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STABLE HOMOTOPY TYPES OF STUNTED LENS SPACES MOD 4

Dedicated to Professor Hideki Ozeki on his 60th birthday

SUSUMU KÔNO

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1. Introduction

Let $L^n(q)=S^{2n+1}/(\boldsymbol{Z}/q)$ be the $(2n+1)$-dimensional standard lens space mod q. As defined in [8], we set

$$(1.1)\qquad \begin{aligned} L_q^{2n+1} &= L^n(q)\,, \\ L_q^{2n} &= \{[z_0, \cdots, z_n]\in L^n(q)\,|\,z_n \text{ is real}\geqq 0\}\ . \end{aligned}$$

The stable homotopy types (S-types) of stunted lens spaces L_q^m/L_q^n have been studied by several authors (e.g. [7], [8], [9], [10], [11] and [12]). For the case $q=2$, D.M. Davis and M. Mahowald have completed the classification of the stable homotopy types of stunted real projective spaces in [7]. Their result shows that we can use structures of J-groups of suspensions of stunted real projective spaces to obtain the necessary conditions for stunted real projective spaces $RP(m)/RP(n)$ and $RP(m+t)/RP(n+t)$ to have the same stable homotopy type as follows: if $RP(m)/RP(n)$ and $RP(m+t)/RP(n+t)$ have the same stable homotopy type, then there exists a non-negative integer N such that

$$\tilde{J}(S^j(RP(m)/RP(n))) \cong \tilde{J}(S^{j-t}(RP(m+t)/RP(n+t)))$$

for each integer j with $j\geqq N$ (see [13]). For the case where q is an odd prime, T. Kobayashi has obtained some necessary conditions for stunted lens spaces L_q^m/L_q^n and L_q^{m+t}/L_q^{n+t} to have the same stable homotopy type (cf. [10]). The conditions are also sufficient if $k=[m/2]-[(n+1)/2]\not\equiv 0 \pmod{(q-1)}$ or $n+1\equiv 0 \pmod{2q^{[k/(q-1)]}}$. We can use structures of J-groups of suspensions of stunted lens spaces mod q to obtain the conditions (see [14]). The object of this paper is to study the stable homotopy types of stunted lens spaces L_q^m/L_q^n for $q=4$ or 8.

In order to state our results, we prepare functions $h_1, h_2, \alpha, \beta_1, \beta_2$ and γ_1 defined by

$$(1.2)\qquad h_1(k) = \begin{cases} [k/4]+[(k+7)/8]+[(k+4)/8] & (k\geqq 2) \\ 0 & (1\geqq k\geqq 0)\,. \end{cases}$$

(1.3) $$h_2(k)=\begin{cases}[k/4]+[(k+7)/8]+[k/8]+1 & (k\geqq 4)\\ h_1(k) & (3\geqq k\geqq 0)\,.\end{cases}$$

(1.4) $$\alpha(k,n)=\begin{cases}1 & (n\equiv 0 \pmod 2 \text{ and } k\equiv 1 \pmod 8\,,\\ & \quad\text{or}\quad k=2([n/2]-[(n-1)/2]))\\ 0 & (\text{otherwise})\,.\end{cases}$$

(1.5) $\beta_1(k, n)$ is equal to the corresponding integer in the following table:

k (mod 8) \ n (mod 4)	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	0	1	1	0	0	1	1
2	0	0	0	1	0	0	1	1
3	0	0	0	0	0	0	0	0

(1.6) $\beta_2(k, n)$ is equal to the corresponding integer in the following table:

k (mod 8) \ n (mod 8)	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	0	1	1	0	0	1	1
2	0	0	1	1	0	0	1	1
3	0	0	0	0	1	1	0	0
4	0	0	0	0	1	1	0	0
5	0	0	2	2	1	1	1	1
6	0	0	1	2	1	1	1	1
7	0	0	0	0	0	0	0	0

(1.7) $\gamma_1(m, n)$ is equal to the corresponding integer in the following table:

n (mod 8) \ $m-n$	1	2	3	4	5	6
0	0	0	1	0	1	0
1	0	1	1	1	1	2
2	0	0	1	0	2	1
3	0	1	1	1	1	1
4	0	0	1	0	1	0
5	0	1	1	1	1	1
6	0	0	1	0	1	1
7	0	1	1	1	2	2

Let $\nu_p(s)$ denote the exponent of the prime p in the prime power decomposition of s.

