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ON EXISTENCE OF KÄHLER METRICS WITH CONSTANT SCALAR CURVATURE

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1. Introduction and Statements of Results

Let N be a compact Kählerian manifold, let Ω be a Kähler class on N, and let Ω^+ be the set of Kähler forms representing Ω. On Ω^+, consider the functional Φ_Ω that assigns to a Kähler form the square of the L^2-norm of the scalar curvature. A critical point of Φ_Ω is called an *extremal* Kähler metric. Any Kähler metric with constant scalar curvature is extremal. Conversely, the variational appraoch can be used to find metrics with constant scalar curvature.

We begin with an existence theorem for extremal metrics. Recall that a Kähler metric is called a *generalized Einstein-Kähler* metric if the eigenvalues of the Ricci tensor are constant, see [27]. For example, a product of Einstein-Kähler metrics is a generalized Einstein-Kähler metric. If M is homogeneous under the action of a compact Lie group, then every Kähler class on M is represented by a generalized Einstein-Kähler metric.

Theorem 1. *Let (M,g_M) be a generalized Einstein-Kähler manifold with non-negative Ricci curvature, and let (L,h) be a holomorphic Hermitian line bundle such that the eigenvalues of $c_1(L,h)$ with respect to g_M are constant on M. Suppose $\hat{L}$ is a Kählerian compactification of $L^0=(L\backslash zero\ section)$, and Ω is a Kähler class on $\hat{L}$ which is represented by a metric of 'special type'* (see Section 2). *Then $\hat{L}$ admits an extremal metric representing Ω. This metric is unique up to the action of the connected automorphism group* $\mathrm{Aut}^0(\hat{L})$.

This may be taken as a generalization of the existence theorem of Koiso and Sakane for Einstein-Kähler metrics, since if $c_1(\hat{L})>0$ and the Futaki character of $c_1(\hat{L})$ vanishes, then an extremal metric in the anticanonical class is necessarily Einstein. We interpret vanishing of a Futaki character as a condition that the scalar curvature of an extremal metric be constant, rather than as a condition for extendability of a constant-curvature metric to $\hat{L}$ as in [20].

We note here two particular cases, emphasizing that in Corollary 1.2 the base space is not assumed to be homogeneous.

Corollary 1.1. *If N is compact Kählerian, and a maximal compact subgroup $K \subset \mathrm{Aut}^0(N)$ acts holomorphically on N with real cohomogeneity one, then N admits an extremal metric in each Kähler class.*

Kähler manifolds with real hypersurface orbits have been extensively studied [1,15]. Any such manifold is almost-homogeneous with respect to the complexification K^C of the compact group K, and the exceptional set is a smooth complex submanifold with two K-homogeneous components.

Corollary 1.2. *If $\hat{L}$ is of the form $\boldsymbol{P}(L \oplus 1)$ for a line bundle L over a product of Ricci-positive Einstein-Kähler manifolds with $b_2=1$, then $\hat{L}$ admits an extremal metric in each Kähler class.*

For metrics of special type, the Futaki character reduces to a single real integral and thus yields a tractable condition for an extremal metric to have constant scalar curvature. Sufficient conditions for the vanishing of this integral can be expressed in terms of curvature of the bundle L. As a first result we have the following.

Theorem 2. *Let $\hat{L} = \boldsymbol{P}(L \oplus 1)$ be as in* Corollary 1.2. *If $c_1(L,h)$ is neither definite nor semidefinite, then $\hat{L}$ admits a Kähler metric with constant scalar curvature.*

REMARK 1.1. In fact, the proof shows that the set of Kähler classes containing a metric with constant scalar curvature is a real-algebraic hypersurface in the Kähler cone $H^{1,1}_{+}(N,\boldsymbol{R})$.

In Theorem 2 we need assume only the following: The base space M is a product of Ricci-nonnegative Einstein-Kähler manifolds (M_i,ω_i) whose Kähler forms are integral, the first Chern form of the line bundle (L,h) is a linear combination of the pullbacks of these Kähler forms and is indefinite.

Theorem 2 can be strengthened to apply to Kählerian manifolds N obtained from $\hat{L}$ by (partially) blowing down the zero and infinity sections of $\hat{L}$. Let D_0 and D_∞ be the images in N of the zero and infinity sections of $\hat{L}$ (respectively), and let d_0 and d_∞ be their complex codimensions. The zero and infinity sections of $\hat{L}$ are biholomorphic to M, and blowing down ϖ: $\hat{L} \to N$ gives rise to fibrations

$$\boldsymbol{P}^{d_0-1} \hookrightarrow M \xrightarrow{\pi_0} D_0, \qquad \boldsymbol{P}^{d_\infty-1} \hookrightarrow M \xrightarrow{\pi_\infty} D_\infty.$$

The restriction of L to a fibre of π_0 is the tautological bundle $\mathcal{O}_{\boldsymbol{P}^{d_0-1}}(-1)$, and a similar assertion is true for the restriction of L^{-1} to a fibre of π_∞. Thus $c_1(L,h)$ has at least d_0-1 negative eigenvalues and at least $d_\infty-1$ positive eigenvalues.

Theorem 3. *With the above notation, assume that $c_1(L,h)$ has at least d_0 negative eigenvalues and at least d_∞ positive eigenvalues. Then N admits a Kähler metric with constant scalar curvature.*

We are now ready to state the main result of this paper, which is a partial converse to a theorem of Lichnerowicz. For convenience, we say a Lie group G is *reductive* if any (finite-dimensional) representation of G is completely reducible. If G is a complex Lie group, then G is isogenous to $H \times \mathrm{Alb}(G)$, where $\mathrm{Alb}(G)$ is the Albanese torus of G and H is algebraic; G is reductive in our sense if and only if H is reductive in the usual sense. Thus our usage is an extension of the usual concept of reductivity to complex Lie groups which may not be algebraic.

Theorem 4. *Let N be a compact almost-homogeneous Kählerian manifold as in Corollary* 1.1. *If $\mathrm{Aut}^0(N)$ is reductive, then N admits a Kähler metric with constant positive scalar curvature.*

1.1. Organization of the Paper In Section 2 we give a detailed exposition of the construction in [20]. We introduce the concept of a special-type metric and give various properties, especially the data needed to construct them, their components in local coordinates, and the components of their Ricci tensors.

In Section 3 we review the definition and elementary properties of extremal metrics, and give a necessary and sufficient condition for a metric of special type to be extremal. We then calculate the scalar curvature of a special-type metric and show that a partcular choice of defining data gives an extremal Kähler metric on $\hat{L}$. The proof of the last assertion characterizes functions which arise as the scalar curvature of a special-type metric, see Proposition 3.2. In Section 4 we indicate the proofs of the corollaries of Theorem 1.

In Section 5, we quickly review obstructions to existence of Kähler metrics with constant scalar curvature, due to Lichnerowicz, Calabi and Futaki, and note the well-known fact, due to Calabi [8], that vanishing of the Futaki character implies existence of a constant scalar curvature metric in the presence of an extremal metric. The Futaki character measures the amount by which the functional Φ_Ω fails to achieve the Cauchy-Schwarz lower bound for an extremal metric, see Remark 5.1. Consequently, a manifold admitting extremal metrics cannot have a vanishing Futaki character *and* non-reductive automorphism group. This observation is addressed in more detail in [16].

We prove Theorem 2, then sketch the calculations needed to prove Theorem 3. Finally we prove Theorem 4 by linking reductivity of $\mathrm{Aut}^0(N)$—for N almost-homogeneous—to existence of sufficiently many positive and negative eigenvalues of the class $c_1(L^0, h)$, where the open orbit of N is regarded as the total space of principal $\boldsymbol{C}^\times$-bundle L^0 equipped with a K-invariant Hermitian metric h.

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2. Metrics of Special Type

The results of this section are primarily due to Koiso and Sakane. When a result has previously been stated in the form given here, we have included a citation. Results stated without a citation are implicit in [20, 14, 19]. Some of the details do not seem to exist in written form, particularly in the form used here, so we have included them.

Let M be a compact (connected) Kählerian manifold, p: $L \to M$ a holomorphic line bundle, L^0 the complement of the zero section, and $\hat{L}$ a Kählerian compactification of L^0. Let h be a Hermitian metric on L, and let s: $L^0 \to (0,\infty)$ be the associated norm function. Assume that s extends to a continuous function s: $\hat{L} \to [0,\infty]$, and that $\hat{L}\backslash L^0$ is a disjoint union of two complex submanifolds of $\hat{L}$.

The group $\boldsymbol{C}^\times$ acts naturally on L^0. Let S generate the S^1-action, so that $\exp 2\pi S = \mathrm{Id}$, and let $H = -JS$.

Lemma 2.1. *As functions on* L^0, $ds(H) = s$.

Proof. The group $\boldsymbol{R}^+$ acts on L^0 by scalar multiplication along the fibres. For a point $\boldsymbol{z} \in L^0$, we have

$$H(\boldsymbol{z}) = \frac{d}{dx}\bigg|_{x=1} x\cdot\boldsymbol{z}, \quad Hs(\boldsymbol{z}) = \frac{d}{dx}\bigg|_{x=1} \|x\cdot\boldsymbol{z}\| = \|\boldsymbol{z}\| = s(\boldsymbol{z}). \tag{1}$$

Q.E.D.

The map (p,s): $L^0 \to M \times (0,\infty)$ factors through the quotient map π: $L^0 \to L^0/S^1$, yielding a diffeomorphism $L^0/S^1 \simeq M \times (0,\infty)$.

Fix $R \in (0,\infty)$ and let τ: $(0,\infty) \to (0,R)$ be an increasing diffeomorphism. Assume that $\tau(s)$ and $\tau(1/s)$ extend smoothly over 0 and satisfy

$$\tau'(0) = \lim_{s\to 0}\tau'(s) \neq 0, \qquad \tau'(\infty) = \lim_{r\to 0}\frac{1}{r^2}(1/r) \neq 0.$$

Let g_t be a one-parameter family of Riemannian metrics on M. A Hermitian metric on L^0 is of *special type* if there exists a Riemannian metric

$$g(x,s) = g_{\tau(s)}(x) + d\tau(s)^2 \tag{2}$$

on $L^0/S^1 \simeq M \times (0,\infty)$ such that the projection π is a Riemannian submersion. By abuse of language, we say a metric on $\hat{L}$ is of special type if the restriction to L^0 is of special type.

