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CHARACTERIZATION OF CONDITIONAL EXPECTATIONS FOR *M*-SPACE-VALUED FUNCTIONS

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Introduction Let $(\Omega, \mathcal{A}, \mu)$ be a probability space, E a Banach space. We consider constant-preserving contractive projections of $L_1(\Omega, \mathcal{A}, \mu, E)$ into itself. If $E=R$ or E is a strictly-convex Banach space, then it is known (Ando [2], Douglas [3] and Landers and Rogge [6]) that such operators coincide precisely with the conditional expectation operators. If $E=L_1(X, S, \lambda, R)$, where (X, S, λ) is a localizable measure space, then the author [8] proved that such operators which are translation invariant coincide with the conditional expectation operators. If $E=L_\infty(X, S, \lambda, R)$, where (X, S, λ) is a measure space, and the dimension of E is bigger than 2, then author [9] proved that such operators coincide with the conditional expectation operators. On the other hand if $E=L_\infty(X, S, \lambda, R)$ and the dimension of E is 2, then the author [9] proved that such operators can be expressed as a linear combination of two conditional expectation operators. In this paper we deal with the case that E is an M-space. An L_∞-space is an M-space, and hence this paper contains the result of the author [9] as a special case. If E is an M-space, whose dimension is bigger than 2, then such operators coincide with conditional expectation operators.
If E is an M-space with unit, i.e., the unit ball in E has a least upper bound, then we can prove many of lemmas in this paper by easier way. In this paper we do not assume that E is an M-space with unit.

1. Definitions and properties of M-spaces.

Let E be a real linear space and N the class of natural numbers and R the class of real numbers.

Definition 1.1. A lattice $(E, \leqq)$ is an ordered linear space such that
(1) $a\leqq a$ for any $a\in E$;
(2) if $a, b\in E$, $a\leqq b$ and $b\leqq a$, then $a=b$;
(3) if $a, b, c\in E$ and $a\leqq b$ and $b\leqq c$, then $a\leqq c$;
(4) if $a\leqq b$, then $a+c\leqq b+c$ for any $c\in E$;
(5) if $0\leqq a$ in E, then $0\leqq ka$ in E for any $k\geqq 0$ in R;
(6) $\sup\{a, b\}$ and $\inf\{a, b\}$ exist for any $a, b\in E$.

In a lattice we write $a\vee b=\sup\{a, b\}$, $a\wedge b=\inf\{a, b\}$, $a^+=a\vee 0$, $a^-=(-a)\vee 0$ and $|a|=a\vee(-a)$ for any $a, b\in E$. Let $E^+=\{a\in E; a\geqq 0\}$. Note that $a\wedge b=0$ implies that $a, b\in E^+$. If $a\in E^+$ and $a\neq 0$, then we write $a>0$. We also use $\vee$ and $\wedge$ for real numbers, and hence $k\vee h=\sup\{k, h\}$ and $k\wedge h=\inf\{k, h\}$ for $k, h\in R$.

DEFINITION 1.2. An M-space $(E, \leqq, \|\ \|)$ is a normed lattice such that
(1) $(a\vee b)\wedge c=(a\wedge c)\vee(b\wedge c)$ for any $a, b, c\in E$;
(2) E is complete under $\|\ \|$;
(3) $\|a\vee b\|=\|a\|\vee\|b\|$ for any $a, b\in E^+$;
(4) If $a, b\in E$ and $|a|\leqq b$, then $\|a\|\leqq\|b\|$. In particular $\|\,|a|\,\|=\|a\|$.

Lemma 1.1. *If E is an M-space, then there exist a Hausdorff compact space X, a linear operator T of E into $C(X)$ and a linear subspace F of $C(X)$ which satisfy the following conditions, where $C(X)$ is the class of real-valued continuous functions on X with the norm $\|d\|=\sup\{|d(x)|; x\in X\}$ for $d\in C(X)$.*
(1) *$d\vee e\in F$ for $d, e\in F$, where $\vee$ is defined by*

$$(d\vee e)(x)=\sup\{d(x), e(x)\}.$$

(2) *T is a one-to-one operator onto F such that*

$$T(a\vee b)=T(a)\vee T(b)$$

and

$$\|T(a)\|=\|a\|.$$

For the proof see Aliprantis and Bourkinshaw [1] p. 75.

Let $E_h=\{a^*; a^*$ is a linear functional of E into R, $\|a^*\|\leqq 1$, i.e., $|a^*(a)|\leqq\|a\|$ for $a\in E$ and $a^*(a\vee b)=a^*(a)\vee a^*(b)$ for $a, b\in E\}$.

Lemma 1.2. *For any $a\in E$ there exists $a^*\in E_h$ such that $|a^*(a)|=\|a\|$.*

Proof. By Lemma 1.1 $T(a)\in C(X)$ and $\|a\|=\|T(a)\|$. We can choose $x\in X$ such that $|T(a)(x)|=\|T(a)\|$. We define a^* by $a^*(b)=T(b)(x)$ for any $b\in E$. Then a^* is linear and

$$|a^*(a)|=|T(a)(x)|=\|T(a)\|=\|a\|.$$

By the definition of a^*

$$\begin{aligned}a^*(b\vee c)&=T(b\vee c)(x)=(T(b)\vee T(c))(x)=(T(b)(x))\vee(T(c)(x))\\&=a^*(b)\vee a^*(c).\end{aligned}$$

Therefore $a^*\in E_h$. Q.E.D.