Theorem 1. *If* $\nu_2(t)\geqq h_1(m-2[(n+1)/2])+1-\alpha(m-2[(n+1)/2], n)$, *then* L_4^m/L_4^n *and* L_4^{m+t}/L_4^{n+t} *have the same stable homotopy type.*

Theorem 2. (1) *If* L_4^m/L_4^n *and* L_4^{m+t}/L_4^{n+t} *have the same stable homotopy type, then*

$$\nu_2(t)\geqq[m/2]-[(n+1)/2]+\beta_1(m-2[(n+1)/2], n).$$

(2) *Suppose* $h_1(m-2[(n+1)/2])-\alpha(m-2[(n+1)/2], n)\geqq i\geqq 3$ *and* $\max\{\nu_2(n+1), \nu_2(m+1)\}\geqq i$. *If* L_4^m/L_4^n *and* L_4^{m+t}/L_4^{n+t} *have the same stable homotopy type, then* $\nu_2(t)\geqq i+1$.
(3) *Suppose* $n<m\leqq n+6$. *If* L_4^m/L_4^n *and* L_4^{m+t}/L_4^{n+t} *have the same stable homotopy type, then*

$$\nu_2(t)\geqq[m/2]-[(n+1)/2]+\gamma_1(m, n).$$

REMARK. It follows from Theorems 1 and 2, that we have obtained necessary and sufficient conditions for spaces L_4^m/L_4^n and L_4^{m+t}/L_4^{n+t} to have the same stable homotopy type if one of the following conditions is satisfied:
(1) $n<m\leqq 2[(n+1)/2]+3$,
(2) $\beta_1(m-2[(n+1)/2], n)=1$,
(3) $\max\{\nu_2(n+1), \nu_2(m+1)\}\geqq h_1(m-2[(n+1)/2])-\alpha(m-2[(n+1)/2], n)$ and $m\geqq n+5$.

Theorem 3. *If* $\nu_2(t)\geqq h_2(m-2[(n+1)/2])+1-\alpha(m-2[(n+1)/2], n)$, *then*

L_8^m/L_8^n and L_8^{m+t}/L_8^{n+t} have the same stable homotopy type.

Theorem 4. (1) *Suppose $m \geqq n+5$. If L_8^m/L_8^n and L_8^{m+t}/L_8^{n+t} have the same stable homotopy type, then*

$$\nu_2(t) \geqq [m/2]-[(n+1)/2]+\beta_2(m-2\,[(n+1)/2], n)\,.$$

(2) *Suppose $h_2(m-2[(n+1)/2])-\alpha(m-2[(n+1)/2], n) \geqq i \geqq 3$ and $\max\{\nu_2(n+1), \nu_2(m+1)\} \geqq i$. If L_8^m/L_8^n and L_8^{m+t}/L_8^{n+t} have the same stable homotopy type, then $\nu_2(t) \geqq i+1$.*

(3) *Suppose $n<m \leqq n+6$. If L_8^m/L_8^n and L_8^{m+t}/L_8^{n+t} have the same stable homotopy type, then*

$$\nu_2(t) \geqq [m/2]-[(n+1)/2]+\gamma_2(m, n),$$

where $\gamma_2(m, n)$ is the integer defined by

$$\gamma_1(m, n) = \begin{cases} 1 & (n \equiv 0 \pmod 8) \text{ and } m=n+6) \\ \gamma_1(m, n) & (\text{otherwise}). \end{cases}$$

REMARK. It follows from Theorems 3 and 4, that we have obtained necessary and sufficient conditions for spaces L_8^m/L_8^n and L_8^{m+t}/L_8^{n+t} to have the same stable homotopy type if one of the following conditions is satisfied:

(1) $n<m \leqq 2\,[(n+1)/2]+3$,
(2) $n \equiv 0 \pmod 8$ and $m=n+6$,
(3) $\beta_2(m-2\,[(n+1)/2], n)=2$ or $2\,[(n+1)/2] \equiv [m/2]-[(n+3)/2] \equiv 2 \pmod 4$,
(4) $\max\{\nu_2(n+1), \nu_2(m+1)\} \geqq h_2(m-2\,[(n+1)/2])-\alpha(m-2\,[(n+1)/2], n)$ and $m \geqq n+5$.

This paper is organized as follows. In section 2 we prepare some lemmas and recall known results. We prove Theorems 1, 2, 3 and 4 in the final section.

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2. Preliminaries

In this section we prepare some lemmas and recall known results which are needed to prove Theorems 1, 2, 3 and 4.

Lemma 2.1. *Let m, n, k and s be non-negative integers with $2^s \leqq k < 2^{s+1}$. Assume that $\binom{m}{i} \equiv \binom{n}{i} \pmod 2$ for $1 \leqq i \leqq k$. Then $m \equiv n \pmod{2^{s+1}}$.*

Proof. Suppose that $m=\sum_{j=0}^N a_j 2^j$, $n=\sum_{j=0}^N b_j 2^j$ and $i=\sum_{j=0}^N c_{i,j} 2^j$ ($1 \leqq i \leqq k$), where a_j, b_j and $c_{i,j}$ are non-negative integers with $a_j \leqq 1$, $b_j \leqq 1$ and $c_{i,j} \leqq 1$ ($1 \leqq i \leqq k$, $1 \leqq j \leqq N$). Then we have $\binom{m}{i} \equiv \Pi_{j=0}^N \binom{a_j}{c_{i,j}} \pmod 2$ and $\binom{n}{i} \equiv \Pi_{j=0}^N$

$\binom{b_j}{c_{i,j}}$ (mod 2) for $1 \leqq i \leqq k$. It follows from the hypothesis that we have $a_j = b_j$ for $0 \leqq j \leqq s$; that is, $m \equiv n \pmod{2^{s+1}}$. q.e.d.