Put $t = \tau(s)$: $\hat{L} \to [0,R]$, and note that $t(\boldsymbol{z})$ is the distance from the point $\boldsymbol{z} \in L^0$ to the submanifold $\{s=0\}$ with respect to the metric.

We say a function $\hat{f}$: $\hat{L} \to \boldsymbol{R}$ *depends only on* s if there is a function f: $(0,\infty) \to \boldsymbol{R}$ with $\hat{f} = f(s)$.

Lemma 2.2. *The function $dt(H)$ depends only on s. If we put $dt(H) = u(t)$, then $u'(0) = 1$ and $u'(R) = -1$.*

Proof. By Lemma 2.1, $dt(H) = \tau'(s)Hs = s\tau'(s)$. Since t depends only on s, we may write $dt(H) = u(t)$ meaningfully. To prove the assertions about derivatives, differentiate $u(t) = s\tau'(s)$ with respect to s, obtaining

$$u'(t) = \frac{\tau'(s) + s\tau''(s)}{\tau'(s)} \tag{3}$$

Since $\tau'(0) \neq 0$, $u'(0) = 1$. To treat the case $s = \infty$, write $\tau(1/r)$ as a Taylor series about $r = 0$:

$$\tau(1/r) = R + b_1 r + b_2 r^2 + o(r^2)$$

Differentiating with respect to r at $r=0$ gives

$$-b_1=\lim_{r\to 0}\frac{1}{r^2}\tau'(1/r), \tag{4}$$

which exists and is non-zero by hypothesis. Differentiating again,

$$2b_2=\lim_{r\to 0}\frac{1}{r}\left(\frac{2}{r^2}\tau'(1/r)+\frac{1}{r^3}\tau''(1/r)\right).$$

Since this limit exists, the term in parentheses must approach zero as $r\to 0$. In particular,

$$\lim_{r\to 0}\frac{1}{r^3}\tau''(1/r)=-2\lim_{s\to 0}\frac{1}{r^2}\tau'(1/r)=2b_1. \tag{5}$$

As in equation (3),

$$\left.\frac{du}{dt}\right|_{t=R}=\lim_{s\to\infty}\frac{\tau'(s)+s\tau''(s)}{\tau'(s)}=\lim_{r\to 0}\left(\frac{\tau'(1/r)}{\tau'(1/r)}+\frac{\tau''(1/r)}{r\tau'(1/r)}\right).$$

Using equations (4) and (5), this equals

$$1+\lim_{r\to 0}\frac{\tau''(1/r)}{r^3}\,\frac{r^2}{\tau'(1/r)}=1+2b_1\left(\frac{-1}{b_1}\right)=-1. \tag{6}$$

Q.E.D.

Lemma 2.3.([20]) *A metric g of special type is Kähler if and only if*
1. *g_t is a Kähler metric on M for all $t\in(0,R)$, and*
2. *$\frac{d}{dt}g_t=-u(t)B$, where B is the 2-tensor associated to the curvature form $\rho=2\pi c_1(L,h)$ of the bundle metric h, i.e. $B(X,Y)=\rho(JX,Y)$.*

Proof. ([20], pp. 166–7.) For a vector field X on M of type (1,0), define the horizontal lift $\tilde{X}$ by

$$p_*(\tilde{X})=X, \qquad \tilde{X}s=(J\tilde{X})s=0,$$

so the horizontal lift is $\boldsymbol{C}^\times$-invariant. Let ω_t and ω denote the Kähler forms of g_t and g respectively. By invariance of the horizontal lift under the $\boldsymbol{C}^\times$-action,

$$(d\omega)(H,JH,\tilde{X}) = 0 = (d\omega)(JH,\tilde{X},Y).$$

For vector fields X, Y, and Z on M,

$$(d\omega)(\tilde{X},\tilde{Y},\tilde{Z}) = (d\omega_t)(X,Y,Z).$$

Finally, $[\tilde{X},\tilde{Y}] - \widetilde{[X,Y]} = -B(X,JY)(JH)$, so

$$\begin{aligned}(d\omega)(H,\tilde{X},\tilde{Y}) &= H\big(g_t(X,JY)\big) + B(X,JY)g_t(JH,JH)\\ &= u\frac{d}{dt}\,g_t(X,JY) + u^2B(X,JY).\end{aligned}$$

Q.E.D.

REMARK 2.1. If p: $L \to M$ is a holomorphic line bundle over a compact Kähler manifold, then any representative ρ of the class $2\pi c_1(L)$ is the curvature form of a Hermitian fibre metric, see for example [18].

Lemma 2.4. *Let g_M be a Kähler metric on M and let B be the curvature tensor of (L,h). Define $b>0$ by*

$$2b = \int_0^R u(x)dx.$$

Let U: $[0,R] \to [-b,b]$ be the antiderivative of u given by

$$U(w) = -b + \int_0^w u(x)dx.$$

Assume $g_M \pm bB$ is a Kähler metric. Then the special-type metric

$$g = dt^2 + (dt\circ J)^2 + p^*g_M - U(t)p^*B \tag{7}$$

is Kähler on L^0.

Proof. The metric $g_M - U(t)B + dt^2$ on $M\times(0,\infty)$ makes π into a Riemannian submersion, and the conditions of Lemma 2.3 are satisfied.

Q.E.D.

Introduce the function $\varphi:[-b,b] \to \boldsymbol{R}$ by $\varphi\big(U(t)\big) = u^2(t)$.

Lemma 2.5. *The function φ satisfies the following properties.*

1. *$\varphi(w)\geq 0$, with equality if and only if $w=\pm b$.*
2. *$\varphi'(-b)=2$, $\varphi'(b)=-2$, and φ extends smoothly over $\pm b$.*
3. *For any smooth function f: $[-b,b]\to \boldsymbol{R}$,*

$$H(f\circ \hat{U})=(\varphi\cdot f')(\hat{U}),$$

where $\hat{U}=U(t)$: $\hat{L}\to[-b,b]$.

Proof. The first statement follows at once from $\varphi(\hat{U})=g(H,H)$. The second follows from Lemma 2.2. The third follows from

$$H\hat{U}=U'(\tau(s))\tau'(s)Hs=u(t)\ s\tau'(s)=u^2(t)=\varphi(\hat{U}). \tag{8}$$

Q.E.D.

Proposition 2.1. *Let $\hat{L}$ be a compactification of L^0, and let h be a Hermitian fibre metric on L with norm function s and curvature tensor B. Suppose the following data are given: a Kähler metric g_M on M, a number $b>0$ with $g_M\pm bB$ positive-definite, a function φ:$[-b,b]\to \boldsymbol{R}$ satisfying the first two conditions of Lemma 2.5, and a function $\hat{U}$: $\hat{L}\to[-b,b]$ depending only on s and satisfying the third condition of Lemma 2.5. Then there exists a special-type metric of the form (7) on L^0.*

Proof. Given the above data, define t: $L^0\to(0,R)$ by

$$t(z)=\int_{-b}^{\hat{U}(z)}\frac{dx}{\sqrt{\varphi(x)}}, \tag{9}$$

and put $t=\tau(s)$ as usual. The integral is bounded because φ has simple zeros at $\pm b$. We claim the metric

$$g=dt^2+(dt\circ J)^2+p^*g_M-\hat{U}p^*B$$

is Kähler. If we define U: $(0,R)\to(-b,b)$ by $\hat{U}=U(t)$, then it suffices to show $U'=u$ by Lemma 2.3, where $u(t)=dt(H)=s\tau'(s)$. Differentiating equation (9) by the vector field H gives

$$s\tau'(s)=\frac{1}{\sqrt{\varphi(\hat{U})}}U'(t)\ s\tau'(s), \tag{10}$$

which implies $U'(t)=\sqrt{\varphi(\hat{U})}$, while $\varphi(\hat{U})=H\hat{U}=U'(t)\ s\tau'(s)$ by condition 3. Combining these, $U'(t)=\sqrt{\varphi(\hat{U})}=s\tau'(s)=u(t)$. Q.E.D.

Proposition 2.2. *Suppose* φ: $[-b,b]\to \boldsymbol{R}$ *satisfies the first two conditions of* Lemma 2.5. *Then there exists a function* $\hat{U}$: $\hat{L}\to[-b,b]$, *unique up to the action of* $\boldsymbol{R}^+$, *satisfying condition* 3 *of Lemma* 2.5.

Proof. Let s be the norm function of a Hermitian fibre metric on L. For $r\in(0,\infty)$ and $y\in(-b,b)$, the equation

$$F(r,y)=\log r-\int_0^y \frac{dx}{\varphi(x)}=0 \tag{11}$$

defines y as a function of r by the Implicit Function Theorem, and y: $(0,\infty)\to(-b,b)$ is an increasing diffeomorphism since the integrand is positive on $(-b,b)$ and has simple poles at $\pm b$. Moreover, y extends continuously over $r=0$ and $r=\infty$. Put $\hat{U}=y(s)$: $\hat{L}\to[-b,b]$. We claim that

$$H\hat{U}=\varphi(\hat{U}).$$

This follows from

$$\log s=\int_{-b}^{y(s)}\frac{dx}{\varphi(x)}$$

by differentiating with respect to H and using Lemma 2.1, since

$$H\hat{U}=y'(s)Hs=sy'(s)=\varphi(y(s))=\varphi(\hat{U}).$$

Q.E.D.

In short, a special-type Kähler metric on $\hat{L}$ is determined by the base metric g_M, the fibre metric h, a number $b>0$, and a smooth function φ: $[-b,b]\to\boldsymbol{R}$ satisfying the first two conditions of Lemma 2.5, cf.[19].

We continue with the exposition of [20]. On a trivializing neighborhood for L, there exist local coordinates $z^0,\cdots z^m$ such that $z^1,\cdots z^m$ are coordinates on M and z^0 is a fibre coordinate with $\partial/\partial z^0=H-\sqrt{-1}S$. We say such a coordinate system is *adapted* to L. Let ∂_α, $0\leq\alpha\leq m$, denote partial differentiation.

Lemma 2.6. ([20]) *With respect to an adapted coordinate system, the components of the metric g are given by*

$$g_{0\bar{0}}=2u^2(t), \qquad g_{0\bar{\beta}}=2u(t)\partial_{\bar{\beta}}t, \qquad g_{\alpha\bar{\beta}}=(g_M)_{\alpha\bar{\beta}}-U(t)B_{\alpha\bar{\beta}}+2\partial_\alpha t\partial_{\bar{\beta}}t \tag{12}$$

for $1\leq\alpha,\ \beta\leq m$.