Lemma 1.3. *Let $a\in E$ and $b, c, d\in E^+$. Then*

(1) $(a\wedge b)\vee -b=(a\vee -b)\wedge b$,
(2) $((-a)\wedge b)\vee -b=-((a\wedge b)\vee -b)$,
(3) $((a\wedge b)\vee -b)^{+}=a^{+}\wedge b$,
(4) $((a\wedge b)\vee -b)^{-}=a^{-}\wedge b$,
(5) $|(a\wedge b)\vee -b|=|a|\wedge b$
and
(6) $(b+c)\wedge d\leqq b\wedge d+c\wedge d$.

Proof. Since $b\in E^{+}$, for any $a\in E$

$$(a\wedge b)\vee -b=(a\wedge b)\vee((-b)\wedge b)=(a\vee -b)\wedge b\,,$$

and hence we have (1). Since a is arbitrary, by (1)

$$((-a)\wedge b)\vee -b=((-a)\vee -b)\wedge b=-((a\wedge b)\vee -b)\,,$$

which implies (2). Since $b\in E^{+}$, we have

$$\begin{aligned}((a\wedge b)\vee -b)^{+}&=((a\wedge b)\vee -b)\vee 0=(a\wedge b)\vee 0\\ &=(a\vee 0)\wedge(b\vee 0)=a^{+}\wedge b\,,\end{aligned}$$

which implies (3). By (2) and (3)

$$\begin{aligned}((a\wedge b)\vee -b)^{-}&=(-((a\wedge b)\vee -b))^{+}=(((-a)\wedge b)\vee -b)^{+}\\ &=(-a)^{+}\wedge b=a^{-}\wedge b\,,\end{aligned}$$

which implies (4). Since $a^{+}\wedge a^{-}=0$, by (3) and (4)

$$\begin{aligned}|(a\wedge b)\vee -b|&=a^{+}\wedge b+a^{-}\wedge b=(a^{+}\wedge b)\vee(a^{-}\wedge b)\\ &=(a^{+}\vee a^{-})\wedge b=|a|\wedge b\,.\end{aligned}$$

For the proof of (6) see Fremlin [4] p.14. Q.E.D.

Lemma 1.4. *For any $a, b\in E$ and $c, d\in E^{+}$ we have*
(1) $\|(a\wedge c)\vee -c\pm(b\wedge c)\vee -c\|\leqq\|a\pm b\|$,
(2) $\|c+a^{-}\|\leqq\|c\|\vee\|a-c\|$
and
(3) $\|c-d\|\leqq\|c\|\vee\|d\|$.
If in addition $|a|\wedge c=0$, then
(4) $\|a+c\|=\|a\|\vee\|c\|$.

Proof. By Lemma 1.2 there exists $a^{*}\in E^{*}$ such that
(5) $\|(a\wedge c)\vee -c\pm(b\wedge c)\vee -c\|=|a^{*}((a\wedge c)\vee -c\pm(b\wedge c)\vee -c)|$.
We may assume that $a^{*}(c)\geqq 0$.
By the definition of E_{h}
(6) $|a^{*}((a\wedge c)\vee -c\pm(b\wedge c)\vee -c)|=|(a^{*}(a)\wedge a^{*}(c))\vee -a^{*}(c)$

$\pm(a^*(b)\wedge a^*(c))\vee -a^*(c)|$.

Since $a^*(a)$, $a^*(b)\in R$, $a^*(c)\geqq 0$ and $||a^*||\leqq 1$, we have

(7) $|(a^*(a)\wedge a^*(c))\vee -a^*(c)\pm(a^*(b)\wedge a^*(c))\vee -a^*(c)|$
$\leqq |a^*(a)\pm a^*(b)|\leqq ||a\pm b||$.

By (5), (6) and (7) we have (1).

$$c\vee |a-c|\geqq c\vee(c-a) = c+(0\vee(-a)) = c+a^-\geqq 0\,,$$

and hence by Definition 1.2 (4) we have

$$||c\vee |a-c|\;||\geqq ||c+a^-||\,.$$

By Definition 1.2 (3) and (4)

$$||c\vee |a-c|\;|| = ||c||\vee||\;|a-c|\;|| = ||c||\vee||a-c||\,,$$

and hence we have (2).

Since c, $d\in E$ implies that $|c-d|\leqq c\vee d$, by Definition 1.2(3) and (4) we have

$$||c-d||\leqq ||c\vee d|| = ||c||\vee||d||\,.$$

If $|a|\wedge c=0$, then

$$|a+c| = |a|+c = |a|\vee c\,.$$

Therefore by Definition 1.2(3) and (4)

$$||a+c|| = ||\;|a+c|\;|| = ||\;|a|\vee c|| = ||a||\vee||c||\,. \qquad \text{Q.E.D.}$$

Lemma 1.5. *For any b, $c\in E^+$ with $b\wedge c=0$ and $x\in E$*

(1) $||x+b\pm c||\geqq ||(x\wedge b)\vee -b+b||\vee||(x\wedge c)\vee -c\pm c||$

Proof. Since $b\wedge c=0$ implies that $c\wedge 2b=0$,

$$b = b-c\wedge 2b = (b-c)\vee -b\leqq((b\pm c)\wedge b)\vee -b\leqq b\,.$$

Therefore

(2) $((b\pm c)\wedge b)\vee -b = b$.

Since $b\wedge(c\pm c)\leqq b\wedge 2c=0$, we have

(3) $((b\pm c)\wedge c)\vee -c = ((b\wedge(c\mp c))\pm c)\vee -c = (\pm c)\vee -c = \pm c$.