Let $q \geqq 2$ be an integer and $q_0, q_1, \cdots, q_n$ be integers relatively prime to q. Consider the $(\boldsymbol{Z}/q)$-action on the unit sphere $S^{2n+1} \subset \boldsymbol{C}^{n+1}$ given by

$$\exp(2\pi\sqrt{-1}/q)(z_0, \cdots, z_n) = (z_0 \cdot \exp(2q_0\pi\sqrt{-1}/q), \cdots, z_n \cdot \exp(2q_n\pi\sqrt{-1}/q)).$$

Then $S^{2n+1}(q; q_0, \cdots, q_n)$ denotes the space S^{2n+1} with this action,

$$L_q^{2n+1}(q_0, \cdots, q_n) = S^{2n+1}(q; q_0, \cdots, q_n)/(\boldsymbol{Z}/q)$$

and $L_q^{2n}(q_0, \cdots, q_n)$ is the subspace of $L_q^{2n+1}(q_0, \cdots, q_n)$ defined by

$$L_q^{2n}(q_0, \cdots, q_n) = \{[z_0, \cdots, z_n] \in L_q^{2n+1}(q_0, \cdots, q_n) \mid z_n \text{ is real} \geqq 0\}.$$

For $0 \leqq n < m \leqq 2l+1$, we set

$$L_q^m/L_q^n(q_0, \cdots, q_l) = L_q^m(q_0, \cdots, q_l)/L_q^n(q_0, \cdots, q_l),$$

which is called a stunted lens space mod q. Then we have

(2.2) $L_q^m/L_q^n(1, \cdots, 1) = L_q^m/L_q^n$.

Considering the $(\boldsymbol{Z}/q)$-action on $S^{2l+1}(q; q_0, \cdots, q_l) \times \boldsymbol{C}^k$ given by

$$\begin{aligned} &\exp(2\pi\sqrt{-1}/q)(z, w_1, \cdots, w_k) \\ &= (\exp(2\pi\sqrt{-1})/q) \cdot z, w_1 \cdot \exp(2a_1\pi\sqrt{-1}/q), \cdots, w_k \cdot \exp(2a_k\pi\sqrt{-1}/q)) \end{aligned}$$

for $(z, w_1, \cdots, w_k) \in S^{2l+1}(q; q_0, \cdots, q_l) \times \boldsymbol{C}^k$, we have a complex k-dimensional vector bundle

$$\eta(a_1, \cdots, a_k) \colon (S^{2l+1}(q; q_0, \cdots, q_l) \times \boldsymbol{C}^k)/(\boldsymbol{Z}/q) \to L_q^{2l+1}(q_0, \cdots, q_l).$$

We use same symbol for the restriction of $\eta(a_1, \cdots, a_k)$ to $L_q^n(q_0, \cdots, q_l)$ ($n \leqq 2l+1$) and denote the complex line bundle $\eta(1)$ by η. Then we have

(2.3) $\eta(a_1, \cdots, a_k) \cong \eta^{a_1} \oplus \cdots \oplus \eta^{a_k}$.

Let $\xi \colon E(\xi) \to X$ be a real vector bundle over a finite CW-complex X with disk bundle $B(\xi)$ and sphere bundle $S(\xi)$. Then the Thom complex X^ξ of ξ is defined as the quotient space $B(\xi)/S(\xi)$. Define $f \colon S^{2n+1} \times D^{2k} \to S^{2k+2n+1}$ by

$$f(z_0, \cdots, z_n, w) = (w, (1-\|w\|^2)^{1/2} z_0, \cdots, (1-\|w\|^2)^{1/2} z_n)$$

for $(z_0, \cdots, z_n, w) \in S^{2n+1} \times D^{2k}$, where $D^{2k} \subset \boldsymbol{C}^k$ is the unit disk. Then we have

(2.4) *Let $a_1, \cdots, a_k$ be integers relatively prime to q. Then f induces the following homeomorphisms.*

(1) $(L_q^{2n+1}(q_0, \cdots, q_n))^{r(\eta(a_1,\cdots,a_k))} \approx L_q^{2k+2n+1}/L_q^{2k-1}(a_1, \cdots, a_k, q_0, \cdots, q_n)$.