On a fibre of L^0 we may assume that $\partial_\alpha t=0$ for $1\leq\alpha\leq m$. Under this assumption we have, *on a fibre*,

Lemma 2.7. ([20]) *Let $\hat{\phi}=\phi(t)$: $L^0\to\boldsymbol{R}$ be a function depending only on t. Then*

$$\partial_0\partial_0\hat{\phi}=u(t)(u\phi')'(t), \qquad \partial_0\partial_{\bar{\beta}}\phi=0, \qquad \partial_\alpha\partial_{\bar{\beta}}\hat{\phi}=-\frac{1}{2}(u\phi')(t)B_{\alpha\bar{\beta}}. \tag{13}$$

Define $\hat{q}=q(t)=\det(\mathrm{I}-U(t)g_M^{-1}B)$, so $\det(\mathrm{g})=2u^2(t)\hat{q}\ \det g_M$, *and put*

$$\Upsilon(t)=\log(u^2q)(t):\ \hat{L}\to\boldsymbol{R}.$$

The components of the Ricci tensor are given, on the fibre, by

$$r_{0\bar{0}}=-u(t)(u\Upsilon')'(t), \qquad r_{0\bar{\beta}}=0, \qquad r_{\alpha\bar{\beta}}=(r_M)_{\alpha\bar{\beta}}+\frac{1}{2}(u\Upsilon')(t)B_{\alpha\bar{\beta}}, \tag{14}$$

where r_M is the Ricci tensor of g_M.

Proof. See [20], pp. 168-9. Q.E.D.

For subsequent calculations it is substantially more convenient to express the metric and Ricci tensor in terms of the parameter U. Observe that on $M\times(0,\infty)\simeq L^0/S^1$ we have

$$\pi_*H=s\frac{d}{ds}=u(t)\frac{d}{dt}=\varphi(\hat{U})\frac{d}{dU}.$$

On a fibre where $\partial_\alpha s=0$ for $1\leq\alpha\leq m$,

$$g_{0\bar{0}}=2\varphi(\hat{U}), \qquad g_{0\bar{\beta}}=0, \qquad g_{\alpha\bar{\beta}}=(g_M)_{\alpha\bar{\beta}}-\hat{U}B_{\alpha\bar{\beta}} \tag{15}$$

If $\hat{f}=f(\hat{U})$: $\hat{L}\to\boldsymbol{R}$ is a function depending only on $\hat{U}$, then (on a fibre)

$$\partial_0\partial_{\bar{0}}\hat{f}=H((\varphi f')(\hat{U})), \qquad \partial_0\partial_{\bar{\beta}}\hat{f}=0, \qquad \partial_\alpha\partial_{\bar{\beta}}\hat{f}=-\frac{1}{2}(\varphi f')(\hat{U})B_{\alpha\bar{\beta}}. \tag{16}$$

Define Q: $[-b,b]\to\boldsymbol{R}$ by $Q(\hat{U})=q(t)=\det(\mathrm{I}-\hat{U}g_M^{-1}B)$, and put

$$\Upsilon(t)=\hat{\Psi}=\Psi(\hat{U})=\log(\varphi Q)(\hat{U}).$$

The components of the Ricci tensor are given, on the fibre, by

$$r_{0\bar{0}}=-H(\varphi\Psi('\hat{U})),\qquad r_{0\bar{\beta}}=0,\qquad r_{\alpha\bar{\beta}}=(r_M)_{\alpha\bar{\beta}}+\frac{1}{2}(\varphi\Psi')(\hat{U})B_{\alpha\bar{\beta}}. \tag{17}$$

Lemma 2.8. *If $\hat{f}=f(\hat{U})$: $\hat{L}\to \boldsymbol{R}$ depends only on $\hat{U}$, then*

$$\int_{\hat{L}}\hat{f}\,\mathrm{dvol}(g)=2\pi\,\mathrm{Vol}(M,g_M)\int_{-b}^{b}f(x)Q(x)dx. \tag{18}$$

Proof. Compute the iterated integral, using $\det(g_t)=Q(\hat{U})\det(g_M)$.
Q.E.D.

Finally, we state a result giving necessary and sufficient conditions for extendability of special-type metrics to $\hat{L}$. It was first proven in [20], in the course of proving their Theorem 4.1.

Proposition 2.3. *Let $g=dt^2+(dt\circ J)^2+p^*g_M-\hat{U}p^*B$ be a special-type metric on L^0 such that the associated function φ satisfies the first two conditions of Lemma 2.5. Assume there exists a special-type Kähler metric $\hat{g}$ on $\hat{L}$ with*

$$\hat{g}|_{L^0}=d\zeta^2+(d\zeta\circ J)^2+p^*g_M-\hat{V}p^*B,$$

where the functions $\hat{U}$ and $\hat{V}$ have the same range. Then there exists a function $\hat{\phi}$: $\hat{L}\to\boldsymbol{R}$ depending only on s such that

$$\hat{g}+\partial\bar{\partial}\hat{\phi}=g$$

on L^0, and the metric g extends to a Kähler metric on $\hat{L}$.

Proof. Put $\hat{U}=U(t)$, $\hat{V}=V(\zeta)$, and let $v=V'>0$, $\psi(\hat{V})=v^2(\zeta)$. Note that ζ depends only on s, and is 'increasing with respect to s'. We may regard t as a function of ζ and U as a function of V, so that

$$\frac{dt}{d\zeta}=\frac{u(t)}{v(\zeta)},\qquad \frac{dU}{dV}=\frac{\varphi(U)}{\psi(V)}. \tag{19}$$

By Lemma 2.5, $\psi(V)=2(V+b)+O((V+b)^2)$ near $V=-b$, and an analogous equation holds for φ. Since

$$\int \frac{dV}{\psi(V)} = \int \frac{dU}{\varphi(U)},$$

U extends to a smooth function on $[0,R]$. Moreover, $\hat{U} - \hat{V}$ vanishes on the submanifolds $\{s=0\}$ and $\{s=\infty\}$. Thus there exists a smooth function ϕ: $[-b,b] \to \boldsymbol{R}$ satisfying

$$\phi'(\hat{V}) = 2\frac{\hat{U} - \hat{V}}{\psi(\hat{V})}. \tag{20}$$

Put $\hat{\phi} = \phi(\hat{V})$. We will prove the following assertions:

1. $\hat{g} + \partial\bar{\partial}\hat{\phi} = g$ on $\hat{L}^0$, and
2. $\hat{g} + \partial\bar{\partial}\hat{\phi}$ is positive-definite on $\hat{L} \backslash L^0$.

Compute on a fibre with respect to an adapted coordinate system where $\partial_\alpha s = 0$. By equations (15), (16), and (17), and by the definition of ϕ,

$$\begin{aligned}
\hat{g}_{0\bar{0}} + \hat{\phi}_{0\bar{0}} &= 2\psi(\hat{V}) + H\big((\psi\phi')(\hat{V})\big) \\
&= 2\psi(\hat{V}) + 2H(\hat{U} - \hat{V})\big) = 2\varphi(\hat{U}) = g_{0\bar{0}}, \\
\hat{g}_{\alpha\bar{\beta}} + \hat{\phi}_{\alpha\bar{\beta}} &= (g_M)_{\alpha\bar{\beta}} - \hat{V}B_{\alpha\bar{\beta}} - \frac{1}{2}(\psi\phi')(\hat{V})B_{\alpha\bar{\beta}} \\
&= (g_M)_{\alpha\bar{\beta}} - \hat{U}B_{\alpha\bar{\beta}} = g_{\alpha\bar{\beta}}.
\end{aligned}$$

To prove the second assertion, compare $g = \hat{g} + \partial\bar{\partial}\hat{\phi}$ and $\hat{g}$ on tubular neighborhoods of the submanifolds $\{s=0\}$ and $\{s=\infty\}$. Since $\hat{\phi}$ depends only on $\hat{V}$, which is constant on the 'ends' of $\hat{L}$, g and $\hat{g}$ coincide on the tangent bundles, and differ on the normal bundles by a constant factor of

$$\frac{g(H,H)}{\hat{g}(H,H)} = \frac{\varphi(\hat{U})}{\psi(\hat{V})} = \left(\frac{s\tau'(s)}{s\zeta'(s)}\right)^2,$$

which converges to a non-zero value as $s \to 0$ or $s \to \infty$. This completes the proof of the proposition. Q.E.D.

We summarize the results of this section as follows.

Theorem 5. *Suppose $\hat{L}$ is a compactification of L^0, and the natural $\boldsymbol{C}^\times$-action extends to $\hat{L}$. Assume $\hat{L}$ admits a Kähler metric $\hat{g}$ which is of the form*

$$d\zeta^2 + (d\zeta \circ J)^2 + p^* g_M - V(\zeta) p^* B$$

when restricted to L^0, *and the range of* V *is* $[-b,b]$. *Given any smooth function* φ: $[-b,b] \to \boldsymbol{R}$ *satisfying the first two conditions of* Lemma 2.5, *there is a Kähler metric* g *on* $\hat{L}$ *which is of the form* (7) *when restricted to* L^0. *Moreover* $g(H,H) = \varphi(U(t)) = HU(t)$.

Proof. By Proposition 2.2, there is a function $\hat{U}$ as in condition 3 of Lemma 2.5. By Proposition 2.1, there is a special-type metric g on L^0, of the form (7) with $\hat{U} = U(t)$, and $g(H,H) = \varphi(U) = H\hat{U}$. By Proposition 2.3, this metric extends to $\hat{L}$. Q.E.D.

3. Extremal Metrics

We recall the definition of extremality for Kähler metrics. Let N be a compact Kählerian manifold of complex dimension m. For Ω a fixed Kähler class on N, let Ω^+ denote the set of Kähler forms in Ω. Define the functional Φ_Ω: $\Omega^+ \to \boldsymbol{R}$ by

$$\Phi_\Omega(\omega) = \|\sigma_\omega\|_{L^2}^2 = \int_N \sigma_\omega^2 \frac{\omega^m}{m!},$$

where $\omega \in \Omega^+$ and σ_ω is the scalar curvature of ω. After Calabi [7], we say ω is an *extremal* metric if ω is a critical point of the functional Φ_Ω. By the Cauchy-Schwarz inequality,

$$\Phi_\Omega \geq \frac{(2\pi m c_1(N) \cup \Omega^{m-1}[N])^2}{m!\,\Omega^m[N]},$$

and this bound is achieved precisely when there exists an $\omega \in \Omega^+$ with constant scalar curvature. We often write $\Omega^m[N] = \mathrm{Vol}_\Omega(N)$.