By (2), (3) and Lemma 1.4 (1)

$$\begin{aligned}&||x+b\pm c||\\ &\quad\geqq ||(x\wedge b)\vee -b+((b\pm c)\wedge b)\vee -b||\vee||(x\wedge c)\vee -c+((b\pm c)\wedge c)\vee -c||\\ &\quad\geqq ||(x\wedge b)\vee -b+b||\vee||(x\wedge c)\vee -c\pm c|| \qquad \text{Q.E.D.}\end{aligned}$$

2. A characterization of conditional expectation. Let $(\Omega, \mathcal{A}, \mu)$ be a probability space and for any $A\in\mathcal{A}$ we denote by I_A the indicator function of

A. Let $L_1(\Omega, \mathcal{A}, \mu, E)$ be the class of E-valued Bochner integrable functions, which is a Banach spase with the norm $\|\ \ \|_L$ defined by

$$\|f\|_L = \int \|f(\omega)\|\, d\mu \quad \text{for any} \quad f \in L_1(\Omega, \mathcal{A}, \mu, E)\,.$$

Let $L_1(\Omega, \mathcal{A}, \mu, E^+) = \{f \in L_1(\Omega, \mathcal{A}, \mu, E);\ f(\omega) \in E^+(a.e.\omega)\}$. For any $f \in L_1(\Omega, \mathcal{A}, \mu, E)$ and $a \in E$ we define $f+a$ by

$$(f+a)(\omega) = f(\omega)+a\,.$$

For any $\psi \in L_1(\Omega, \mathcal{A}, \mu, R)$ and $a \in E$ we define $\psi\, a$ by $(\psi\, a)(\omega) = \psi(\omega)\, a$. Then $\|\psi\, a\|_L = \|a\|\, \|\psi\|_L$. For the definition and properties of Bochner integral, see Hille and Phillips [5].

DEFINITION 2.1. For a σ-subalgebra $\mathcal{B}$ of $\mathcal{A}$, a function g is called the conditional expectation of f given $\mathcal{B}$ if g is measurable with respect to $\mathcal{B}$, and

$$\int_B g d\,\mu = \int_B f d\,\mu \quad \text{for each} \quad B \in \mathcal{B}\,,$$

where the integral is the Bochner integral. We denote by $f^{\mathcal{B}}$ the conditional expectation of f given $\mathcal{B}$.

DEFINITION 2.2. Let P be a linear operator of $L_1(\Omega, \mathcal{A}, \mu, E)$ into itself. P is said to be *contractive* if

$$\|P\| = \sup\{\|P(f)\|_L\,; f \in L_1(\Omega, \mathcal{A}, \mu, E) \quad \text{and} \quad \|f\|_L = 1\} \leqq 1\,,$$

P is *constant-preserving* if $P(I_\Omega\, a) = I_\Omega\, a$ for each $a \in E$ and P is called a *projection* if $P \circ P = P$, where I_Ω is the indicator function of Ω.

Lemma 2.1. *For each $f \in L_1(\Omega, \mathcal{A}, \mu, E)$ the conditional expectation of f exists uniquely up to almost everywhere and the conditional expectation operator $(\ \ )^{\mathcal{B}}$ is a constant-preserving contractive projection for each σ-subalgebra $\mathcal{B}$ of $\mathcal{A}$.*

For the proof see Schwartz [10].

Lemma 2.2. *If P is a constant-preserving contractive projection of $L_1(\Omega, \mathcal{A}, \mu, R)$ into itself, then there exists a σ-subalgebra $\mathcal{B}$ of $\mathcal{A}$ such that $P(f) = f^{\mathcal{B}}$ for any $f \in L_1(\Omega, \mathcal{A}, \mu, R)$.*

For the proof see Douglas [3]. Note that this Lemma is for the real-valued functions.

Lemma 2.3. *If a^* is a bounded linear operator of E into R and $f \in L_1(\Omega, \mathcal{A}, \mu, E)$, then we have*

$$a^*\Big(\int f(\omega)\, d\mu\Big) = \int a^*(f(\omega))\, d\mu\,.$$

For the proof see Hille and Phillips [5].

Lemma 2.4. *Let Q be a constant-preserving contractive projection of $L_1(\Omega, \mathcal{A}, \mu, E)$ into itself. If $a, b, c \in E^+$ with $a \wedge b = b \wedge c = c \wedge a = 0$ and $b > 0$, then for any $\psi \in L_1(\Omega, \mathcal{A}, \mu, R)$ we have*

$$(1) \qquad (Q(\psi\, a)(\omega) \wedge c) \vee -c = 0 \ (a.e.\omega)$$

Proof. If $a=0$ or $c=0$, then this Lemma is trivial. So we may assume that $||a|| = ||b|| = ||c|| = 1$. First we assume that $|\psi(\omega)| \leqq 1$ $(a.e.\omega)$.
Let $e = \int |(Q(\psi\, a) \wedge c) \vee -c|\, d\mu = \int |Q(\psi\, a)| \wedge c\, d\mu$, where the last equation comes from Lemma 1.3 (5).
Suppose that $e > 0$. Then there exist $k \in R^+$ and $d^* \in E_h$ such that $d^*(ke) = ||ke|| = 1$. Let $d = ke \vee c$, then $||d|| = 1$.
Since $e \leqq c$, $a \wedge d = d \wedge b = 0$.
Since $d^*(c) \leqq ||c|| = 1$,