(2) $(L_q^{2n}(q_0, \cdots, q_n))^{r(\eta(a_1,\cdots,a_k))} \approx L_q^{2k+2n}/L_q^{2k-1}(a_1, \cdots, a_k, q_0, \cdots, q_n)$.

(3) $(L_q^{2n+1}(q_0, \cdots, q_n))^{r(\eta(a_1,\cdots,a_k))}/S^{2k} \approx L_q^{2k+2n+1}/L_q^{2k}(a_1, \cdots, a_k, q_0, \cdots, q_n)$.

(4) $(L_q^{2n}(q_0, \cdots, q_n))^{r(\eta(a_1,\cdots,a_k))}/S^{2k} \approx L_q^{2k+2n}/L_q^{2k}(a_1, \cdots, a_k, q_0, \cdots, q_n)$.

We define the function $h(q, k)$ by setting

(2.5) $h(q, k) = \mathrm{ord}\langle J(r(\eta)-2)\rangle$,

where $J(r(\eta)-2)$ is the image of $r(\eta)-2 \in \widetilde{KO}(L_q^k)$ by the J-homomorphism $J\colon \widetilde{KO}(L_q^k) \to \tilde{J}(L_q^k)$.

Remark. The function $h(q, k)$ have been determined completely (cf. [8]).

Spaces X and Y are said to have the same stable homotopy type ($X \underset{S}{\simeq} Y$) if there exist non-negative integers a and b such that $S^a X$ and $S^b Y$ have the same homotopy type. For stunted generalized lens spaces, we have

(2.6) (1) *If* $mn \equiv 0 \pmod 2$, *then* $L_q^m/L_q^n(q_0, \cdots, q_{[m/2]})$ *and* L_q^m/L_q^n *have the same stable homotopy type. In particular,* $L_q^m(q_0, \cdots, q_{[m/2]})$ *and* L_q^m *have the same stable homotopy type.*

(2) *Let* $a_1, \cdots, a_k, b_1, \cdots, b_k, q_0, \cdots, q_n, r_0, \cdots, r_n$ *and* a *be integers relatively prime to* q.

i) *Assume* $q_0 \cdots q_n \equiv \pm a^{n+1} r_0 \cdots r_n \pmod q$. *Then*

$$L_q^{2n+2k+1}/L_q^{2k-1}(a_1, \cdots, a_k, q_0, \cdots, q_n)$$

and $L_q^{2n+2k+1}/L_q^{2k-1}(b_1, \cdots, b_k, r_0, \cdots, r_n)$ *have the same stable homotopy type.*

ii) *If* $k \equiv 0 \pmod{h(q, 2n+1)}$, *then the spaces*

$$L_q^{2n+2k+1}/L_q^{2k-1}(a_1, \cdots, a_k, q_0, \cdots, q_n)$$

and $L_q^{2n+2k+1}/L_q^{2k-1}$ *have the same stable homotopy type.*

Proof. Set $X = L_q^{2n+1}(q_0, \cdots, q_n)$. According to [17], we have

$$K(X) \cong \boldsymbol{Z}[\eta]/(\eta^q - 1, (\eta-1)^{n+1})$$

and $\mathrm{ord}\, \tilde{K}(X) = q^n$. If a is an integer relatively prime to q, then

$$\begin{aligned}\bigcap_f \Big(\sum_j j^{f(j)}(\psi^j - 1)\,\tilde{K}(X)\Big) &= \sum_j \Big(\bigcap_e j^e(\psi^j - 1)\,\tilde{K}(X)\Big)\\ &\supset \bigcap_e a^e(\psi^a - 1)\,\tilde{K}(X) = (\psi^a - 1)\,\tilde{K}(X)\\ &\ni (\psi^a - 1)(\eta - 1) = \eta^a - \eta\,.\end{aligned}$$

Since the Adams operations are compatible with the real restriction $r\colon K\to KO$ [4],

$$r(\eta^a-\eta)=r(\eta^a)-r(\eta)\in\bigcap_f\big(\sum_j j^{f(j)}(\psi^j-1)\widetilde{KO}(X)\big).$$

By [2], [3] and [19], this implies that

$$J(r(\eta^a)-2)=J(r(\eta)-2)$$

in $\tilde{J}(X)$. If $a_1,\cdots,a_k$ are integers relatively prime to q,

$$J(r(\eta(a_1,\cdots,a_k))-2k)=J(k(r(\eta))-2k)$$

in $\tilde{J}(X)$. Suppose that $a_1,\cdots,a_k$, $b_1,\cdots,b_k$, $q_0,\cdots,q_n$, $r_0,\cdots,r_n$ and a be integers relatively prime to q. According to [5, Proposition (2.6)], we have

$$L_q^{2n+2k+1}/L_q^{2k-1}(a_1,\cdots,a_k,q_0,\cdots,q_n)\underset{S}{\simeq}L_q^{2n+2k+1}/L_q^{2k-1}(a,\cdots,a,q_0,\cdots,q_n)$$

and

$$L_q^{2n+2k+1}/L_q^{2k-1}(b_1,\cdots,b_k,r_0,\cdots,r_n)\underset{S}{\simeq}L_q^{2n+2k+1}/L_q^{2k-1}(1,\cdots,1,r_0,\cdots,r_n).$$