Calabi computed the Euler-Lagrange equation for the problem of minimizing Φ_Ω. To explain this, we recall that the *complex gradient* (with respect to ω) of a smooth function F is the vector field of type (1,0) associated to the (0,1)-form $\bar{\partial} F$. In local; coordinates z^α, if $\omega = \sqrt{-1}\Sigma g_{\alpha\bar{\beta}} dz^\alpha \wedge dz^{\bar{\beta}}$ and $(g^{\bar{\beta}\alpha})$ is the inverse matrix of $(g_{\alpha\bar{\beta}})$, then

$$\operatorname{grad} F = \sum_{\alpha\bar{\beta}} g^{\bar{\beta}\alpha} \frac{\partial F}{\partial z_{\bar{\beta}}} \frac{\partial}{\partial z^\alpha}.$$

Theorem 6. ([7]) *If* N *is compact, a metric* $\omega \in \Omega^+$ *is extremal if and only if the gradient field of the scalar curvature* σ_ω *is holomorphic.*

3.1. Extremality of Special-Type Metrics. From now on, we assume the following. For $i=1,\cdots,k$, (M_i,ω_i) is a compact, Ricci-nonnegative Einstein-Kähler manifold of complex dimension l_i. Assume ω_i is indivisible and integral. There exist integers $\kappa_i \geq 0$ with $c_1(M_i,\omega_i)=\kappa_i\omega_i$. This restriction on the sign of the curvature will be necessary for the proof of Proposition 3.1. Put $M=M_1\times\cdots\times M_k$, and by abuse of notation let ω_i denote the pullback by projection on M_i. Let p: $(L,h)\to M$ be the holomorphic Hermitian line bundle with first Chern form $c_1(L,h)=\Sigma_i n_i\omega_i$. Fix positive real numbers a_i and b with $a_i \pm bn_i>0$, and let g_M be the Kähler metric whose Kähler form is $\Sigma_i a_i\omega_i$. The eigenvalues of r_M (the Ricci tensor of g_M) and B with respect to g_M are constant on M; specifically, the eigenvalues of r_M are $\kappa_i/a_i \geq 0$, of multiplicity l_i, and the eigenvalues of B are n_i/a_i, also of multiplicity l_i. In other words, the line bundle L is a tensor product of Einstein-Hermitian line bundles. The function $Q(w)=\det(I-wg_M^{-1}B)$ is given by

$$Q(w)=\prod_{i=1}^{k}\left(1-\frac{n_i}{a_i}w\right)^{l_i}. \tag{21}$$

Recall that $Q(\hat{U})=\hat{q}$, as in Lemma 2.7. Introduce the function

$$G(w)=\operatorname{tr}_{g_M-wB} r_M=\sum_{i=1}^{k}\frac{\kappa_i l_i}{a_i-n_i w}. \tag{22}$$

Q and GQ are defined on all of $\boldsymbol{R}$, and are positive on $(-b,b)$.

Lemma 3.1. *Suppose $g=dt^2+(dt\circ J)^2+p^*g_M-\hat{U}p^*B$ on L^0. On L^0, the scalar curvature $\hat{\sigma}$ of g is*

$$\hat{\sigma}=G(\hat{U})-\frac{1}{2Q(\hat{U})}(\varphi Q)''(\hat{U}). \tag{23}$$

Proof. Choose an adapted coordinate system near $\boldsymbol{z}$ so that $\partial_\alpha t=0$ for $\alpha=1,\cdots,m$ on the fibre containing $\boldsymbol{z}$. By equation (15), g is a block diagonal matrix

$$2\varphi(\hat{U})\oplus\bigoplus_{i=1}^{k}(a_i-n_i\hat{U})(g_i)_{\alpha\bar{\beta}}$$

on the fibre, where g_i is the Einstein-Kähler metric on M_i. By equation (17), the Ricci tensor of g is the block-diagonal matrix

$$-\varphi(\varphi\Psi')'(\hat{U})\oplus\bigoplus_{i=1}^{k}\Big(\kappa_i+\frac{n_i}{2}(\varphi\Psi')(\hat{U})\Big)(g_i)_{\alpha\bar{\beta}},$$

where $\Psi=\log(\varphi Q)$ as usual. Taking the trace of r with respect to g and using

$$\sum_{i=1}^{k}\frac{n_i l_i}{a_i-n_i\hat{U}}=-(\log Q')(\hat{U}),\qquad (\varphi\Psi')'+(\varphi\Psi')(\log Q)'=\frac{1}{Q}(\varphi Q)'',$$

proves equation (23). Q.E.D.

In particular, if a special-type metric on L^0 extends smoothly to a compactification $\hat{L}$, then the right-hand side of equation (23) must converge as $\hat{U}\to\pm\mathrm{b}$. This is vacuously true if $\hat{L}\backslash L^0$ is of pure codimension one, since in this case $a_i>|n_i|$.

Lemma 3.2. *Suppose g is a metric of special type on $\hat{L}$. g is extremal if and only if the scalar curvature of g is of the form $\sigma_0+\lambda\hat{U}$ for some constants σ_0 and λ.*

Proof. By Theorem 6, the gradient of $\hat{\sigma}=\sigma(\hat{U})$ is a global holomorphic vector field on $\hat{L}$. The gradient field of the function $\hat{U}$ is $H-\sqrt{-1}S$. Thus on L^0

$$\mathrm{grad}\sigma(\hat{U})=\sigma'(\hat{U})\mathrm{grad}\hat{U}=\sigma'(\hat{U})(H-\sqrt{-1}S). \tag{24}$$

This extends to a holomorphic vector field on $\hat{L}$ if and only if $\sigma'(\hat{U})$ is a global holomorphic function, i.e. a constant. Q.E.D.

3.2. Proof of Theorem 1. Let $Q,GQ\colon[-b,b]\to\boldsymbol{R}$ be defined by equations (21) and (22).

Proposition 3.1. *Given constants σ_0 and λ, define $\varphi\colon[-b,b]\to R$ by*

$$(\varphi Q)(w)=2(w+b)Q(-b)-2\int_{-b}^{w}(\sigma_0+\lambda x-G(x))(w-x)Q(x)dx. \tag{25}$$

For a suitable choice of σ_0 and λ, this function φ satisfies the first two conditions of Lemma 2.5.

Proof. $\varphi(-b)=0$ and $\varphi'(-b)=2$ are immediate. $\varphi(b)=0$ is equivalent to

$$0=2bQ(-b)-\int_{-b}^{b}(\sigma_o+\lambda x-G(x))(b-x)Q(x)dx. \tag{26}$$

Under this hypothesis, $(\varphi Q)'(b)=\varphi'(b)Q(b)$, so $\varphi'(b)=-2$ is equivalent to

$$-2Q(b)=2Q(-b)-2\int_{-b}^{b}(\sigma_0+\lambda x-G(x))Q(x)dx. \tag{27}$$

Define constants α, β, and γ by

$$\alpha=\int_{-b}^{b}Q(x)dx,\ \beta=\int_{-b}^{b}xQ(x)dx,\ \gamma=\int_{-b}^{b}x^2Q(x)dx.$$

Equations (26) and (27) are rewritten as

$$\sigma_0\alpha+\lambda\beta=Q(b)+Q(-b)+\int_{-b}^{b}G(x)Q(x)dx,$$

$$\sigma_0\beta+\lambda\gamma=b\big(Q(b)-Q(-b)\big)+\int_{-b}^{b}xG(x)Q(x)dx.$$

This system always has a solution (σ_0,λ) since $\alpha\gamma-\beta^2>0$. For this choice of σ_0 and λ, the function φ defined by equation (25) satisfies $\varphi(b)=0$, $\varphi'(b)=-2$.

It remains to show that the function φ defined by equation (25) is positive on the open interval $(-b,b)$. To preserve continuity of the argument we defer this to Appendix A. The idea is to show that the polynomial $(\varphi Q)''$ does not have enough roots on the interval $(-b,b)$ for φQ to become negative. Since $Q>0$ on $(-b,b)$, so is φ. Q.E.D.

Using the function φ whose existence is guaranteed by Proposition 3.1, construct a special-type Kähler metric g as in Theorem 5. By Lemma 3.1, the scalar curvature of g is $\sigma_0+\lambda\hat{U}$, where (σ_0,λ) is as in the proof of Proposition 3.1. By Lemma 3.2, g is extremal. This completes the proof of Theorem 1. Q.E.D.

We note the following result, which seems to be of sufficient interest to be stated separately. Our setup is as in Theorem 1.

Proposition 3.2. *Given a smooth function σ: $[-b,b]\to\boldsymbol{R}$, define the function φ_σ: $[-b,b]\to\boldsymbol{R}$ by*

$$\varphi_\sigma(w)=\frac{2}{Q(w)}\left((w+b)Q(-b)-\int_{-b}^{w}\big(\sigma(x)-G(x)\big)(w-x)Q(x)dx\right). \tag{28}$$

Suppose the function $\varphi=\varphi_\sigma$ satisfies the conditions

1. $\varphi>0$ *on* $(-b,\ b)$, *and*
2. $\varphi(b)=0,\ \varphi'(b)=-2.$

Then $\sigma(\hat{U})$ is the scalar curvature of a special-type metric on $\hat{L}$, where the function $\hat{U}$: $L\rightarrow[-b,b]$ is as in Proposition 2.2.

REMARK 3.1. The endpoint conditions $\varphi(b)=0$ and $\varphi'(b)=-2$ are equivalent to the system

$$b\big(Q(b)-Q(-b)\big)=\int_{-b}^{b}\big(\sigma(x)-G(x)\big)xQ(x)dx,$$

$$Q(b)+Q(-b)=\int_{-b}^{b}\big(\sigma(x)-G(x)\big)Q(x)dx.$$

The total scalar curvature of a Kähler metric depends only on the choice of Kähler class. Computing this value for a special-type metric gives, by Lemmas 2.8 and 3.1,

$$\int_{\hat{L}}\hat{\sigma}\ \mathrm{dvol}(g)=2\pi\ \mathrm{Vol}(M,g_M)\left(Q(b)+Q(-b)+\int_{-b}^{b}G(x)Q(x)dx\right).$$

In words, the special-type metric defined by φ extends to $\hat{L}$ if and only if the fibres 'close up', i.e. $\varphi(b)=0$, and the total scalar curvature takes the 'correct' value, i.e. $\varphi(b)=-2$.

