

SPLITTING A 4-MANIFOLD WITH INFINITE CYCLIC FUNDAMENTAL GROUP

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1. Introduction.

Throughout the paper, n -dimensional connected orientable topological manifolds will be understood as n -manifolds, unless otherwise stated. In this paper we show the following splitting theorem for closed 4-manifolds with infinite cyclic fundamental groups:

Theorem 1.1. *Every closed connected orientable 4-manifold M with $\pi_1(M) \cong Z$ is homeomorphic to the connected sum $S^1 \times S^3 \# M_1$ for a closed simply connected 4-manifold M_1 .*

The idea of the proof is to investigate when a homology cobordism between closed 4-manifolds with infinite cyclic fundamental groups is an h -cobordism which is always a product cobordism by Freedman [2], for we can construct a homology cobordism between M and $S^1 \times S^3 \# M_1$ by a method similar to Kervaire's surgery argument [11]. Freedman showed in [1] that any two closed oriented simply connected 4-manifolds M_1, M'_1 are orientation-preservingly homeomorphic if and only if the intersection forms on $H_2(M_1; Z), H_2(M'_1; Z)$ are isomorphic and the Kirby-Siebenmann invariants $ks(M_1), ks(M'_1) (\in Z_2)$ are equal, and in this case there is an orientation-preserving homeomorphism $M_1 \cong M'_1$ inducing the isomorphism of the intersection forms. By combining this classification of Freedman with the above splitting theorem, we have a similar characterization for closed oriented 4-manifolds with infinite cyclic fundamental groups:

Corollary 1.2. *Two closed connected oriented 4-manifolds M, M' with $\pi_1(M) \cong \pi_1(M') \cong Z$ are orientation-preservingly homeomorphic if and only if the intersection forms on $H_2(M; Z), H_2(M'; Z)$ are isomorphic and $ksM = ksM'$, and in this case there is an orientation-preserving homeomorphism $M \cong M'$ inducing the isomorphism of the intersection forms.*

As a next observation, let \tilde{M} be the infinite cyclic (= universal) covering space of a 4-manifold M with $\pi_1(M) \cong Z$. Let $\Lambda = Z[Z] = Z[t, t^{-1}]$ be the group ring of Z . In [3], Freedman-Quinn showed that every nonsingular Hermitian Λ -form on a free Λ -module of finite rank is realizable as a Λ -intersection form on $H_2(\tilde{M}; Z)$ (which is a free Λ -module) of a closed 4-manifold M with $\pi_1(M) \cong Z$. Applying the splitting theorem to this M , we obtain the following result in algebra:

Corollary 1.3. *Every non-singular Hermitian Λ -form on a free Λ -module of finite rank is Λ -isomorphic to a trivial Λ -extension of a non-singular symmetric bilinear form on a free abelian group of finite rank.*

It is known that a (null-homotopic) locally flat 2-knot K in a simply connected 4-manifold M_1 has a tubular neighborhood $N(K) \cong S^2 \times D^2$ (cf. [3, 9.3]), and it is trivial (i.e., bounds a bicollared 3-disk in M_1) if and only if the exterior $E = M_1 - \text{int}N(K)$ is homeomorphic to a connected sum $S^1 \times D^3 \# M_1$ (cf. Gluck [5, (17.1)]). The following unknotting criterion for a 2-knot is proved by Freedman [2] when $M_1 = S^4$, and a weaker but more general result is given by Matumoto [12]:

Corollary 1.4. *A locally flat 2-knot K in a closed simply connected 4-manifold M_1 is trivial if and only if $\pi_1(M_1 - K) \cong Z$.*

This follows by applying the splitting theorem to a closed 4-manifold M with $\pi_1(M) \cong Z$, obtained from M_1 by a surgery replacing $N(K)$ with $D^3 \times S^1$, because any two simple loops representing a generator of $\pi_1(M)$ and their tubular neighborhoods are ambient isotopic in M , respectively. Proof of the splitting theorem (Theorem 1.1) is given in Section 2. In Section 3, we note a smooth structure on $M = S^1 \times S^3 \# M_1$ for a simply connected 4-manifold M_1 . Splitting a closed 4-manifold with infinite cyclic fundamental group is a prelude to the topological classification problem for bounded 4-manifolds with infinite cyclic fundamental groups. The typical problem is the unknotting problem for a closed connected orientable surface in S^4 , which is solved in a joint work with Hillman [6].