$$d^*(d) = d^*(ke \vee c) = d^*(ke) \vee d^*(c) = 1\,.$$

Let $f(\omega) = (Q(\psi\, a)(\omega) \wedge b) \vee -b$
and

$$g(\omega) = (Q(\psi\, a)(\omega) \wedge d) \vee -d\,.$$

By Lemma 1.3 (5) $|g(\omega)| = |Q(\psi\, a)(\omega)| \wedge d$, and hence by Lemma 2.3 we have

$$\begin{aligned}(2) \qquad 1 = d^*(ke) &= kd^*\left(\int |Q(\psi\, a)| \wedge c\, d\mu\right) \\ &\leqq kd^*\left(\int |Q(\psi\, a)| \wedge d\, d\mu\right) \\ &\leqq kd^*\left(\int |g|\, d\mu\right) \\ &= k \int d^*(|g|)\, d\mu\,,\end{aligned}$$

where the last equation comes from Lemma 2.3.
Since $|\psi(\omega)| \leqq 1$ $(a.e.\omega)$ and $a \wedge b = b \wedge d = d \wedge a = 0$ with

$$||a|| = ||b|| = ||d|| = 1\,,$$

by Lemma 1.4 (4)

$$||\psi(\omega)\, a + b \pm d|| = ||\psi(\omega)\, a|| \vee ||b|| \vee ||d|| = 1 \ (a.e.\omega)\,.$$

Q is constant-preserving and contractive, and hence

$$(3) \qquad 1 = \int ||\psi\, a + b \pm d||\, d\mu \geqq \int ||Q(\psi\, a) + b \pm d||\, d\mu\,.$$

By Lemma 1.5 we have

$$\int \|Q(\psi_r a)+b\pm d\|\, d\mu \geqq \int \|f+b\| \vee \|g\pm d\|\, d\mu . \tag{4}$$

By the property of integral we have

$$\begin{aligned}\int \|f+b\| \vee \|g\pm d\|\, d\mu \\ \geqq \int \|f+b\|\, d\mu \vee \int \|g\pm d\|\, d\mu \\ \geqq \int \|f+b\|\, d\mu \wedge \int \|g\pm d\|\, d\mu \\ \geqq \left\|\int f d\mu+b\right\| \wedge \left\|\int g\, d\mu \pm d\right\| .\end{aligned} \tag{5}$$

Therefore by (3), (4) and (5)

$$1 \geqq \int \|g\pm d\|\, d\mu \geqq \left\|\int g\, d\mu \pm d\right\| .$$

Since $\left\|\int g d\mu+d\right\|+\left\|\int g d\mu-d\right\| \geqq 2\,\|d\|=2$, we have

$$\left\|\int g d\mu \pm d\right\| = 1 . \tag{6}$$

Similarly we can prove that

$$\left\|\int f d\mu+b\right\| = 1 . \tag{7}$$

Therefore by (3), (4), (5), (6) and (7)

$$\begin{aligned}\|g(\omega)+d\| &= \|f(\omega)+b\| \\ &= \|g(\omega)-d\| .\end{aligned}$$

Since

$$\|g(\omega)+d\|+\|g(\omega)-d\| \geqq 2\,\|d\| = 2 ,$$

by (5) we have

$$\|g(\omega)+d\| = \|g(\omega)-d\| = 1 \ (a.e.\omega) . \tag{8}$$

By the definition of $g(\omega)$ we have $d-g(\omega)$, $d+g(\omega) \geqq 0$ $(a.e.\omega)$, and hence by (8)

$$\begin{aligned}&\|d+|g(\omega)|\,\| \\ &= \|(d-g(\omega)) \vee (d+g(\omega))\| \\ &= \|d-g(\omega)\| \vee \|d+g(\omega)\| = 1 \ (a.e.\omega) .\end{aligned}$$

Since $d^*(d)=1$,

$$1+d^*(|g(\omega)|) = d^*(d+|g(\omega)|)$$
$$\leqq ||d+|g(\omega)|\,|| \leqq 1 \ (a.e.\omega) .$$

Therefore we have

$$d^*(|g(\omega)|) = 0 \ (a.e.\omega) ,$$

which contradicts (2). (1) remains valid for any bounded function ψ. Since an arbitrary function can be approximated by bounded functions, by Lemma 1.4(1) we can prove (1). Q.E.D.

Lemma 2.5. *Suppose that there exist $a, b, c \in E$ with $a\wedge b=b\wedge c=c\wedge a=0$ and $a, b, c>0$. Then*

(1) $$Q(\psi\, a)(\omega) \in E^+(a.e.\omega) \quad \text{for any} \quad \psi \in L_1(\Omega, \mathcal{A}, \mu, R^+) .$$

In particular if $0\leqq\psi(\omega)\leqq 1$ $(a.e.\omega)$, then $0\leqq Q(\psi\, a)(\omega)\leqq a$ $(a.e.\omega)$.

