ can be defined. To begin with we shall show (6.8). (6.6) yields

$$\begin{aligned} w_1(s)=e^{-sA}x+\tilde{h}_1(s)&+K(f-A_1y(\cdot-h))(s)\\ &+L_1(f-A_1y(\cdot-h))(s). \end{aligned}$$

We put $h_1=\tilde{h}_1$. Then (6.7) yields (6.9) in case of $n=1$. Therefore, from the above, (6.8) is proved in case of $n=1$. Next let w_{n-1} satisfy (6.8) or (6.9) and (6.10) with $n-1$ in place of n for $2 \leq n \leq N-1$. We investgate (6.9) and (6.10) for $n=2$. From (6.6) it follows

$$w_2(s)=e^{-sA}w_2(0)+\tilde{h}_2(s)+(Ef(\cdot+h))(s)-(EA_1w_1)(s).$$

From (6.8) we see

$$\begin{aligned} &(EA_1w_1)(s)=(EA_1e^{-\cdot A}x)(s)+(EA_1h_1)(s)\\ &+(EA_1K(f-A_1y(\cdot-h)))(s)+(EA_1L_1(f-Ay(\cdot-h)))(s) \end{aligned}$$

$$=I_0+I_1+I_2+I_3.$$

From 2) in Lemma 6.4 and (6.10) there exists a number α_2 such that $0<\alpha_2<1$ and $A^{\alpha_2}I_1\in C_0$. Since KA_1 is continuous from $L^2_{loc}((0,h];D(A))$ to $L^2_{loc}((0,h];D(A))$ and $|(ALA_1h_1)(t)|_H\leq C\cdot t^{-1+\alpha_1}$ and from $EA_1=KA_1+LA_1$it follows that $EA_1h_1\in L^2_{loc}((0,h];D(A))$. Putting $h_n=\tilde{h}_2-I_1$ we obtain that h_2 satisfies (6.10) in case of $n=2$. Since

$$I_2=(KA_1)K(f-A_1y(\cdot-h))(s)+LA_1K(f-A_1y(\cdot-h))(s),$$
$$I_3=(KA_1)L_1(f-A_1y(\cdot-h))(s)+LA_1L_1(f-A_1y(\cdot-h))(s)$$

it follws

$$I_2+I_3=(KA_1)^1K(f-A_1y(\cdot-h))(s)+L_2(f-A_1y(\cdot-h))(s).$$

Thus, we get the representation of (6.9) in case of $n=2$ from $I_0=EA_1e^{-\cdot A}x$. Inductively we shall show (6.9) and (6.10). Let n be a natural number larger than 2. (6.6) yields

$$w_n(s)=e^{-sA}w_n(0)+\tilde{h}_n(s)+(Ef(\cdot+(n-1)h)(s)-(EA_1w_{n-1})(s).$$

From the assumption of induction we get

$$\begin{aligned}&(EA_1w_{n-1})(s)=(EA_1h_{n-1})(s)\\&+\sum_{k=2}^{n-1}(-1)^{n-1-k}((EA_1)^{n-k}\{Ef(\cdot+(k-1)h)(s)+e^{-\cdot A}w_k(0)\})(s)\\&+(-1)^{n-2}((EA_1)^{n-1}e^{-\cdot A}x)(s)+\\&(-1)^{n-2}EA_1\{(KA_1)^{n-2}K(f(\cdot)-A_1y(\cdot-h))+L_{n-1}(f(\cdot)-A_1y(\cdot-h))\}(s)\\&=I_4+I_5+I_6+I_7.\end{aligned}$$

Putting $h_n=\tilde{h}_n-I_4$ and noting (6.7) and 2) of Lemma 6.4 we know that h_n satisfies (6.10). Using a similar method to the proof of (6.9) in case of $n=2$ we also get (6.9) in this case.