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2. Proof of the splitting theorem. For a compact n -manifold W

with $H_1(W; Z) \cong Z$, we have $H_{n-1}(W, \partial W; Z) \cong Z$ by Poincaré duality. A bicollared proper connected $(n-1)$ -submanifold U of W is simply called a leaf in W if U represents a generator of $H_{n-1}(W, \partial W; Z)$. Note that U lifts trivially to the infinite cyclic cover \tilde{W} of W . We use the following sufficient condition for a homology cobordism between closed 4-manifolds with infinite cyclic fundamental groups to be an h-cobordism:

Lemma 2.1. *Let W be a homology cobordism with $\pi_1(W) \cong Z$ between closed orientable 4-manifolds M and M' with $\pi_1(M) \cong \pi_1(M') \cong Z$. If there is a leaf U in W with $V = U \cap M$ a leaf in M such that a lift $\tilde{I}: (U, V) \rightarrow (\tilde{W}, \tilde{M})$ of the inclusion $I: (U, V) \subset (W, M)$ to the infinite cyclic covering (\tilde{W}, \tilde{M}) of (W, M) induces the trivial homomorphism $\tilde{I}_* = 0: H_2(U, V; Q) \rightarrow H_2(\tilde{W}, \tilde{M}; Q)$, then W is an h-cobordism and hence a product cobordism.*

Proof. We use the dualities on the infinite cyclic covering, stated in [9], where we denote $\text{Ext}^q(\cdot, \Lambda)$ by E^q . Since $H_*(W, M; Z) = H_*(W, M'; Z) = 0$, we see from the Wang exact sequence that $\tilde{H}_* = H_*(\tilde{W}, \tilde{M}; Z)$ and $\tilde{H}'_* = H_*(\tilde{W}, \tilde{M}'; Z)$ are finitely generated torsion Λ -modules whose integral torsion parts are finite, for $t-1$ induces an automorphism on \tilde{H}_* and \tilde{H}'_* (cf. [9, §3]). Then $\tilde{H}_* \otimes Q$ and $\tilde{H}'_* \otimes Q$ are finitely generated over Q and we see from [8], [10] that $\tilde{I}_*: H_2(U, V; Q) \rightarrow H_2(\tilde{W}, \tilde{M}; Q) = \tilde{H}_2 \otimes Q$ is onto. Since $\tilde{I}_* = 0$, we have $\tilde{H}_2 \otimes Q = 0$. Then $E^1 E^1 \tilde{H}_2 = 0$, for the integral torsion part of \tilde{H}_2 is finite (cf. [9, §3]). By the first duality of [9], we have a non-singular Λ -sesquilinear pairing $E^1 E^1 \tilde{H}_p \times E^1 E^1 \tilde{H}'_r \rightarrow Q(\Lambda)/\Lambda$ for all p, r with $p+r=4$. Noting that $\tilde{H}_p = \tilde{H}'_p = 0$ for $p \leq 1$ and $E^1 E^1 \tilde{H}_2 = 0$, we see that $E^1 E^1 \tilde{H}_* = E^1 E^1 \tilde{H}'_* = 0$. Next, since \tilde{H}_* and \tilde{H}'_* are torsion Λ -modules, the second duality of [9] implies that there is a non-singular t -isometric pairing $E^2 E^2 \tilde{H}_p \times E^2 E^2 \tilde{H}'_s \rightarrow Q/Z$ for all p, s with $p+s=3$. Nothing that $\tilde{H}_p = \tilde{H}'_p = 0$ for $p \leq 1$, we see that $E^2 E^2 \tilde{H}_* = E^2 E^2 \tilde{H}'_* = 0$. By [9, 3.7], we have $\tilde{H}_* = \tilde{H}'_* = 0$. Hence $(W; M, M')$ is an h-cobordism which is a product cobordism by [2]. This completes the proof.