Proposition 3.2. Characterizes the functions which can arise as the scalar curvature of a special-type metric. The condition $\varphi_\sigma>0$ is an open condition and amounts to positive-definiteness of the resulting symmetric two-tensor.

The boundary conditions are the more interesting from a geometric point of view. For example, if one takes σ to be a constant, then the boundary conditions at b cannot be simultaneously satisfied unless the Futaki character vanishes. In [14, 20] the constant is dictated by the Kähler class, and the condition $\mathscr{F}(H)=0$ is the integrability condition which guarantees that $\varphi(b)=0$. If one wants σ to be of the form $a+cx$ on $[-b,b]$, then the choice of Kähler class determines a and c uniquely, cf. [12], Theorem 3.3.3.

In this sense, existence of Kähler metrics with constant scalar curvature is overdetermined, which interprets the necessity and sufficiency of the condition that the Futaki character vanshes. Existence of extremal metrics is, in this sense, unobstructed.

If N is the total space of a vector bundle or disc bundle over a product of Einstein-Kähler manifolds, there are no non-trivial boundary conditions. Given some mildly restrictive conditions on the curvature of N (regarded as a vector bundle), one can construct complete Einstein-Kähler metrics of special type on N. This has been accomplished by other techniques, see [6, 29, 31] for example.

3.3. Historical Remarks. The first examples of (non-Einstein) extremal Kähler metrics were obtained by Calabi [7]. He solved a differential equation for the Kähler potential of an extremal metric g on $\boldsymbol{P}^1$ bundles over $\boldsymbol{P}^m$. His method relies on the fact that the total space of any (negative) holomorphic $\boldsymbol{C}^{\times}$-bundle over $\boldsymbol{P}^m$ is covered by $\boldsymbol{C}^{m+1}\backslash\{0\}$, and thus does not generalize easily.

Sakane [30] proved existence of non-homogeneous Einstein-Kähler metrics on certain Fano manifolds of the form $\hat{L}$, under the assumption that the Futaki invariant vanishes. This result was generalized jointly with Koiso [20] to include the manifolds considered in our Theorem 1.

At about the same time Koiso and Sakane announced their result, Mabuchi [25] obtained a separate proof of existence of Einstein-Kähler metrics on the same class of Fano manifolds. His proof comes from symplectic geometry, and interprets the Futaki character in these terms. It also uses the fact that every holomorphic line bundle over a homogeneous Kähler manifold is homogeneous, so any automorphism of the base lifts to a bundle automorphism.

In [19], Remark 2.3, Koiso claimed that $\hat{L}$ admits an extremal metric in the *anticanonical* class, provided $\hat{L}$ is Fano.

LeBrun and Simanca [22, 23] have recently obtained beautiful results on existence of extremal metrics under small deformations of the Kähler class and/or the complex structure. They are able to give many new examples of extremal Kähler metrics on certain complex surfaces. Their results also put our examples into a more general context.

In principle one could search for Einstein-Kähler metrics directly by the present method, though the proofs of [20] and [25] are better adapted to the Einstein case. In particular, it is convenient to let the range of the function U and the choice of base metric g_M be determined by the anticanonical polarization $\Omega=2\pi c_1(M)$.

4. Examples

Suppose a compact Lie group K acts on a Kähler manifold N with real hypersurface orbits. Then N is almost-homogeneous (see [15], Theorem 3.2), and the union L^0 of the principal K-orbits is naturally a principal $\boldsymbol{C}^{\times}$-bundle over a homogeneous Kähler manifold M. Each Kähler class on N is represented by a K-invariant metric (by averaging), and this metric is of special type if one chooses the fibre metric h to be K-invariant. The remaining hypotheses of Theorem 1 are clear since M is K-homogeneous. Thus N admits an extremal metric in each Kähler class.[1]

If the compactification $\hat{L}$ *of* L^0 is the associated projective line bundle, then homogeneity of the base space is not required to prove existence. Since there are examples (due to Siu [33], Nadel [28], and Tian [34, 35]) of positive Einsten-Kähler manifolds with no holomorphic vector fields, one gets extremal metrics with non-constant curvature on manifolds with a one-dimensional space of holomorphic vector fields.

For $i=1,\cdots,k$, let (M_i,g_i) be a Ricci-positive Einstein-Kähler manifold of dimension l_i, with $b_2(M_i)=h^{1,1}(M_i)=1$. Assume further that the Kähler form ω_i is indivisible and integral. This implies that $\mathrm{Pic}(M_i)\simeq\boldsymbol{Z}$ is generated by the holomorphic Hermitian line bundle with first Chern form ω_i. Define $\kappa_i>0$ by $c_1(M_i,\omega_i)=\kappa_i\omega_i$. Put $M=M_1\times\cdots\times M_k$, and by abuse of notation let ω_i denote the pullback under projection to M_i. Let p: $(L,h)\to M$ be the holomorphic Hermitian line bundle with $c_1(L,h)=\Sigma\, n_i\omega_i$, and put $E=L\oplus\boldsymbol{1}$. Let $\hat{p}$: $\hat{L}=\boldsymbol{P}(E)\to M$, denote by $\tau\subset\hat{p}^*(E)\to\hat{L}$ the tautological bundle, and let $\zeta=c_1(\tau)\in H^2(\hat{L},\boldsymbol{Z})$. By the Leray-Hirsch Theorem, we have an explicit description of the cohomology ring of $\hat{L}$.

Lemma 4.1. *As an $H^2(M,\boldsymbol{Z})$-module, $H^2(L,\boldsymbol{Z})$ is generated by the class ζ, subject to the single relation*

$$\zeta^2-\hat{p}^*c_1(L)\zeta=0. \tag{29}$$

As remarked above, extendability of a special-type metric on L^0 to $\hat{L}$ is easy to verify. This follows immediately from Proposition 2.3 and the following:

Lemma 4.2. *Fix a Kähler class ω on $\hat{L}$. There exists a special-type*

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[1]The author would like to acknowledge Z. D. Guan for pointing out that the positivity proof for φ goes through essentially unchanged if there is a torus factor in M.

Kähler metric representing Ω, whose restriction to a fibre is the Fubini-Study metric.

Proof. Choose an adapted local coordinate system for L^0, so $z^0=se^{i\theta}$ is the fibre coordinate. It is straightforward to check that if $u(t)=b\sin(t/b)$, then the map

$$z^0 \mapsto (u(t)\cos\theta,\ u(t)\sin\theta,\ b\cos(t/b))$$

is an isometric embedding from the fibre to the sphere of radius $b>0$ in $\boldsymbol{R}^3$, where $t=\tau(s)$ as usual.

By Lemma 4.1, a Kähler class Ω on L is determined by the Kähler class of the metric $g_M=\Sigma_i a_i\omega_j$ and by $\Omega[F]=4\pi b>0$, where F is a fibre of $\hat{L}$. By scaling the metric, we may assume $b=1$. Taking

$$\tau(s)=\cos^{-1}\left(\frac{s^2-1}{s^2+1}\right)=2\tan^{-1}s,$$

we get $u(t)=s\tau'(s)=\sin t$ as desired. Q.E.D.

REMARK 4.1. The restriction of a special-type metric to the closure of a fibre of L^0 is an S^1-invariant metric on a two-shpere, which embeds in $\boldsymbol{R}^3$ as a surface of revolution if and only if $|u'(t)|<1$ for $t\in(0,R)$. To see this, observe that the function t: $\hat{L}\to[0,R]$ is the distance to the zero section. The function $2\pi u(t)$ is the length of the S^1-orbit of a point at distance t from the zero section since $u(t)=\|H\|=\|S\|$. If a typical fibre embeds isometrically in $\boldsymbol{R}^3$, then the generating curve is parametrized by $(\eta(t),u(t))$, $t\in[0,R]$, where $\eta'(t)^2+u'(t)^2=1$.

We remark that our existence proof fails for arbitrary almost-homogeneous spaces. Even in dimension two there are equivariant compactifications of $\boldsymbol{C}^\times\times\boldsymbol{C}^\times$ on which a maximal compact group has complex hypersurface orbits. The simplest are the blowing-up of the complex projective plane at two or three points.

Recent work of LeBrun and Simanca [22,23] shows that for any Kählerian manifold with fixed complex structure, the set of extremal classes (i.e. Kähler classes containing an extremal metric) is open in the set of all Kähler classes.

If three non-collinear points are blown up on $\boldsymbol{P}^2$, the results of [22] imply existence of many extremal metrics since this surface admits an Einstein-Kähler metric by a theorem of Siu [33]. In fact, LeBrun and Simanca's results show that the set of Kähler classes containing a constant curvature representative is a real-analytic hypersurface in the set of extremal classes near the anticanonical class.

For the below-up of $\boldsymbol{P}^2$ at two points or three collinear points nothing is known; the results of [23] do not guarantee the set of extremal classes is non-empty, and these manifolds admit no Einstein-Kähler metric. On the other hand, it would be extremely surprising if the blow-up of $\boldsymbol{P}^2$ at two points admits no extremal metric.

5. Metrics With Constant Curvature

It is of interest to find simple necessary and sufficient conditions for existence of Kähler metrics with constant scalar curvature on a compact manifold N. Regarding necessity, there is a theorem due to Lichnerowicz relating the structure of $\mathrm{Aut}^0(N)$ with existence of a constant curvature metric. Recall our convention or reductivity of groups, which does not assume the group to be algebraic.

Theorem 7. ([24]) *Let N be a compact manifold admitting a Kähler metric with constant scalar curvature. Then the connected automorphism group* $\mathrm{Aut}^0(N)$ *is reductive.*

Proof. See [8], Theorem 1, or [12], Theorem 2.3.6. Q.E.D.

There is a sharper necessary condition due to Calabi and Futaki. For a given Kähler class Ω, there is a Lie algebra character

$$\mathscr{F}_\Omega\colon H^0(N, T^{1,0}N) \to \boldsymbol{C}$$

which vanishes if Ω contains a metric with constant scalar curvature. The trade-off is that this character is not easily computed in general. (However, if $\Omega = 2\pi c_1(N) > 0$, then there is a localization formula for $\mathscr{F}_\Omega$, see[14].)