Proof. We may suppose that $0\leqq\psi(\omega)\leqq 1$ (a.e.ω) and $||a||=||b||=1$. Let $e=\int Q(\psi\, a)^- d\mu$. We suppose that $e>0$. Then there exists $k>0$ such that $||ke||=1$. Let $d=ke$. Since $a\wedge b=0$, by Lemma 2.4

$$(Q(\psi\, a)(\omega)\wedge b)\vee -b = 0 .$$

Hence by Lemma 1.3 (4), (5)

(2) $$|Q(\psi\, a)(\omega)|\wedge b = Q(\psi\, a)(\omega)^-\wedge b = 0 \ (\text{a.e.}\omega) .$$

Therefore

(3) $$d\wedge b = ke\wedge b = 0 .$$

Since $a\wedge b=0$, by Lemma 1.4(3) and Definition 1.2

(4) $$||\psi(\omega)\, a-d+b|| \leqq ||\psi(\omega)\, a+b||\vee||d||$$
$$= ||b||\vee||\psi(\omega)\, a||\vee||d|| = 1 \ (\text{a.e.}\omega) .$$

By (2), (3) and Lemma 1.3 (6) we have

$$|Q(\psi\, a)(\omega)-d|\wedge b \leqq |Q(\psi\, a)(\omega)|\wedge b+d\wedge b = 0 ,$$

and hence by Lemma 1.4 (4) and the fact that $||b||=||d||=1$

(5) $$||Q(\psi\, a)(\omega)-d+b|| = ||Q(\psi\, a)(\omega)-d||\vee||b||$$
$$= ||Q(\psi\, a)(\omega)-d||\vee||d|| .$$

By Lemma 1.4 (2)

(6) $$||Q(\psi\, a)(\omega)-d||\vee||d|| \geqq ||Q(\psi\, a)(\omega)^- + d|| .$$

Since Q is constant-preserving and contractive, by (4),(5) and (6),

$$\begin{aligned} 1 &\geqq \int \|\psi a-d+b\|\, d\mu \geqq \int \|Q(\psi a)-d+b\|\, d\mu \\ &\geqq \int \|Q(\psi a)^{-}+d\|\, d\mu \geqq \left\| \int Q(\psi a)^{-} d\mu+d \right\| \\ &= \|e+d\| = \|((1/k)+1)\, d\|>1\,, \end{aligned}$$

which leads to a contradiction, and hence $e=0$. Therefore

$$Q(\psi a)(\omega)\in E^{+}\ (a.e.\omega)\,.$$

Let $\phi(\omega)=1-\psi(\omega)$. Then similarly we can prove that

$Q(\phi a)(\omega)\in E^{+}$. Since Q is constant-preserving,

$$Q(\psi a)(\omega)+Q(\phi a)(\omega) = a\,.$$

Hence we have

$$0\leqq Q(\psi a)(\omega)\leqq a\ (a.e.\omega)\,. \qquad \text{Q.E.D.}$$

Lemma 2.6. *Suppose that there exist $a, b, c\in E^{+}$ with $a\wedge b=b\wedge c=c\wedge a=0$ and $\|a\|=\|b\|=\|c\|=1$. If $d\in E^{+}$ and $d^{*}\in E_{h}$ with $d^{*}(d)=\|d\|$, then for any $\psi\in L_{1}(\Omega, \mathcal{A}, \mu, R^{+})$ we have*

$$d^{*}(Q(\psi d)(\omega)) = \|Q(\psi d)(\omega)\|\ (a.e.\omega)\,,$$
$$\|Q(\psi d)\|_{L} = \|\psi d\|_{L}$$

and

$$\|Q(\psi a)(\omega)\| = \|Q(\psi b)(\omega)\|\ (a.e.\omega)\,.$$

Proof. First we assume that $0\leqq\psi(\omega)\leqq 1$ $(a.e.\omega)$ and $\|d\|=1$. Let $\phi(\omega)=1-\psi(\omega)$. Since $\|d^{*}\|\leqq 1$, we have

$$d^{*}(Q(\psi d)(\omega))\leqq\|Q(\psi d)(\omega)\|\ (a.e.\omega) \tag{4}$$

and

$$d^{*}(Q(\phi d)(\omega))\leqq\|Q(\phi d)(\omega)\|\ (a.e.\omega)\,. \tag{5}$$

Q is constant-preserving, and hence

$$\begin{aligned} &d^{*}(Q(\psi d)(\omega))+d^{*}(Q(\phi d)(\omega)) \\ &\quad = d^{*}(Q(I_{\Omega} d)(\omega)) = d^{*}(d) = 1\,. \end{aligned} \tag{6}$$

Since Q is contractive,

$$\begin{aligned} &\int \|Q(\psi d)\|\, d\mu+\int \|Q(\phi d)\|\, d\mu\leqq\int \|\psi d\|\, d\mu+\int \|\phi d\|\, d\mu \\ &\quad = \|d\| = 1\,. \end{aligned} \tag{7}$$

By (4), (5), (6) and (7) we have

(8) $$d^*(Q(\psi d)(\omega)) = \|Q(\psi d)(\omega)\| \ (a.e.\omega)$$

and

(9) $$\int \|Q(\psi d)\| \, d\mu = \int \|\psi d\| \, d\mu .$$

It is easy to show that (8) and (9) remain true for any bounded function $\psi \in L_1(\Omega, \mathcal{A}, \mu, R^+)$. Since any $\psi \in L_1(\Omega, \mathcal{A}, \mu, R^+)$ can be approximated by a sequence of bounded functions, (8) and (9) are true for ψ. We have proved (1) and (2). By Lemma 2.5 $0 \leqq Q(\psi a)(\omega) \leqq a$ and $0 \leqq Q(\psi b)(\omega) \leqq b$, and hence by the relation $a \wedge b = 0$ we have

$$Q(\psi a)(\omega) \wedge Q(\psi b)(\omega) = 0 \ (a.e.\omega) .$$

By Lemma 1.4 (4)

(10) $$\begin{aligned} \int \|Q(\psi a)\| \vee \|Q(\psi b)\| \, d\mu &= \int \|Q(\psi a) + Q(\psi b)\| \, d\mu \\ &\leqq \int \|\psi a + \psi b\| \, d\mu = \int \|\psi a\| \vee \|\psi b\| \, d\mu \\ &= \int \|\psi a\| \, d\mu = \int \|\psi b\| \, d\mu . \end{aligned}$$

(9) remains true for $d = a$ or b, and hence by (10) we have

$$\|Q(\psi a)(\omega)\| = \|Q(\psi b)(\omega)\| \ (a.e.\omega) .$$ Q.E.D.