The following lemma is more or less known in principle, but for convenience we give a proof:

Lemma 2.2. *Let f be a bicollared embedding of $S^1 \times D^2$ into a (possibly non-compact) orientable 4-manifold M such that $f(S^1 \times 0)$ is null-homotopic in M . When M is spin, we assume that $f(S^1 \times p)$ for a point $p \in \partial D^2 = S^1$ is a longitude of the solid torus $f(S^1 \times D^2)$ induced from a spin structure on M . When M is not spin, we assume that there is a spherical element in $H_2(M; Z)$ with odd self-intersection number. Then f extends to a bicollared embedding $f^+: D^2 \times D^2 \rightarrow M$.*

Proof. If necessary, by replacing M with a punctured submanifold of M , we can assume that M is a smooth 4-manifold (cf. [3]). Since $f(S^1 \times 0)$ is null-homotopic in M , we have an immersed disk D in M such that $D \cap f(S^1 \times D^2) = f(S^1 \times 0) = \partial D$. Let D_p be an immersed disk in M with $D_p \cap f(S^1 \times D^2) = f(S^1 \times p) = \partial D_p$, obtained from the singular disk $D \cup f(S^1 \times [0, p])$ by pushing the part $f(S^1 \times [0, p])$ in the positive normal direction of $f(S^1 \times D^2)$ in M , where $[0, p]$ denotes the interval in D^2 with endpoints $0, p$. The intersection number of D and D_p in M is denoted by $I_f(D)$ since it depends only on D and f . We first show that we can take $I_f(D)$ to be even. When M is spin, $I_f(D)$ is always even by our assumption on the solid torus $f(S^1 \times D^2)$. When M is not spin and $I_f(D)$ is odd, we take a connected sum in M of D and an immersed 2-sphere with odd self-intersection number. The resulting immersed disk, written again as D , has that $I_f(D)$ is even. Note that the singularity of D consists of double points, say p_i , $i=1, 2, \dots, s$, each of which is locally represented as the vertex of a cone over a Hopf link. We choose mutually disjoint simple arcs a_i , $i=1, 2, \dots, s$, in D such that a_i joins p_i and a point in ∂D and then take a small disk neighborhood Δ_i of a_i in D . By sliding the arc $\Delta_i \cap \partial D$ to the arc $\partial \Delta_i - \text{int}(\Delta_i \cap \partial D)$ along Δ_i for each i , the embedding f is ambient isotopic to an embedding $f': S^1 \times D^2 \rightarrow M$ such that there exists a smoothly embedded disk D' in M with $f'(S^1 \times D^2) \cap D' = f'(S^1 \times 0) = \partial D'$. Hence there is a smoothly embedded disk in M , written again as D , such that $f(S^1 \times D^2) \cap D = f(S^1 \times 0) = \partial D$. Since the slide of $f(S^1 \times 0)$ along each Δ_i contributes $I_f(D)$ by ± 2 , we see that $I_f(D)$ is even for the embedded disk D . For each integer m , let $f_m: S^1 \times D^2 \rightarrow M$ be a smooth embedding with $f_m(S^1 \times D^2) = f(S^1 \times D^2)$ such that $f_m(S^1 \times p)$ is homologous to $f(S^1 \times p) + mf(p \times S^1)$ in $H_1(f(S^1 \times S^1); \mathbb{Z})$. Then we have $I_{f_m}(D) = 0$ for some even integer m . Since there is an embedded $S^1 \times S^2$ in M such that $f(S^1 \times D^2) \subset S^1 \times S^2$ and $f(S^1 \times 0) = S^1 \times *$ for a point $*$ in S^2 , we see from Gluck's observation [5] that f_m is ambient isotopic to f in $S^1 \times S^2$ and hence in M for any even integer m . Thus, we have a smoothly embedded disk D with $D \cap f(S^1 \times D^2) = f(S^1 \times 0) = \partial D$ and $I_f(D) = 0$. This means that f extends to a bicollared embedding $f^+: D^2 \times D^2 \rightarrow M$. This completes the proof.