Suppose that $q_0\cdots q_n\equiv\pm a^{n+1}r_0\cdots r_n \pmod q$. Identify $S^{2n+2k+1}$ with the iterated join

$$S^1*\cdots*S^1=\{\lambda_0 z_0+\cdots+\lambda_{n+k}z_{n+k}\mid\sum_j\lambda_j=1,\lambda_j\geqq 0,z_j\in S^1\}.$$

Choose integers $\bar{q}_i\ (0\leqq i\leqq n)$ with $q_i\bar{q}_i\equiv 1\pmod q$. Denote the generator $\exp(2\pi\sqrt{-1}/q)$ of $\boldsymbol{Z}/q$ by g. Then, the map

$$f\colon S^{2n+2k+1}(q;a,\cdots,a,q_0,\cdots,q_n)\to S^{2n+2k+1}(q;1,\cdots,1,r_0,\cdots,r_n)$$

defined by

$$f=1*\cdots*1*(ar_0\bar{q}_0)*\cdots*(ar_n\bar{q}_n),$$

is a map of degree $a^{n+1}r_0\cdots r_n\bar{q}_0\cdots\bar{q}_n\equiv\pm 1\pmod q$ with $f\circ g=g^a\circ f$. Modify f to get a map h of degree ± 1 with $h\circ g=g^a\circ h$ and

$$h\mid S^{2k-1}(q;a,\cdots,a)=f\mid S^{2k-1}(q;a,\cdots,a)$$

(see the proof of [6, (29.4)] for the detail). Then h induces a homotopy equivalence

$$\bar{h}\colon L_q^{2n+2k+1}/L_q^{2k-1}(a,\cdots,a,q_0,\cdots,q_n)\to L_q^{2n+2k+1}/L_q^{2k-1}(1,\cdots,1,r_0,\cdots,r_n).$$

This completes the proof of (2) i).

Now we turn to the proof of (1). Since $\tilde{J}(X)$ has a finite order, we may assume that $k>1$. Let a be an integer with $aq_0\cdots q_n\equiv 1\pmod q$. Then

$$L_q^{2n+2k+1}/L_q^{2k}(a_1, \cdots, a_k, q_0, \cdots, q_n) \underset{S}{\simeq} L_q^{2n+2k+1}/L_q^{2k}(1, \cdots, 1, a, q_0, \cdots, q_n).$$

By the proof above, there exists an equivariant map

$$h\colon S^{2n+2k+1}(q; 1, \cdots, 1) \to S^{2n+2k+1}(q; 1, \cdots, 1, a, q_0, \cdots, q_n)$$

of degree 1, which induces homotopy equivarences

$$\bar{h}\colon L_q^{2n+2k+1}/L_q^{2k} \to L_q^{2n+2k+1}/L_q^{2k}(1, \cdots, 1, a, q_0, \cdots, q_n)$$

and $\bar{h}|L_q^{2n+2k}/L_q^{2k}\colon L_q^{2n+2k}/L_q^{2k} \to L_q^{2n+2k}/L_q^{2k}(1, \cdots, 1, a, q_0, \cdots, q_n)$. Thus $L_q^{2n+2k+1}/L_q^{2k}(a_1, \cdots, a_k, q_0, \cdots, q_n)$ is stably homotopically equivalent to $L_q^{2n+2k+1}/L_q^{2k}$ and $L_q^{2n+2k}/L_q^{2k}(a_1, \cdots, a_k, q_0, \cdots, q_n)$ is stably homotopically equivalent to L_q^{2n+2k}/L_q^{2k}. The equivariant map

$$f = 1*\cdots*1*q_0*\cdots*q_n\colon S^{2n+2k+1}(q; 1, \cdots, 1) \to S^{2n+2k+1}(q; 1, \cdots, 1, q_0, \cdots, q_n)$$

induces a homotopy equivalence

$$\bar{f}|L_q^{2n+2k}/L_q^{2k-1}\colon L_q^{2n+2k}/L_q^{2k-1} \to L_q^{2n+2k}/L_q^{2k-1}(1, \cdots, 1, q_0, \cdots, q_n).$$

Thus $L_q^{2n+2k}/L_q^{2k-1}(a_1, \cdots, a_k, q_0, \cdots, q_n)$ is stably homotopically equivalent to L_q^{2n+2k}/L_q^{2k-1}. This completes the proof of (1).