Lemma 6.5 *Let $f(\cdot)-A_1y(\cdot-h)\in F_{-1}$. Then the weak solution w_n of* (6.5) *satisfies*

$$w_n(t)-e^{-tA}w_n(0)-(-1)^{n-1}(KA_1)^{n-1}K(f-A_1y(\cdot-h))(t)\in C_0.$$

Proof. Using a similar method to (6.1) we get

$$|((EA_1)^{n-1}e^{-\cdot A}x)(t)|_H\leq Const|x|_H \text{ for any } x\in H,$$

$$\lim_{t \to 0} ((EA_1)^{n-1} e^{-\cdot A} x)(t) = 0 \text{ for any } x \in D(A).$$

Thus it follows $\lim_{t \to 0} ((EA_1)^{n-1} e^{-\cdot A} x)(t) = 0$ for any $x \in H$. From 1) and 3) of Lemma 6.4 it follows

$$\lim_{s \to 0} |A^{1/2} \sum_{k=2}^{n-1} (-1)^{n-k} ((EA_1)^{n-k} \{Ef(\cdot + (k-1)h + e^{-\cdot A} w_k(0)\})(s)|_H = 0.$$

Therefore combining the above results, Lemma 6.3 and (6.8), (6.9) we complete the proof of the lemma.

Proposition 6.6 *Let* $f(\cdot)$ *belong to* F_{-1} *and* $f \in L^2_{loc}((0,T]; H)$. *The following two conditions are equivalent.*

1) *A weak solution of* (0.1) *is continuous on* $[0,mh)$ *but, at* $t = mh$, *this solution is discontinuous in* H.
2) $f(\cdot) - A_1 y(\cdot - h) \in F_{m-1}$ *but* $f(\cdot) - A_1 y(\cdot - h) \notin F_m$.

Proof. Lemma 6.5 imlies of the proposition.

Proof of Theorem 2. From Proposition 6.6 we obtain that the weak solution of (0.1) is continuous on $[0,t]$. Thus the proof of Theorem 2 is complete.

7. Appendix

To begin with we give the proof of Remak 2.

The frist half of our statement is trivial. Then we show the latter half of our statement.

1). Let f belong to $L^1(0,h; H) \cap L^2_{loc}((0,h]; H)$. We shall show $f \in F_n$ for any natural number n. For the sake of simplicity we consider the case $n=2$. Let α, β and θ be positive numbers satisgying $\alpha + \beta + \theta = 2$ and $\alpha < 1$, $\beta < 1$, $\theta < 1$. With the aid of a similar method to (6.1) and (6.2) it follows that

$$|e^{-(t-s)A} A_1 e^{-(s-\tau)A} A_1 e^{-(\tau-\xi)A}|_{H \to H} \leq Const \cdot (t-s)^{-\alpha} (s-\tau)^{-\beta} (\tau - \xi)^{-\theta}. \tag{7.1}$$

Noting the above inequality and changing the order of integration we have

$$|((KA_1)^2Kf)(t)|_H \le Const\cdot\int_0^t |f(\xi)|_H d\xi.$$

Thus we get $f\in F_2$.

2). Let f be a function satisfying the assumption (0.2). We shall show $f\in F_n$. For the sake of simplicity let $n=2$. From (7.1) it follows

$$|((KA_1)^2Kf)(t)|_H \le$$
$$\le Const\cdot\int_{t/2}^t\int_{s/2}^s\int_{\tau/2}^\tau (t-s)^{-\alpha}(s-\tau)^{-\beta}(\tau-\xi)^{-\theta}\xi^{-1/2}\cdot\xi^{1/2}|f(\xi)|)_H d\xi d\tau ds$$
$$=I(t).$$

If $\theta<1/2$, using Schwarz's inequality, we get

$$I(t)\le \mathrm{Const}\cdot(\int_{t/2}^t \xi|f(\xi)|_H^2 d\xi)^{1/2}.$$

Hence $f\in F_2$.

Their works in [1], [7], [8], [10] and [11] provide us with the information to study the inculusion relation between the function spaces F_i. We will show somewhere else by an example that F_i is in general a proper subset of F_{i-1} for any $i=0, 1,\cdots, N-1$.

3). Finally we show that there exists a function which belongs to Q but not to F_{-1}.

Supposing for instance that A^{-1} is completely continuous let $\{\lambda_i\}$ be a set of eigenvalues of A such that

$$(7.2)\qquad \begin{cases} 0<\lambda_1<\lambda_2<\cdots<\lambda_n<\cdots, & \lim_{\epsilon\to 0}\lambda_n=\infty, \\ \sum_{n=1}^{\infty}\{1-(\lambda_n/\lambda_{n+1})^2\}<\infty. \end{cases}$$

Let φ_j be normalized eigenvector of A corresponding to λ_j .