We are in a position to prove Theorem 1.1 (the splitting theorem).

Proof of Theorem 1.1: Let $S^1 \times D^3$ be a neighborhood of a loop representing a generator of $\pi_1(M)$ in M . Let M_1 be the manifold obtained from M by the surgery replacing $S^1 \times D^3$ with $D^2 \times S^2$. Let $K = 0 \times S^2 \subset M_1$ be a 2-knot and $X = M_1 - K$. Then M_1 is simply connected and $\pi_1(X) \cong \mathbb{Z}$. Let V be a Seifert hypersurface for K in M_1 . We obtain

a 3-disk D^3 from V by surgery on 2-handles $h_s^2, s=1,2, \dots, n$, disjointedly attached to $\text{int}V$, where when M_1 is spin, this surgery should be taken the spin surgery on a spin structure on V induced from a spin structure on M_1 (cf. [7]). Let A_s be the attaching solid torus of h_s^2 to $\text{int}V$. Note that each core of A_s is null-homotopic in X since it is null-homologous in X and $\pi_1(X)$ is abelian. Since there is a natural isomorphism $H_2(X;Z) \cong H_2(M_1;Z)$, we have that X is spin if and only if M_1 is spin. When M_1 is not spin, there is an element in $H_2(X;Z)$ with odd self-intersection number by the universal coefficient theorem, which is spherical by the Hopf theorem, for $\pi_1(X) \cong Z$. Applying Lemma 2.2 inductively, we can embed the 2-handles $h_s^2, s=1,2, \dots, n$, into X disjointedly by embeddings extending the inclusions $A_s \subset \text{int}V \subset X, s=1,2, \dots, n$. By an isotopic deformation of the h_s^2 's relative to the A_s 's, we can assume that $T_s = V \cap h_s^2 - A_s$ is a union of solid tori in $\text{int}V$. Let D_0^3 be a 3-disk in $\text{int}V - \bigcup_{s=1}^n (A_s \cup T_s)$ and $K_0 = \partial D_0^3 \subset M_1$. For any numbers $t_i, i=1,2$, with $0 < t_1 < t_2 < 1$, the 4-manifold

$$U = V \times [0, t_1] \cup \left(\bigcup_{s=1}^n A_s \times [t_1, t_2] \right) \cup \left(\bigcup_{s=1}^n h_s^2 \times t_2 \right) \cup D_0^3 \times [t_1, 1]$$