Finally, we prove the part ii) of (2). Suppose that $k \equiv 0 \pmod{h(q, 2n+1)}$. Since the order of $J(r(\eta)-2) \in \tilde{J}(X)$ coincides with $h(q, 2n+1)$, we have

$$\begin{aligned}
&L_q^{2n+2k+1}/L_q^{2k-1}(a_1, \cdots, a_k, q_0, \cdots, q_n)\\
&\underset{S}{\simeq} L_q^{2n+2k+1}/L_q^{2k-1}(1, \cdots, 1, q_0, \cdots, q_n)\\
&\approx (L_q^{2n+1}(q_0, \cdots, q_n))^{r(k\eta)}\\
&\underset{S}{\simeq} (L_q^{2n+1}(q_0, \cdots, q_n))^{2k}\\
&\simeq S^{2k} L_q^{2n+1}(q_0, \cdots, q_n) \vee S^{2k}\\
&\underset{S}{\simeq} S^{2k} L_q^{2n+1} \vee S^{2k}\\
&\simeq (L_q^{2n+1})^{2k} \underset{S}{\simeq} (L_q^{2n+1})^{r(k\eta)} \approx L_q^{2n+2k+1}/L_q^{2k-1},
\end{aligned}$$

by [5, Proposition (2.6)], (2.4) and (2.6) (1). This completes the proof of the part ii) of (2). q.e.d.

Let X and Y be pointed CW-complexes. The stable homotopy group from X to Y is defined by

(2.7) $\{X, Y\} = \lim_{n\to\infty} [S^n X, S^n Y]_0$.

The following assertion is proved by making use of Puppe exact sequences.

(2.8) (1) *If* $k>m$, *then* $\{L_q^m/L_q^n(q_0, \cdots, q_{[m/2]}), S^k\} \cong 0$.

(2) *If* $k \leqq n$, *then* $\{S^k, L_q^m/L_q^n(q_0, \cdots, q_{[m/2]})\} \cong 0$.

(3) *Suppose that* $[m/2] > [(n+1)/2]$. *Then we have*

i) $\{S^{n+1}, L_q^m/L_q^n(q_0, \cdots, q_{[m/2]})\} = \langle \{i_{n+1}\} \rangle \cong \begin{cases} \boldsymbol{Z} & (n\text{: odd}) \\ \boldsymbol{Z}/q & (n\text{: even}), \end{cases}$

where i_{n+1} *is the composition*

$$S^{n+1} \approx L_q^{n+1}/L_q^n(q_0, \cdots, q_{[m/2]}) \subset L_q^m/L_q^n(q_0, \cdots, q_{[m/2]}) .$$

ii) $\{L_q^m/L_q^n(q_0, \cdots, q_{[m/2]}), S^m\} = \langle \{p_m\} \rangle \cong \begin{cases} \boldsymbol{Z} & (m\text{: odd}) \\ \boldsymbol{Z}/q & (m\text{: even}), \end{cases}$

where p_m *is the composition*

$$L_q^m/L_q^n(q_0, \cdots, q_{[m/2]}) \to L_q^m/L_q^{m-1}(q_0, \cdots, q_{[m/2]}) \approx S^m .$$

The following lemma implies that the family of stunted lens spaces is closed under S-duality.

Lemma 2.9. *Suppose that* $k = 2[m/2] + 1 - 2[(n+1)/2] \geqq 3$, $N \equiv 0 \pmod{2h(q, k)}$ *and* $N > m+1$. *Then the S-dual of*

$$L_q^m/L_q^n(a_1, \cdots, a_{[(n+1)/2]}, q_0, \cdots, q_{[k/2]})$$

is $L_q^{N-n-2}/L_q^{N-m-2}(b_1, \cdots, b_{[(N-m-1)/2]}, q_0, \cdots, q_{[k/2]})$.

Proof. If $nm \equiv 1 \pmod 2$, then the S-dual of

$$\begin{aligned} X &= L_q^m/L_q^n(a_1, \cdots, a_{[(n+1)/2]}, q_0, \cdots, q_{[k/2]}) \\ &\underset{S}{\simeq} (L_q^k(q_0, \cdots, q_{(k-1)/2}))^{((n+1)/2)r(\eta)} \end{aligned}$$

is $(L_q^k(q_0, \cdots, q_{(k-1)/2}))^{-\tau-((n+1)/2)r(\eta)}$, where τ denotes the tangent bundle of $L_q^k(q_0, \cdots, q_{(k-1)/2})$ (cf. [5, Theorem (3.5)]). Since

$$\tau \oplus 1 \cong r(\eta(q_0, \cdots, q_{(k-1)/2})) ,$$

it follows from [5, Proposition (2.6)] and (2.6) that

$$\begin{aligned} &(L_q^k(q_0, \cdots, q_{(k-1)/2}))^{-\tau-((n+1)/2)r(\eta)} \\ &\underset{S}{\simeq} (L_q^k(q_0, \cdots, q_{(k-1)/2}))^{(N/2)r(\eta)-((k+1)/2)r(\eta)-((n+1)/2)r(\eta)} \\ &= (L_q^k(q_0, \cdots, q_{(k-1)/2}))^{((N-m-1)/2)r(\eta)} \\ &\underset{S}{\simeq} L_q^{N-n-2}/L_q^{N-m-2}(b_1, \cdots, b_{(N-m-1)/2}, q_0, \cdots, q_{(k-1)/2}) = Y . \end{aligned}$$