Lemma 2.7. *Suppose that there exist $a, b, c \in E$ such that $a, b, c > 0$ and $a \wedge b = b \wedge c = c \wedge a = 0$. If $\psi, \phi \in L_1(\Omega, \mathcal{A}, \mu, R)$ satisfy the condition*

(1) $0 \leqq \psi(\omega) \leqq 1 \ (a.e.\omega)$ *and* $\phi(\omega) \|a\| = \|Q(\psi a)(\omega)\| \ (a.e.\omega)$, *then* $\|Q(\phi a)(\omega)\| = \phi(\omega) \|a\|$.

Proof. We assume that $\|a\| = \|b\| = 1$. By (1) and Lemma 2.5 we have

(2) $$0 \leqq Q(\psi b)(\omega) \leqq b \ (a.e.\omega) ,$$

and hence $0 \leqq \phi(\omega) \leqq 1 \ (a.e.\omega)$.
Therefore by Lemma 2.5 we have

(3) $$0 \leqq Q(\phi a)(\omega) \leqq a \ (a.e.\omega) .$$

Since $a \wedge b = 0$, by (1), (2), (3) and Lemma 1.4 we have

(4) $$\begin{aligned} \|Q(\psi b)(\omega) - Q(\phi a)(\omega)\| &= \|Q(\psi b)(\omega)\| \vee \|Q(\phi a)(\omega)\| \\ &= \phi(\omega) \vee \|Q(\phi a)(\omega)\| \ (a.e.\omega) \end{aligned}$$

and

$$(5) \qquad \|Q(\psi b)(\omega)-\phi(\omega)a\| = \|Q(\psi b)(\omega)\| \vee \|\phi(\omega)a\| = \phi(\omega) \ (a.e.\omega).$$

Since Q is a contractive projection,

$$\int \|Q(\psi b)-\phi a\| \, d\mu \geqq \int \|Q(\psi b)-Q(\phi a)\| \, d\mu ,$$

and hence by (4) and (5) we have

$$\int \phi \, d\mu \geqq \int \phi \vee \|Q(\phi a)\| \, d\mu ,$$

which implies that

$$\phi(\omega) \leqq \|Q(\phi a)(\omega)\| \ (a.e.\omega).$$

By Lemma 2.6

$$\|Q(\phi a)\|_L = \|\phi a\|_L = \|\phi\|_L .$$

Therefore we have

$$\phi(\omega) = \|Q(\phi a)(\omega)\| \ (a.e.\omega). \qquad \text{Q.E.D.}$$

Lemma 2.8. *If there exist $a, b, c \in E$ with $a \wedge b = b \wedge c = c \wedge a = 0$ and $a, b, c > 0$, then there exists a σ-subalgebra $\mathcal{B}$ of $\mathcal{A}$ such that $\|Q(\psi a)(\omega)\| = \psi^{\mathcal{B}}(\omega)\,\|a\|$ for any $\psi \in L_1(\Omega, \mathcal{A}, \mu, R^+)$.*

Proof. We may suppose that $\|a\|=1$. Let $a^* \in E_h$ such that $a^*(a)=1$. Define an operator P of $L_1(\Omega, \mathcal{A}, \mu, R)$ into itself by $P(\psi)(\omega) = a^*(Q(\psi a)(\omega))$ for any $\psi \in L_1(\Omega, \mathcal{A}, \mu, R)$. Since a^* and Q are linear operators, P is a linear operator. Since Q is constant-preserving, we have

$$(1) \qquad P(I_\Omega)(\omega) = a^*(Q(I_\Omega a)(\omega)) = a^*(a) = I_\Omega(\omega).$$

If $\psi(\omega) \geqq 0$, then by Lemma 2.6

$$\|Q(\psi a)(\omega)\| = a^*(Q(I_\Omega a)(\omega)) = P(\psi).$$

Since Q is contractive and $\|a^*\| \leqq 1$,

$$(2) \qquad \int |P(\psi)| \, d\mu = \int |a^*(Q(\psi a)(\omega))| \, d\mu \leqq \int \|Q(\psi a)\| \, d\mu \leqq \int \|\psi a\| \, d\mu = \int |\psi| \, d\mu .$$

Let

$$(3) \qquad \phi(\omega) = \|Q(\psi a)(\omega)\| = P(\psi)(\omega).$$

If $0 \leqq \psi(\omega) \leqq 1$ $(a.e.\omega)$, then by Lemma 2.7

(4) $$\phi(\omega) = \|(\phi\, a)(\omega)\| = \|Q(\phi\, a)(\omega)\| = P(\phi)(\omega)\,.$$

By (3) and (4)

(5) $$P(\psi) = P(P(\psi))\,.$$

Since P is a linear contractive operator, it is easy to show that (5) remains valid for any $\psi \in L_1(\Omega, \mathcal{A}, \mu, R)$. Therefore by (1), (2), (5) and Lemma 2.2 there exists a σ-subalgebra $\mathcal{B}$ such that

(6) $$P(\psi) = \psi^{\mathcal{B}}\,.$$

By Lemma 2.6 and the definition of P

(7) $$P(\psi)(\omega) = a^*(Q(\psi\, a)(\omega)) = \|Q(\psi\, a)(\omega)\|\,.$$

By (6) and (7) we have proved this Lemma. Q.E.D.