Put $f(t)=\sum_{i=1}^{\infty} f_j(t)$ where

$$f_j(t)=\begin{cases} a_j\varphi_j & \text{if } (\lambda_j^{-1}+\lambda_{j+1}^{-1})/2<t<\lambda_j^{-1}. \\ -a_j\varphi_j & \text{if } \lambda_{j+1}^{-1}\le t\le(\lambda_j^{-1}+\lambda_{j+1}^{-1})/2 \\ 0 & \text{otherwise} \end{cases}$$

Here, $\{a_j\}$ is a sequence of positive numbers such that $(\lambda_j^{-1}-\lambda_{j+1}^{-1})a_j$ converges to 1 as $j\to\infty$. It is trivial that $f\in L^2_{loc}((0,h];\ H)$. For the sake of simplicity we write $c_j=(\lambda_j^{-1}+\ \lambda_{j+1}^{-1})/2$ and $b_j=\lambda_j^{-1}$. Since $|\int_{b_{j+1}}^{c_j} f(s)ds|_H=a_j(\lambda_j^{-1}-\lambda_{j+1}^{-1})/2$, it follows that f is not improperly integrable at $t=0$. Next, let $b_{k+1}<t<b_k$ and $b_{j+1}<\varepsilon<b_j<b_{k+1}$. Then we have

$$A^{-\alpha}\int_\varepsilon^t f(s)ds=A^{-\alpha}\int_\varepsilon^{b_j} f_j(s)ds+A^{-\alpha}\int_{b_{k+1}}^t f_k(s)ds.$$

Since $|A^{-\alpha}\int_\varepsilon^{b_j} f_j(s)ds|_H\le$ Const $(\lambda_j^{-1}-\lambda_{j+1}^{-1})a_j\lambda_j^{-\alpha}$ and $\lambda_j^{-\alpha}(\lambda_j^{-1}-\ \lambda_{j+1}^{-1})a_j$ converges to 0 as $j\to\infty$,we know there exists $A^{-\alpha}\int_{0+}^t f(s)ds$ if $\alpha>0$. Since $A^{-\alpha}\int_{0+}^t f(s)ds=A^{-\alpha}\int_{b_{k+1}}^t f_k(s)ds$ for $t\in[b_{k+1},b_k]$ and $\int_{b_{k+1}}^b f_k(s)ds=0$ we see $\int_{0+}^t f(s)ds\in C((0,h];\ H)$. *Hence* $f\in Q$. Moreover we get

$$\overline{\lim_{t\to 0}}\ |\int_{0+}^t e^{-(t-s)A}f(s)ds|_H>0.$$

Indeed, we have

$$|\int_{b_{b+1}}^{c_j} e^{-(c_j-s)A}f(s)ds|_H>\lambda_j^{-1}\{1-e^{(\lambda_j^{-1}-\lambda_{j+1}^{-1})\lambda_j/2}\}a_j$$

$>$ Const $a_j(\lambda_j^{-1}-\lambda_{j+1}^{-1})\to$ Const >0 as $j\to\infty$. On the other hand we get

$$|\int_{0+}^{b_{j+1}} e^{-(c_j-s)A}\ f(s)ds|_H^2$$

$$= \sum_{k=j+1}^{\infty} \left| \int_{c_k}^{b_k} a_k e^{-(c_j - s)\lambda_k} \, ds - \int_{b_{k+1}}^{c_k} a_k e^{-(c_j - s)\lambda_k} \, ds \right|_H^2$$

$$\leq \text{Const} \cdot \sum_{k=j+1}^{\infty} \{a_k(\lambda_k^{-1} - \lambda_{k+1}^{-1})^2 \lambda_k e^{-c_j \lambda_k}\}^2 = I_j.$$

From (7.2) we have $\lim_{j \to \infty} I_j = 0$. Thus our claim is showed. If α is a sufficiently small poisitive number, then we have $\sum_{j=1}^{\infty} \lambda_j^{-\alpha} = \infty$.. From $\int_{b_{j+1}}^{b_j} |A^{-\alpha} f(s) ds|_H \, ds = a_j(\lambda_j^{-1} - \lambda_{j+1}^{-1})\lambda_j^{-\alpha}$, we obtain $A^{-\alpha} f \notin L^1(0,h; H)$.

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