This implies that there are positive integers a, b and M such that the M-dual of S^aX is S^bY. Let

$$u\colon S^aX \wedge S^bY \to S^M$$

be a M-duality pairing. Then the homomorphism

$$\Gamma_u^i\colon \{S^i, S^a X\} \to \{S^{i+b} Y, S^M\}$$

defined by $\Gamma_u^i(\{f\}) = \{S^l u \circ (f \wedge 1)\}$ for $f\colon S^{l+i} \to S^{l+a} X$, is an isomorphism. It follows from (2.8) that $M = a+b+N-1$ and

$$\{S^{n+1}, X\} = \langle \{i_{n+1}\} \rangle \cong \{Y, S^{N-n-2}\} = \langle \{p_{N-n-2}\} \rangle \cong \boldsymbol{Z}\,.$$

Hence, by the isomorphism

$$\{S^{n+1}, X\} \cong \{S^{a+n+1}, S^a X\} \xrightarrow{\Gamma_u^{a+n+1}} \{S^{a+b+n+1}\, Y, S^{a+b+N-1}\} \cong \{Y, S^{N-n-2}\},$$

$\{i_{n+1}\}$ corresponds to either $\{p_{N-n-2}\}$ or $-\{p_{N-n-2}\}$. This implies that there exists a homotopy commutative diagram

$$\begin{array}{ccc} S^{l+a+n+1} \wedge S^b Y & \xrightarrow{1 \wedge S^b(p_{N-n-2})} & S^{l+a+n+1} \wedge S^{b+N-n-2} \\ \Big\downarrow {\scriptstyle S^{l+a}(i_{n+1}) \wedge 1} & & \approx \Big\downarrow {\scriptstyle v} \\ S^{l+a} X \wedge S^b Y & \xrightarrow{\quad S^l u \quad} & S^{l+a+b+N-1}\,, \end{array}$$

where v and $S^l u$ are $(l+a+b+N-1)$-duality pairings. It follows that $C_{S^{l+a}(i_{n+1})}$ is the $(l+a+b+N)$-dual of $C_{S^b(p_{N-n-2})}$. Since

$$C_{S^{l+a}(i_{n+1})} \simeq S^{l+a}(L_q^m / L_q^{n+1}(a_1, \cdots, a_{(n+1)/2}, q_0, \cdots, q_{[k/2]}))$$

and

$$C_{S^b(p_{N-n-2})} \simeq S^{b+1}(L^{N-n-3} / L^{N-m-2}(b_1, \cdots, b_{(N-m-1)/2}, q_0, \cdots, q_{[k/2]}))\,,$$

this implies that L_q^m / L_q^{n+1} is the S-dual of L^{N-n-3} / L^{N-m-2}. In the similar way it is shown that $L_q^{N-n-2} / L_q^{N-m-1}$ is the S-dual of L_q^{m-1} / L_q^n. Using this fact, in the similar way it is shown that L_q^{m-1} / L_q^{n+1} is the S-dual of $L_q^{N-n-3} / L_q^{N-m-1}$. q.e.d.

REMARK. The partial results for the case where q is a prime of this lemma have been obtained in [18].

It follows from (2.6) and Lemma 2.9 that, in the following cases,

$$L_q^m / L_q^n(a_1, \cdots, a_{[(n+1)/2]}, q_0, \cdots, q_{[m/2]-[(n+1)/2]})$$

and L_q^m / L_q^n have the same stable homotopy type:

(1) $q = 2, 3, 4$ or 6,

(2) $mn \equiv 0 \pmod 2$,

(3) $q_0 \cdots q_{[m/2]-[(n+1)/2]} \equiv \pm a^{[m/2]-[(n-1)/2]} \pmod q$,

(4) $n+1\equiv 0 \pmod{2h(q, m-n-1)}$,
(5) $m+1\equiv 0 \pmod{2h(q, m-n-1)}$.

QUESTION. Is it true that $L_q^m/L_q^n(q_0, \cdots, q_{[m/2]})$ and L_q^m/L_q^n have the same stable homotopy type for any case? If it is not, then how many stable homotopy types are there for fixed m, n and q?

From now on, we restrict ourselves to standard stunted lens spaces. According to [5, Propositions (2.6) and (2.9)], (2.4) and Lemma 2.9, we obtain the following.