We recall the definition of $\mathscr{F}_\Omega$. Fix any Kähler form $\omega \in \Omega$, and let ρ and $\mathscr{H}\rho$ denote the Ricci form and the harmonic part of the Ricci form respectively. There is a smooth function $F_\omega\colon N \to \boldsymbol{R}$, unique up to an added constant, with

$$\rho - \mathscr{H}\rho = \sqrt{-1}\partial\bar{\partial}F_\omega. \tag{30}$$

For a holomorphic vector field X, define

$$\mathscr{F}_\Omega(X) = \int_N (XF_\omega)\frac{\omega^m}{m!}. \tag{31}$$

Proposition 5.1. *The functional $\mathscr{F}_\Omega$ is independent of the choice of $\omega \in \Omega$. Thus it is invariant under the coadjoint action of* $\mathrm{Aut}^0(N)$, *i.e. is a Lie algebra character.*

Proof. See for instance [12], Theorem 3.2.1, or [14], Theorem 2.3. Q.E.D.

Proposition 5.2. *If Ω contains a metric with constant scalar curvature, then $\mathscr{F}_\Omega$ vanishes identically. Conversely, if $\mathscr{F}_\Omega$ vanishes, then an extremal metric in Ω has constant scalar curvature.*

Proof. The first statement follows at once from Proposition 5.1, since the scalar curvature of ω is constant if and only if ρ is harmonic, if and only if F_ω is constant.

To prove the second statement, it is actually easier to prove the apparently stronger assertion: If ω is extremal with scalar curvature σ, and $\mathscr{F}_\Omega(\mathrm{grad}\sigma)=0$, then σ is constant. To see this, first observe that

$$\sigma - \mathscr{H}\sigma = \Box F_\omega$$

by taking the trace of equation (30). Writing $F_\omega = F$,

$$\mathscr{F}_\Omega(\mathrm{grad}\sigma) = \int_N (g^{\bar{\beta}\alpha}\sigma_{\bar{\beta}}F_\alpha)\frac{\omega^m}{m!} = -\int_N (\sigma\Box F)\frac{\omega^m}{m!}$$
$$= -\int_N (\Box F)^2\frac{\omega^m}{m!} \leq 0$$

with equality if and only if $\Box F=0$. This property of the Futaki character is due to Calabi [8]. Q.E.D.

REMARK 5.1. Since the integral over N of a Laplacian vanishes,

$$\int_N (\Box F)^2\frac{\omega^m}{m!} = \int_N (\sigma - \mathscr{H}\sigma)\sigma\frac{\omega^m}{m!} = \Phi_\Omega(\omega) - \frac{(2\pi m c_1(N)\cup\Omega^{m-1}[N])^2}{m!\mathrm{Vol}_\Omega(N)}.$$

In [8], Calabi poses the question of whether a sharper lower bound for the functional Φ_Ω can be found. Naturally, this bound should reduce to the Cauchy-Schwarz bound when Ω^+ contains a metric with constant curvature. The above strongly suggests that

$$\frac{(2\pi m c_1(N) \cup \Omega^{m-1}[N])^2}{m!\mathrm{Vol}_\Omega(N)} - \mathscr{F}_\Omega(X_\Omega) \tag{32}$$

is the desired bound, where X_Ω is the vector field whose existence is asserted by [13], see also [12], Theorem 3.3.3. In any case, the functional Φ_Ω has at most one critical value, which is given by equation (32).

REMARK 5.2. If N admits an extremal metric in a Kähler class with vanishing Futaki character, then by the theorem of Lichnerowicz the connected automorphism group of N is reductive. Together with Theorem 1, this gives a simple proof that certain manifolds, e.g. the blowing-up of projective space along a linear subspace, have non-vanishing Futaki character for every Kähler class. For other consequences of this remark, see [16].

5.1. Proofs of Theorems 2 and 3.

For the remainder of Section 5, it is convenient to work with the polynomial

$$Q(x)\prod_{i=1}^{k} a_i^{l_i} = \prod_{i=1}^{k} (a_i - n_i x)^{l_i}.$$

By abuse of notation, we denote this polynomial by Q, also.

For a metric g of special type, the scalar curvature is constant if and only if $\lambda = 0$. By scaling, we may assume $\Omega[F] = 4\pi$, i.e. that $b = 1$. Solve equations (26) and (27) for λ to obtain the following.

Lemma 5.1. *The scalar curvature of g is constant if and only if*

$$\left(Q(1) - Q(-1) + \int_{-1}^{1} xGQ(x)dx\right)\int_{-1}^{1} Q(x)dx - \left(Q(1) + Q(-1) + \int_{-1}^{1} GQ(x)dx\right)\int_{-1}^{1} xQ(x)dx = 0. \tag{33}$$

We first prove Theorem 2 for projective line bundles over $M = M_1 \times M_2$, where $b_2(M_i) = 1$. Let ω_i be an indivisible, integral Einstein-Kähler form on M_i. By abuse of notation we write ω_i for the pullback to M by projection to M_i. Let (L,h) be the holomorphic line bundle with $c_1(L,h) = n_1\omega_1 + n_2\omega_2$. Assume without loss of generality that $n_1 > 0 > n_2$. Put $g_M = a_1\omega_1 + a_2\omega_2$, $a_i > |n_i|$, and let g be the extremal metric guaranteed by Theorem 1.

We wish to solve equation (33). Writing $a_i = t_i + |n_i|$, where $t_i > 0$, we regard the left-hand side of equation (33) as a polynomial $\Lambda(t_1, t_2)$ in t_i. For each fixed $c > 0$, $\Lambda(t, ct)$ is a polynomial of degree $2l_1 + 2l_2 - 1$ in t. We will show the sign of the leading term depends on c. For the remainder of the proof we use an ellipsis mark $(\cdots)$ to denote "terms of lower degree in t". Regarding Q and GQ as functions of x, t, and c,

$$\begin{aligned} Q(x,t,c) &= \big(t + n_1(1-x)\big)^{l_1}\big(ct - n_2(1+x)\big)^{l_2}, \\ GQ(x,t,c) &= \big(t + n_1(1-x)\big)^{l_1 - 1}\big(ct - n_2(1+x)\big)^{l_2 - 1}(\mu - vx + xt), \end{aligned}$$

where $\mu = \kappa_2 l_2 n_1 - \kappa_1 l_1 n_2$, $v = \kappa_1 l_1 n_2 + \kappa_2 l_2 n_1$, $\chi = c\kappa_1 l_1 + \kappa_2 l_2$, c and t are positive, and $x \in [-1, 1]$. The binomial theorem gives

$$\begin{aligned} Q(x,t,c) &= c^{l_2} t^{l_1 + l_2} + \big(c^{l_2} l_1 n_1 (1-x) - c^{l_2 - 1} l_2 n_2 (1+x)\big) t^{l_1 + l_2 - 1} + \cdots, \\ GQ(x,t,c) &= c^{l_2 - 1}(c\kappa_1 l_1 + \kappa_2 l_2) t^{l_1 + l_2 - 1} + \cdots, \\ Q(1,t,c) &= c^{l_2} t^{l_1 + l_2} - 2c^{l_2 - 1} l_2 n_2 t^{l_1 + l_2 - 1} + \cdots, \\ Q(-1,t,c) &= c^{l_2} t^{l_1 + l_2} + 2c^{l_2} l_1 n_1 t^{l_1 + l_2 - 1} + \cdots. \end{aligned}$$

In equation (33), the contribution from the terms

$$\left(\int_{-1}^{1} xGQ(x)dx\right)\int_{-1}^{1} Q(x)dx - \left(\int_{-1}^{1} GQ(x)dx\right)\int_{-1}^{1} xQ(x)dx$$

is of degree $\leq 2(l_1 + l_2 - 1)$ in t, so modulo lower-order terms,

$$\begin{aligned} \Lambda\big(t, ct\big) &\equiv (Q(1) - Q(-1))\int_{-1}^{1} Q(x)dx - \big(Q(1) + Q(-1)\big)\int_{-1}^{1} xQ(x)dx \\ &= -\frac{8}{3} c^{2l_2 - 1}(cl_1 n_1 + l_2 n_2) t^{2l_1 + 2l_2 - 1} + \cdots. \end{aligned} \tag{34}$$

Since $n_1 > 0 > n_2$, there exist positive values of c and t for which $\Lambda(t_1, t_2) = 0$, $t_i > 0$. Each solution gives rise to a Kähler metric with constant scalar curvature, and no two such metrics are homothetic since $b = 1$ for all of them. This establishes Theorem 2 when $b_2(M) = 2$.

REMARK 5.3. If both Q and G are even functions of x, then the scalar curvature is constant. This is the case, for instance, if $M_1 = M_2$, $n_1 = -n_2$, and $a_1 = a_2$. Such a restriction is not necessary: If $M_1 = M_2 =$

$\boldsymbol{P}^1$, $n_1=-n_2=1$, then the scalar curvature is also constant if $a_1a_2=a_1+a_2$.

It is not difficult to see directly that $\lambda=0$ if and only if $\mathscr{F}_\Omega(H)=0$. This also follows immeditely from Proposition 5.2.

The proof of Theorem 2 when $M=M_1\times\cdots\times M_k$ follows easily: For $i=1,\cdots,k$, let ω_i be the pullback of an Einstein-Kähler form on M_i. Let p: $(L,h)\to M$ be the holomorphic line bundle with $c_1(L,h)=\Sigma_i n_i\omega_i$, and as above define Λ, which becomes a polynomial in k variables $t_1,\cdots,t_k$. Assume for convenience that $n_1>0>n_2$. Make the substitution

$$t_i=\frac{n_i}{n_1}t_1 \ \text{ if } n_i>0; \qquad t_i=\frac{n_i}{n_2}t_2 \ \text{ if } n_i<0; \qquad t_i=1 \ \text{ if } n_i=0. \tag{35}$$

This choice of t_i ensures Q has only two real roots. The above argument shows that the scalar curvature vanishes for appropriately chosern t_1 and t_2. Since Λ is a polynomial in k variables and takes both positive and negative values, the zero set is a real hypersurface (even though we have only explicitly shown existence of a one-parameter family of solutions). This completes the proof of Theorem 2. Q.E.D.