is a surgery trace from $V = V \times 0$ to $D^3 = \partial U - \text{int}V$, embedded in $M_1 \times [0, 1]$ with $M_1 \times 0 \cap U = V \times 0$ and $M_1 \times 1 \cap U = D_0^3 \times 1$. Then $C = D^3 - \text{int}D_0^3 \times 1 \cong S^2 \times [0, 1]$ gives a knot cobordism in $M_1 \times [0, 1]$ between the 2-knot $K \times 0 \subset M_1 \times 0$ and the trivial 2-knot $K_0 \times 1 \subset M_1 \times 1$. Let $Y = M_1 \times [0, 1] - C$. To see that $\pi_1(Y) \cong Z$, it suffices to show that $\pi_1(M_1 \times [0, 1] - U) = \{1\}$. Let l be a simple loop in $M_1 \times (0, 1) - U$, which bounds a disk D in $M_1 \times (0, 1)$ meeting U transversely. $L = D \cap U$ is a proper 1-submanifold of U with $\partial L \subset \text{int}C$. Let l_L be a loop in L . Since $U - (L - l_L)$ is simply connected, l_L bounds a disk D_U in $\text{int}U - (L - l_L)$. When we consider a disk D' obtained from D by replacing the disk in D bounded by l_L with D_U and then by pushing D_U into $M_1 \times [0, 1] - U$, we see that the number of loops in $D' \cap U$ is smaller than the number of loops in L . By induction on the number of loops in L , we can find a disk D with $\partial D = l$ such that L has only arc components. Let a be an arc component of L . Let a_s be a simple arc in $\text{int}C$ with $\partial a_s = \partial a$ which does not meet $\partial(L - a)$. Since the loop $a \cup a_s$ bounds a disk D_a in U not meeting $L - a$, we can eliminate a from L by pushing out a bi-collar neighborhood of a in $\text{int}D$ along D_a . By induction on the number of arc components of L , we can find a disk D with $\partial D = l$ and $L = \emptyset$, meaning that $\pi_1(M_1 \times [0, 1] - U) = \{1\}$. Hence $\pi_1(Y) \cong Z$. Then we have a homology cobordism W with $\pi_1(W) \cong Z$ between M and $M_0 = S^1 \times S^3 \# M_1$ by a longitude surgery on $C \subset M_1 \times [0, 1]$. Note that $\pi_1(M) \cong \pi_1(M_0) \cong Z$. W has a leaf \tilde{U} homeomorphic

to a union of U and a $(3\text{-disk}) \times [0,1]$ so that $\hat{U} \cap M$ is a closed leaf \hat{V} in M closing V with a 3-disk and $\hat{U} \cap M_0$ is a factor $1 \times S^3$ of the connected summand $S^1 \times S^3$ of M_0 . We show that a lift $\tilde{I}: (\hat{U}, \hat{V}) \rightarrow (\tilde{W}, \tilde{M})$ of the inclusion $I: (\hat{U}, \hat{V}) \subset (W, M)$ induces a trivial homomorphism $\tilde{I}_* = 0: H_2(\hat{U}, \hat{V}; Q) \rightarrow H_2(\tilde{W}, \tilde{M}; Q)$. Let d_s be a core disk of the 2-handle h_s^2 so that ∂d_s is a core circle of A_s and $k_s = \text{int} d_s \cap V$ is a union of core circles of T_s . Let $c: V \times [-1,1] \rightarrow M_1$ be a bi-collar of V in M_1 with $c(x,0) = x$ for all $x \in V$. We may consider that $c(V \times [-1,1]) \cap d_s = c(\partial d_s \times [0,1]) \cup c(k_s \times [-1,1])$. Let $e_s = \text{cl}((d_s - c(k_s \times [-1,1])))$. Let $D_s = (\partial d_s) \times [0, t_2] \cup d_s \times t_2$ be a proper disk in U . Note that $H_2(\hat{U}, \hat{V}; Q)$ is generated by the disks $D_s, s=1,2,\dots,n$. Let $E_s = (\partial d_s) \times [0, t_1] \cup e_s \times t_1$. For a sufficiently small $\varepsilon > 0$, we consider the following two disks in $M_1 \times [0,1]$:

$$D_s^{+\varepsilon} = E_s \cup c(k_s \times \{-1,1\}) \times [t_1, t_1 + \varepsilon] \cup c(k_s \times [-1,1]) \times (t_1 + \varepsilon)$$

and

$$D_s^{-\varepsilon} = E_s \cup c(k_s \times \{-1,1\}) \times [t_1 - \varepsilon, t_1] \cup c(k_s \times [-1,1]) \times (t_1 - \varepsilon).$$