Lemma 2.9. *Let $a, b, c, d \in E$ with $a, b, c, d > 0$ and $a \wedge b = b \wedge c = c \wedge a = 0$. Then we can choose $a', b', d' \in E^+$, $k \in R$ such that $d = d' + (ka \wedge d) + (kb \wedge d) + (kc \wedge d)$, $a', b' > 0$ and $a' \wedge b' = b' \wedge d' = d' \wedge a' = 0$.*

Proof. We may suppose that $\|a\| = \|b\| = \|c\| = 1$. Let $k = 2\|d\|$, and $a' = ka - ka \wedge d$, $b' = kb - kb \wedge d$ and $d' = d - d \wedge k(a \vee b \vee c)$. Since $\|ka\| = k > \|d\| \geqq \|ka \wedge d\| \vee \|kb \wedge d\|$, we have $a', b' > 0$.
Since $a \wedge b = b \wedge c = c \wedge a = 0$, we have

$$\begin{aligned} d &= d - d \wedge k(a \vee b \vee c) = d - ((ka \wedge d) \vee (kb \wedge d) \vee (kc \wedge d)) \\ &= d - (ka \wedge d + kb \wedge d + kc \wedge d)\,. \end{aligned}$$

By the definitions of k, a', b' and d' we have

$$0 \leqq a' \wedge b' \leqq ka \wedge kb = 0\,.$$

and

$$\begin{aligned} 0 \leqq d' \wedge a' &= (d - d \wedge k(a \vee b \vee c)) \wedge (ka - ka \wedge d) \\ &\leqq (d - ka \wedge d) \wedge (ka - ka \wedge d) = ka \wedge d - ka \wedge d = 0\,. \end{aligned}$$

Similarly we can prove that $b' \wedge d' = 0$. Q.E.D.

Lemma 2.10. *Suppose that there exist $a, b, c \in E$ with $a, b, c > 0$ and $a \wedge b = b \wedge c = c \wedge a = 0$. If $d, e \in E$ and $d \geqq e$, then $Q(\psi\, d)(\omega) \geqq Q(\psi\, e)(\omega)$ (a.e.ω) for any $\psi \in L_1(\Omega, \mathcal{A}, \mu, R^+)$.*

Proof. We may suppose that $d > 0$. Then by Lemma 2.9 there exist $a', b', d' \in E$ such that

(1) $$a', b' > 0\,,$$

(2) $$d = d' + (ka \wedge d) + (kb \wedge d) + (kc \wedge d)$$

and

(3) $$a' \wedge b' = b' \wedge d' = d' \wedge a' = 0\,.$$

If $d' > 0$, then by (1), (3) and Lemma 2.5 we have

(4) $$Q(\psi\, d')(\omega) \in E^+ \ (a.e.\omega)\,.$$

If $d' = 0$, then (4) is trivial.
Since $a \wedge b = b \wedge c = c \wedge a = 0$,

(5) $$(ka \wedge d) \wedge b = (ka \wedge d) \wedge c = b \wedge c = 0\,.$$

If $ka \wedge d > 0$, then by (5) and Lemma 2.5

(6) $$Q(\psi(ka \wedge d))(\omega) \in E^+ \ (a.e.\omega)\,.$$

If $ka \wedge d = 0$, then (6) is trivial.
Similarly we can prove that

(7) $$Q(\psi(kb \wedge d))(\omega) \in E^+$$

and

(8) $$Q(\psi(kc \wedge d))(\omega) \in E^+\,.$$

By (2), (4), (6), (7) and (8) we have

$$Q(\psi\, d)(\omega) \in E^+ \ (a.e.\omega)\,.$$

Since Q is linear, this proves the lemma. Q.E.D.

Lemma 2.11. *Suppose that there exist $a, b, c \in E$ with $a, b, c > 0$ and $a \wedge b = b \wedge c = c \wedge a = 0$. Then for any $d \in E^+$ there exists a σ-subalgebra $\mathcal{B}$ of $\mathcal{A}$ such that*

$$\|Q(\psi\, d)(\omega)\| = \psi^{\mathcal{B}} \|d\| \ (a.e.\omega)$$

for any $\psi \in L_1(\Omega, \mathcal{A}, \mu, R^+)$, where $\mathcal{B}$ is independent of the choice of d.

Proof. We may suppose that $\|a\| = \|d\|$. Then $\|a \vee d\| = \|a\| \vee \|d\| = \|a\| = \|d\|$.
By Lemma 2.10

(1) $$Q(\psi(d \vee a))(\omega) \geqq Q(\psi\, a)(\omega) \vee Q(\psi\, d)(\omega) \geqq 0 \text{ in } E\,.$$

By Lemma 2.8 there exists a σ-subalgebra $\mathcal{B}$ of $\mathcal{A}$ such that

$$\|Q(\psi a)(\omega)\| = \psi^{\mathcal{B}} \|a\|\,.$$

and hence by (1) and Definition 1.2 (4) we have

(2) $$||Q(\psi(a\vee d))(\omega)||\geqq||Q(\psi a)(\omega)|| \\ =\psi^{\mathcal{B}}||a||=\psi^{\mathcal{B}}||a\vee d||\,.$$