We next determine what happens when the zero and infinity sections of $\hat{L}$ are partially blown down. Let D_0 and D_∞ be their images in N, of codimension d_0 and d_∞. Under the hypotheses of Theorem 3, the polynomial Q is given by

$$Q(x)=(1+x)^{d_0-1}(1-x)^{d_\infty-1}\prod_{i=1}^{k}(t_i+|n_i|-n_ix)^{l_i},$$

where $k\geq 2$, and $n_1>0>n_2$ without loss of generality. The function G is given by

$$G(x)=\frac{d_0(d_0-1)}{1+x}+\frac{d_\infty(d_\infty-1)}{1-x}+\sum_{i=1}^{k}\frac{k_il_i}{t_i+|n_i|-n_ix}.$$

A computation analogous to that in equation (34) shows the leading coefficient of $\Lambda(t_1,\cdots,t_k)$ takes both positive and negative values, depending on the choice of t_i. Using equation (35) it suffices to consider the case

$$Q(x,t,c)=(1+x)^{d_0-1}(1-x)^{d_\infty-1}\big(t+n_1(1-x)\big)^{l_1}\big(ct-n_2(1+x)\big)^{l_2},$$

$$G(x,t,c)=\frac{d_0(d_0-1)}{1+x}+\frac{d_\infty(d_\infty-1)}{1-x}+\frac{\kappa_1 l_1}{t+n_1(1-x)}+\frac{\kappa_2 l_2}{ct-n_2(1+x)}.$$

Calculation of the leading term in the left-hand side of equation (33) is expedited by evaluating some integrals that appear repeatedly.

Lemma 5.2. *For non-negative integers m and n,*

$$I(n,m)=\int_{-1}^{1}(1+x)^n(1-x)^m dx=\frac{2^{n+m+1}n!m!}{(n+m+1)!},$$

$$J(n,m)=\int_{-1}^{1}x(1+x)^n(1-x)^m dx=(n-m)\frac{2^{n+m+1}n!m!}{(n+m+2)!}.$$

Proof. The formula for $I(n,m)$ is trivial if $m=0$ or $n=0$. The general case follows from the recursion

$$I(n,m-1)+I(n-1,\mathrm{m})=2I(n-1,m-1).$$

The second equation follows easily from the first together with

$$J(n,m)=I(n+1,m)-I(n,m). \qquad \text{Q.E.D.}$$

Using Lemma 5.2, expand the left-hand side of equation (33), keeping track of the coefficients of $t^{2l_1+2l_2-1}$. A laborious (but otherwise straightforward) calculation gives

$$\Lambda(t,ct)=-\delta c^{2l_2-1}(cn_1 l_1+n_2 l_2)t^{2l_1+2l_2-1}+\cdots, \tag{36}$$

where

$$\delta=\frac{4^{d_0+d_\infty}(d_0-1)!d_0!(d_\infty-1)!d_0!}{(d_0+d_\infty+1)!(d_0+d_\infty-1)!}.$$

Verification of equation (36) is best achieved by separating the cases where the codimension of neither end, exactly one end, or both ends is equal to one. Additionally, one should separately compute the coefficients of $t^{2l_1+2l_2}$ and $t^{2l_1+2l_2-1}$ for the terms

$$\big(Q(1)-Q(-1)\big)\int_{-1}^{1}Q(x)dx-\big(Q(1)+Q(-1)\big)\int_{-1}^{1}xQ(x)dx$$

and

$$\left(\int_{-1}^{1} xGQ(x)dx\right)\int_{-1}^{1} Q(x)dx-\left(\int_{-1}^{1} GQ(x)dx\right)\int_{-1}^{1} xQ(x)dx$$

using Lemma 5.2. Interestingly, equation (36) holds for all $d_0, d_\infty \geq 1$, though the surviving terms from equation (33) differ for the cases $d_0 = d_\infty = 1$, $d_0 > d_\infty = 1$, and $d_0, d_\infty > 1$. This completes the proof of Theorem 3. Q.E.D.

The (constant) scalar curvature of any of these metrics is positive. To see this, compute the average scalar curvature $\mathscr{H}\sigma$. The scalar curvature of g is $\hat{\sigma} = \sigma(\hat{U})$, where

$$\sigma(x) = G(x) - \frac{1}{2Q(x)}(\varphi Q)''(x). \tag{37}$$

By Lemma 2.8, see also Remark 3.1,

$$(\mathscr{H}\sigma)\int_{-b}^{b} Q(x)dx = Q(b) + Q(-b) + \int_{-b}^{b} G(x)Q(x)dx. \tag{38}$$

The right hand side is positive, as is Q, so $\mathscr{H}\sigma > 0$.

5.2. Proof of Theorem 4. We recall the basic structure theorems for Kähler manifolds with real hypersurface orbits. For details, the reader should consult the papers of Ahiezer [1], and Huckleberry and Snow [15].

Let N be a compact, almost-homogeneous Kähler manifold with disconnected exceptional set. A maximal compact group $K \subset \mathrm{Aut}^0(N)$ acts with real cohomogeneity one. There exist a K-homogeneous Kähler manifold M and a homogeneous $\boldsymbol{C}^\times$-bundle p: $L^0 \to M$ such that the open orbit in N is isomorphic to the total space of L^0. The complement of the open orbit is a disjoint union of two K-homogeneous complex submanifolds D_0 and D_∞. Their normal bundles E_0 and E_∞ are homogeneous.

The blow-up of N along $D_0 \cup D_\infty$ is the $\boldsymbol{P}^1$-bundle $\hat{L}$ associated to L_0. The K-action lifts to $\hat{L}$, and the blow-down ϖ: $\hat{L} \to N$ is K-equivariant.

We seek to characterize reductivity of $\mathrm{Aut}^0(N)$ in terms of the curvature of L^0. This is accomplished by relating reductivity with non-existence of sections of E_0 and E_∞, then using the Borel-Weil Theorem to relate curvature and existence of holomorphic sections.

Lemma 5.3. *Let N and $\hat{L}$ be as above. If* $\mathrm{Aut}^0(N)$ *is reductive, then*

$$\mathrm{Aut}^0(N) \simeq \mathrm{Aut}^0(\hat{L}).$$

In particular, the submanifolds D_0 and D_∞ are preserved by $\mathrm{Aut}^0(N)$.

Proof. There is a natural injective group homomorphism

$$\mathrm{Aut}^0(\hat{L}) \hookrightarrow \mathrm{Aut}^0(N) \tag{39}$$

whose image consists of automorphisms of N preserving the submanifolds D_0 and D_∞. If $\mathrm{Aut}^0(N)$ is reductive, then its Lie algebra is the complexification of the Lie algebra of the compact group K. But K preseves D_0 and D_∞, so every element of $\mathrm{Aut}^0(N)$ must also. Thus the map in equation (39) is surjective. Q.E.D.

Lemma 5.4. *Let p: $E \to M$ be a holomorphic vector bundle over a compact Kähler manifold. If $H^0(M,E) \neq 0$, then the total space of E admits a one parameter group of automorphisms which does not preserve the zero section.*

Proof. If ϑ is a section of E, one may define a one-parameter group of automorphisms of the total space of E by "adding ϑ in the fibre direction." More precisely, let $\vartheta \in H^0(M,E)$ be a non-zero section. Then $p^*\vartheta$ is a non-zero section of p^*E. Regarding p^*E as the bundle of vertical tangent vectors of E, we have $p^*E \subset TE$. Then $\Theta_t = \exp(tp^*\vartheta)$ is a one-parameter group of fibre-preserving automorphisms which does not preserve the zero section of E. Q.E.D.

Lemma 5.5. *If* $\mathrm{Aut}^0(N)$ *is reductive, then*

$$H^0(D_0,E_0) = H^0(D_\infty,E_\infty) = 0.$$

Proof. Suppose $H^0(D_0,E_0) \neq 0$. Then there is a one-parameter group Θ_t of automorphisms of E_0 which do not preserve D_0 by Lemma 5.4, and Θ_t induces a one-parameter group of automorphisms on the compact manifold $\boldsymbol{P}(E_0 \oplus 1)$, which is nothing but the blow-up of N along D_∞. Θ_t also preserves the exceptional divisor of the blow-up, so it descends to a one-parameter group of automorphisms of N. This implies $\mathrm{Aut}^0(N)$ is not reductive by Lemma 5.3. A similar statement holds for E_∞. Q.E.D.

Lemma 5.6. *Let $(E,h) \to D$ be a Hermitian holomorphic vector bundle* *of rank r over a compact Kähler manifold, π: $\boldsymbol{P}(E) \to D$ the projection, and let $(L,h') \subset (\pi^*E,\pi^*h)$ be the tautological bundle equipped with the induced Hermitian metric. Assume the eigenvalues of the first Chern forms $c_1(E,h)$ and $c_1(L,h')$ are constant, and that $c_1(L,h')$ has no more than $r-1$ negative eigenvalues. Then $c_1(E,h) \geq 0$.*

Proof. Assume to the contrary, that there exists a tangent vector $v_p \in T_pD$ with $c_1(E,h)(v_p,\bar{v}_p)<0$. We will show that $c_1(L,h')$ has a non-vertical eigenvector with negative eigenvalue.

Let z^α be local coordinates on D near p, let e_i be a local unitary frame of E consisting of eigenvectors of the curvature operator of (E,h), and let e^i by the dual coframe. The curvature of (E,h) is given by

$$R=\sum_i\left(\sum_{\alpha,\bar{\beta}}R_{i\bar{i}\alpha\bar{\beta}}dz^\alpha \wedge dz^{\bar{\beta}}\right)e^i\otimes e_i \in A^{1,1}(\operatorname{End} E),$$

and the first Chern form of (E,h) is given by

$$c_1(E,h)=\frac{\sqrt{-1}}{2\pi}\operatorname{tr} R=\frac{\sqrt{-1}}{2\pi}\sum_{i,\alpha,\bar{\beta}}R_{i\bar{i}\alpha\bar{\beta}}dz^\alpha \wedge dz^{\bar{\beta}}.$$

Without loss of generality, we may assume

$$\sqrt{-1}\left(\sum_{\alpha,\bar{\beta}}R_{1\bar{1}\alpha\bar{\beta}}dz^\alpha \wedge dz^{\bar{\beta}}\right)(v_p,\bar{v}_p)<0.$$

For convenience, we let z^α denote horizontal coordinates on $\boldsymbol{P}(E)$. Let R^* denote the curvature of (π^*E,π^*h) and let R' denote the curvature of (L,h'). Let $q=[e_1(p)]\in \boldsymbol{P}(E_p)$, and let $w_q \in T_q\boldsymbol{P}(E)$ be a lit of v_p. The sections π^*e_i are a local unitary frame for π^*E near q, and the value of the section π^*e_1 at q lies in L_q. Computing the curvature of (L,h) at q, using the principle that "curvature decreses in subbundles," see for example [17] for a precise statement,

$$\begin{aligned}c_1(L,h')(w_q,\bar{w}_q)&=\frac{\sqrt{-1}}{2\pi}\left(\sum_{\alpha,\bar{\beta}}R'_{1\bar{1}\alpha\bar{\beta}}dz^\alpha \wedge dz^{\bar{\beta}}\right)(w_q,\bar{w}_q)\\&\leq\frac{\sqrt{-1}}{2\pi}\left(\sum_{\alpha,\bar{\beta}}R^*_{1\bar{1}\alpha\bar{\beta}}dz^\alpha \wedge dz^{\bar{\beta}}\right)(w_q,\bar{w}_q)\end{aligned}$$

$$=\frac{\sqrt{-1}}{2\pi}\left(\sum_{\alpha,\bar{\beta}} R_{1\bar{1}\alpha\bar{\beta}}dz^{\alpha}\wedge dz^{\bar{\beta}}\right)(v_p,\bar{v}_p)<0.$$

Thus $c_1(L,h)$ has more than $r-1$ negative eigenvalues. Q.E.D.