Clearly, $(D_s^{\pm\varepsilon}, \partial D_s^{\pm\varepsilon}) \subset (Y, X)$. Let $N(C) \cong C \times D^2$ be a tubular neighborhood of C in $M_1 \times [0,1]$ and $P = N(C) - C$. By the uniqueness of a tubular neighborhood, we may consider that $\text{cl}(D_s^{\pm\varepsilon} - E_s)$ is contained in P . Let \tilde{Y} be the infinite cyclic covering space of Y with \tilde{X}, \tilde{P} the lifts of X, P , respectively. Let $\tilde{D}_s^{+\varepsilon}$ and $\tilde{D}_s^{-\varepsilon}$ be lifting disks of $D_s^{+\varepsilon}$ and $D_s^{-\varepsilon}$ to \tilde{Y} , respectively, such that the lifting parts of E_s coincide. Then $[\tilde{D}_s^{+\varepsilon}] = [\tilde{D}_s^{-\varepsilon}]$ in $H_2(\tilde{Y}, \tilde{X} \cup \tilde{P}; Q)$. Since $H_2(\tilde{X} \cup \tilde{P}, \tilde{X}; Q) = 0$, the natural homomorphism $H_2(\tilde{Y}, \tilde{X}; Q) \rightarrow H_2(\tilde{Y}, \tilde{X} \cup \tilde{P}; Q)$ is injective. Hence $[\tilde{D}_s^{+\varepsilon}] = [\tilde{D}_s^{-\varepsilon}]$ in $H_2(\tilde{Y}, \tilde{X}; Q)$. By moves in $M_1 \times [0,1]$ in the direction of the second factor, D_s is ∂ -relatively isotopic to $D_s^{+\varepsilon}$ in Y and $D_s^{-\varepsilon}$ is ∂ -relatively isotopic to $d_s \times 0 \subset X$ in Y . This means that a lifting disk \tilde{D}_s of D_s to \tilde{Y} has $[\tilde{D}_s] = 0$ in $H_2(\tilde{Y}, \tilde{X}; Q)$. This means that $I_*: H_2(\hat{U}, \hat{V}; Q) \rightarrow H_2(\tilde{W}, \tilde{M}; Q)$ is trivial. By Lemma 2.1, W is a product cobordism and there is a homeomorphism $M \cong M_0$. This completes the proof.

3. Note on smooth structure. The following proposition is worthy of note, though it is not hard to prove:

Proposition. *Let $S^1 \times S^3$ have the standard smooth structure, and M_1, M'_1 be simply connected (possibly non-compact) smooth 4-manifolds. If $M = S^1 \times S^3 \# M_1$ and $M' = S^1 \times S^3 \# M'_1$ are diffeomorphic, then M_1 and M'_1 are diffeomorphic.*

Proof. The submanifold $S^1 \times D^3$ of $S^1 \times S^3$ for a 3-ball $D^3 \subset S^3$ is denoted by N or N' when we regard $S^1 \times S^3$ as the direct summand of M or M' , respectively. Then any diffeomorphism $f: M \cong M'$ is smoothly isotopic to a diffeomorphism $f': M \cong M'$ sending N onto N' by the uniqueness of tubular neighborhoods, for $f(S^1 \times O)$ and $S^1 \times O$ are smoothly isotopic in M . Note that any 2-handle surgery of $S^1 \times S^3$ along $S^1 \times p (p \in S^3)$ produces S^4 (cf. [5, (17.1)]), so that any surgery on M or M' replacing N or N' with $D^2 \times S^2$ produces M_1 or M'_1 , respectively. Hence there is a diffeomorphism $M_1 \cong M'_1$. This completes the proof.

By this proposition, we see that when M_1 has infinitely many smooth structures (For example, we can take $M_1 = \mathbb{R}^4$ or $CP^2 \# 9\bar{C}P^2$ by Gompf [4] or Okonek-Van de Ven [13]), M has also infinitely many smooth structures.

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