By Lemma 2.6 and the properties of conditional expectation

$$||Q(\psi(a\vee d))||_L=||\psi(a\vee d)||_L=||\psi^{\mathcal{B}}(a\vee d)||_L\,,$$

and hence by (2) we have

(3) $$||Q(\psi(a\vee d))(\omega)||=\psi^{\mathcal{B}}||a||=\psi^{\mathcal{B}}||d||\,.$$

By (1)

(4) $$||Q(\psi(a\vee d))(\omega)||\geqq||Q(\psi d)(\omega)||\,.$$

By Lemma 2.6

$$||Q(\psi d)||_L=||\psi d||_L=||\psi^{\mathcal{B}} d||_L=||\psi^{\mathcal{B}}||_L\,||d||\,,$$

and hence by (3) and (4)

$$||Q(\psi d)(\omega)||=\psi^{\mathcal{B}}||d||\,.$$

It is clear that $\mathcal{B}$ is independent of the choice of d. Q.E.D.

Lemma 2.12. *If $\dim(E)\geqq 3$, where $\dim(E)$ is the dimension of E as a linear space, then there exist $a, b, c\in E$ such that $a, b, c>0$ and $a\wedge b=b\wedge c=c\wedge a=0$.*

The proof of this lemma is a direct result of Theorem 26.10 of Luxemburg and Zaanen [7]

Theorem 1. *If $\dim(E)\geqq 3$, then there exists a σ-subalgebra $\mathcal{B}$ of $\mathcal{A}$ such that $Q(f)=f^{\mathcal{B}}$ for any $f\in L_1(\Omega, \mathcal{A}, \mu, E)$.*

Proof. Let $\mathcal{B}$ be the σ-subalgebra whose existence was proved in Lemma 2.11. Since the conditional expectation operator $(\ )^{\mathcal{B}}$ and Q are linear bounded operators, it is sufficient to show that for any $d\in E^+$ and $A\in\mathcal{A}$ with $||d||=1$

$$Q(I_A d)=(I_A)^{\mathcal{B}} d\,.$$

Let $e=\int(Q(I_A d)(\omega)\vee(I_A d)^{\mathcal{B}}(\omega)-Q(I_A d)(\omega))\,d\mu(\omega)$. Clearly $e\in E^+$. We suppose that $e>0$. Since $e>0$, by Lemma 1.2 there exists $e^*\in E_h$ such that

$$||e||=|e^*(e)|=e^*(e)\,.$$

By Lemma 2.5

(1) $$0\leqq Q(I_A d)(\omega)\leqq d\,.$$

By the properties of conditional expectation we have

$$0 \leqq (I_A\, d)^{\mathcal{B}}(\omega) \leqq d\,,$$

and hence by (1)

$$0 < e \leqq d\,,$$

by which we have

$$e^*(e) \leqq e^*(d)\,.$$

Therefore we can choose $k \geqq 1$ such that $e^*(ke) = e^*(d)$. Then we have

(2) $$e^*(ke \wedge d) = e^*(ke) \wedge e^*(d) = e^*(d)\,.$$

Since $||e^*|| \leqq 1$,

(3) $$e^*(ke \wedge d) \leqq ||ke \wedge d|| \leqq ||ke|| = ke^*(e) = e^*(d)\,.$$

By (2) and (3) we have

(4) $$e^*(ke \wedge d) = ||ke \wedge d|| = e^*(d)\,.$$

Since $d \geqq ke \wedge d$, by (4) and Lemma 2.6

$$e^*(Q(I_A\, d)(\omega)) \geqq e^*(Q(I_A(ke \wedge d))(\omega)) = ||Q(I_A(ke \wedge d))(\omega)||\,.$$

By Lemma 2.11 and (4)

$$||Q(I_A(ke \wedge d))(\omega)|| = (I_A)^{\mathcal{B}}(\omega)\, ||ke \wedge d|| = (I_A)^{\mathcal{B}}(\omega)\, e^*(d)\,.$$

Therefore

$$\begin{aligned} 0 < e^*(e) &= e^*\Big(\int (Q(I_A\, d)(\omega) \vee (I_A\, d)^{\mathcal{B}}(\omega) - Q(I_A\, d)(\omega))\, d\mu\Big) \\ &= \int (e^*(Q(I_A\, d)(\omega)) \vee (I_A)^{\mathcal{B}}(\omega)\, e^*(d) - e^*(Q(I_A\, d)(\omega)))\, d\mu \\ &= \int (e^*(Q(I_A\, d)(\omega) - e^*(Q(I_A\, d)(\omega))\, d\mu = 0\,, \end{aligned}$$

which is a contradiction. We have proved that $e=0$, and hence we have

(4) $$Q(I_A\, d)(\omega) \geqq (I_A\, d)^{\mathcal{B}}(\omega)\ (a.e.\omega)\,.$$

Similarly we can prove that

(5) $$Q(I_{\Omega-A}\, d)(\omega) \geqq (I_{\Omega-A}\, d)^{\mathcal{B}}(\omega)\ (a.e.\omega)\,.$$

Since Q is constant-preserving,

$$\begin{aligned} Q(I_A\, d)(\omega) + Q(I_{\Omega-A}\, d)(\omega) &= Q(I_\Omega\, d)(\omega) \\ = I_\Omega\, d(\omega) &= (I_A\, d)^{\mathcal{B}}(\omega) + (I_{\Omega-A}\, d)^{\mathcal{B}}(\omega)\,, \end{aligned}$$

and hence by (4) and (5) we have

$$Q(I_A\, d) = (I_A\, d)^{\mathcal{B}}$$ Q.E.D.

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