Proposition 5.3. *Let N be an almost-homogeneous Kähler manifold with two ends D_0 and D_∞. Let E_0 and E_∞ be the normal bundles of D_0 and D_∞ in N, of rank d_0 and d_∞. Blowing up N along $D_0 \cup D_\infty$, we obtain an almost-homogeneous $\boldsymbol{P}^1$-bundle $\hat{L}$ over $\boldsymbol{P}(E_0)\simeq \boldsymbol{P}(E_\infty)\simeq M$, and $L\to \boldsymbol{P}(E_0)$ is the blow-up of E_0 along the zero section, i.e. the tautological bundle of E_0. Equip L with a K-invariant Hermitian metric h. If* $\mathrm{Aut}^0(N)$ *is reductive, then $c_1(L,h)$ has at least d_0 negative eigenvalues and at least d_∞ positive eigenvalues.*

Proof. Suppose $c_1(L,h)$ has exactly d_0-1 negative eigenvalues. The corresponding bundle of eigenspaces is exactly the vertical tangent bundle of $\boldsymbol{P}(E_0)$, that is, the kernel of $(\pi_0)_*$: $T\boldsymbol{P}(E_0)\to TD_0$. By Lemma 5.6, $c_1(E_0)\geq 0$. By the Borel-Weil Theorem, see for example [4], Theorem 24.7, $H^0(D_0,E_0)\neq 0$, that is, E_0 admits a non-zero holomorphic section. Lemma 5.5 now implies $\mathrm{Aut}^0(N)$ is not reductive. Similarly, $c_1(L,h)=-c_1(L^{-1},h^{-1})$ has at least d_∞ positive eigenvalues. Q.E.D.

Theorem 4 follows immediately from Theorem 3 and Proposition 5.3.

REMARK 5.4. A general version of Theorem 4 does not hold. There are examples of complex surfaces with discrete automorphism group, yet admitting no Kähler metric with constant scalar curvature; see [5,22]. Our feeling is that there must be a simple auxilliary condition, such as rigidity of the complex structure or positivity of the first Chern class, under which a version of Theorem 4 holds. The present understanding of existence and obstructedness of extremal Kähler metrics is incomplete. As in the Einstein-Kähler case, it seems likely that the simplest obstructions are related to holomorphic vector fields. In the absence of holomorphic vector fields, there are probably obstructions depending in a very subtle way on the complex structure of the underlying manifold. Fujiki and Schumacher have investigated questions of this sort as well as moduli questions for extremal Kähler metrics, see [9,10] and the references contained therein.

6. Appendix A

We collect here the details of the root-counting argument in the proof of Proposition 3.1. We also make some remarks regarding properties of the polynomial Q in relation to the geometry of $\hat{L}$.

6.1. Remarks on the Function Q. A special-type metric is constructed in part from a path in the space of Riemannian metrics on the base space M, namely

$$g_t = g_M - U(t)B \tag{40}$$

in the notation of Section 2. For each $t \in [0,R]$, one may regard $g_M^{-1} g_t$ as a smooth section of the bundle $\mathrm{End}(TM)$ so that

$$q(t) = \det(g_M^{-1} g_t)$$

is a function on M. Under the hypotheses of Section 3, this function is *constant* for each t. Expressed with respect to the function $\hat{U}$, we have $q(t) = Q(\hat{U})$. Since g_t is positive-definite for all $t \in (0,R)$, Q is a positive function on the open interval $(-b,b)$. The roots of Q are at $p_i = a_i/n_i$ when $n_i \neq 0$.

Denote the submanifold $\{s=0\} \subset \hat{L}$ by D_0, and let d_0 be its complex codimension. Define D_∞ and d_∞ similary. Since $\hat{L}$ is obtained by suitably identifying the normal bundles of D_0 and D_∞, g_t converges to a Hermitian form of rank $m - d_0 + 1$ on M as $t \to 0$, and to a form of rank $m - d_\infty + 1$ as $t \to R$, where m is the complex dimension of M, cf. [14]. Thus

$$Q(w) = \left(1 + \frac{w}{b}\right)^{d_0 - 1}\left(1 - \frac{w}{b}\right)^{d_\infty - 1} P(w),$$

where $P > 0$ on $[-b,b]$. In particular, Q vanishes at an endpoint of the interval $[-b,b]$ precisely when the corresponding component of $\hat{L}\backslash L^0$ has codimension (in $\hat{L}$) greater than one. This puts restrictions on the choice of constants a_i, which simply reflects the fact that a special-type metric

$$dt^2 + (dt \circ J)^2 + \sum_{i=1}^{k} \left(a_i - U(t)n_i\right) p^* \omega_i$$

does not always extend to $\hat{L}$ if the submanifolds D_0 and D_∞ are not divisors.

6.2. Properties of the Function G. Let r_M be the Ricci tensor of g_M, and regard $g_t^{-1}r_M$ as a section of End(TM) for each t. Under the hypotheses of Section 3, the eigenvalues of this bundle endomorphism are non-negative constants on M. It makes sense to take the trace of this bundle endomorphism. If we define the function G: $(-b,b) \to \boldsymbol{R}$ by

$$G(w) = \sum_{i=1}^{k} \frac{\kappa_i l_i}{a_i - n_i w},$$

then $\operatorname{tr}(g_t^{-1}r_M) = G(U(t))$, where t and U are as in Section 2. It is convenient to regard G as defined everywhere except at the roots of Q, and to regard the polynomial GQ as being defined everywhere. Let the roots of Q be indexed so that

$$p_1 \leq p_2 \leq \cdots \leq p_j \leq -b < 0 < b \leq p_{j+1} \leq \cdots \leq p_k.$$

G has a simple pole at $p_i = a_i/n_i$ provided $\kappa_i \neq 0$. We assume this is the case unless specified otherwise.

Lemma 6.1. *Assume all the κ_i have the same sign. For $i=1, \cdots, k-1$, $i \neq j$, the function G maps the interval (p_i, p_{i+1}) onto the real numbers. Moreover, G does not map (p_j, p_{j+1}) onto the real numbers.*

Proof. Assume $\kappa_i > 0$ for definiteness. G is smooth on each interval (p_i, p_{i+1}). Note that p_i and n_i have the same sign since $a_i > 0$, and that n_i and n_{i+1} have the same sign unless $i=j$. Near $w = p_i$, G has a Laurent series expansion

$$G(w) = \frac{c_i}{p_i - w} + \text{ real analytic},$$

where the constants $c_i = \kappa_i l_i / n_i$ are negative for $i = 1, \cdots, j$ and positive for $i = j+1, \cdots, k$. The lemma follows by taking $w \searrow p_i$ and $w \nearrow p_{i+1}$. Q.E.D.

The same conclusion holds for the function $G - f$, where f is an arbitrary continuous function, since f is bounded on each interval(p_i, p_{i+1}). This has a useful consequence.

Lemma 6.2. *Suppose f is a linear function. Then $(G-f)Q$ vanishes at most twice on the open interval (p_j, p_{j+1}).*

Proof. Fix arbitrary constants σ_0 and λ, and let $f(w)=\sigma_0+\lambda w$. If $\kappa_i\neq 0$, $(G-f)Q$ has a root of order l_i-1 at p_i, and has at least one root on (p_i,p_{i+1}), $i\neq j$ by Lemma 6.1. Thus on the complement of $(p_j,p_{j+1})\supseteq(-b,b)$, $(G-f)Q$ has at least

$$\sum_{i=1}^{k}(l_i-1)+(k-2)=\deg\big((G-f)Q\big)-3$$

roots. A similar counting holds if $\kappa_i=0$ for some i, for then $G-f$ is smooth at p_i, while Q has a root of order l_i. By Lemma 6.1, $G-f$ vanishes an even number of times on (p_j,p_{j+1}), counting multiplicity. Since $G-f$ vanishes at most three times on (p_j,p_{j+1}) by the above root count, it can vanish at most twice. Q.E.D.

6.3. Positivity of φ. It suffices to prove that the function φQ is positive on the open interval $(-b,b)$ since $Q>0$ there. Since φQ is a polynomial, this may be done via analysis of the roots of the second derivative. Differentiating equation (25) twice,

$$(\varphi Q)''(w)=2\left(-\sigma_0-\lambda w+\sum_i\frac{\kappa_i l_i}{a_i-n_i w}\right)Q(w).$$

By Lemma 6.2, this vanishes at most twice on $(p_j,p_{j+1})\supseteq(-b,b)$, i.e. φQ has at most two inflection points on the interval $(-b,b)$. This is the first explicit use of the curvature hypotheses on the base space, but requires only that the curvatures all have the same sign (or be zero). As in the proof of Lemma 6.1,

$$\lim_{w\searrow p_j}G(w)-(\sigma_0+\lambda w)=\lim_{w\nearrow p_j+1}G(w)-(\sigma_0+\lambda w)=+\infty,$$

so $(\varphi Q)''>0$ near the ends of (p_j,p_{j+1}). This is the second use of the curvature hypothesis on (M,g_M), and uses non-negativity in an essential way; if $\kappa_i<0$, then the above limits are $-\infty$.

The constants σ_0 and λ are chosen so that $(\varphi Q)(b)=0$. Since φQ is positive near the ends of $(-b,b)$, convex near the ends of $(p_j,p_{j+1})\supseteq(-b,b)$, and has at most two inflection points on (p_j,p_{j+1}), $(\varphi Q)(w)>0$ for $w\in(-b,b)$. This completes the proof of Proposition 3.1.

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