(2.10) *Set* $k=m-2[(n+1)/2]$ *and* $l=2[m/2]-n$.
(1) *If* $t\equiv 0 \pmod{2h(q, k)}$, *then* L_q^m/L_q^n *and* L_q^{m+t}/L_q^{n+t} *have the same stable homotopy type.*
(2) *If* $k\geqq 2$ *and* $n+1\equiv 0 \pmod{2h(q, k)}$, *then* $t\equiv 0 \pmod{2h(q, k)}$ *if and only if* L_q^m/L_q^n *and* L_q^{m+t}/L_q^{n+t} *have the same stable homotopy type.*
(3) *If* $t\equiv 0 \pmod{2h(q, l)}$, *then* L_q^m/L_q^n *and* L_q^{m+t}/L_q^{n+t} *have the same stable homotopy type.*
(4) *If* $l\geqq 2$ *and* $m+1\equiv 0 \pmod{2h(q, l)}$, *then* $t\equiv 0 \pmod{2h(q, l)}$ *if and only if* L_q^m/L_q^n *and* L_q^{m+t}/L_q^{n+t} *have the same stable homotopy type.*

(2.11) (1) ([12, I; Theorem 1.1]) *Let p be a prime and r a positive integer with* $p^r>2$. *Suppose that* $k=m-2[(n+1)/2]\geqq 2$. *Then* $t\equiv 0 \pmod{2p^{[(k-2)/2(p-1)]}}$ *if* $L_{p^r}^m/L_{p^r}^n$ *and* $L_{p^r}^{m+t}/L_{p^r}^{n+t}$ *have the same stable homotopy type.*
(2) *Let* $r\geqq 2$ *be a positive integer and set* $k=m-2[(n+1)/2]$. *Then* $\nu_2(t)\geqq [k/2]+\beta_1(k, n)$ *if* $L_{2^r}^m/L_{2^r}^n$ *and* $L_{2^r}^{m+t}/L_{2^r}^{n+t}$ *have the same stable homotopy type, where* β_1 *is the function defined by* (1.5).
(3) *Suppose that* $q\equiv 0 \pmod 2$ *and* $m\geqq n+2$. *Then* $\nu_2(t)\geqq[\log_2 2(m-n-1)]$ *if* L_q^m/L_q^n *and* L_q^{m+t}/L_q^{n+t} *have the same stable homotopy type.*

Proof. Suppose that $q\equiv 0 \pmod 2$ and $m\geqq n+2$. It is well known that

$$H^*(L_q^m; \boldsymbol{Z}/2)\cong\begin{cases}(\boldsymbol{Z}/2)[u]/(u^{m+1}) & (q\equiv 2 \pmod 4)\\ (\boldsymbol{Z}/2)[u, v]/(u^2, v^{[m/2]+1}, uv^{m-[m/2]}) & (q\equiv 0 \pmod 4)\,,\end{cases}$$

where $\deg u=1$ and $\deg v=2$. The action of the Steenrod squares is given by

$$\begin{cases}Sq^i(u^j)=\binom{j}{i}u^{j+i} & (q\equiv 2 \pmod 4)\\ Sq^{2i}(v^j)=\binom{j}{i}v^{j+i} & (q\equiv 0 \pmod 4)\\ Sq^{2i}(uv^j)=\binom{j}{i}uv^{j+i} & (q\equiv 0 \pmod 4)\\ Sq^{2i+1}=0 & (q\equiv 0 \pmod 4)\,.\end{cases}$$

Assume that L_q^m/L_q^n and L_q^{m+t}/L_q^{n+t} have the same stabel homotopy type. Then $t\equiv 0 \pmod 2$. It follows from the naturality of the Steenrod squares that we have

$$(2.12)\quad \begin{cases} \binom{n+1}{i} \equiv \binom{n+t+1}{i} \pmod 2 & (1\leqq i\leqq m-n-1,\ q\equiv 2 \pmod 4) \\ \binom{[(n+1)/2]}{i} \equiv \binom{[(n+t+1)/2]}{i} \pmod 2 & (2\leqq 2i\leqq m-n-1,\ q\equiv 0 \pmod 4)\,. \end{cases}$$

Let s be the integer with $2^s\leqq m-n-1<2^{s+1}$. By Lemma 2.1, (2.12) implies that $\nu_2(t)\geqq s+1=[\log_2 2(m-n-1)]$. This completes the proof of (3).

(2) It follows from [11, Theorem 1.1] that

$$\nu_2(t)\geqq [m/2]-[(n+1)/2]+C(m, n)$$

if L_{2r}^m/L_{2r}^n and $L_{2r}^{m+t}/L_{2r}^{n+t}$ have the same stable homotopy type, where $C(m, n)$ is the function defined by

$$C(m, n) = \begin{cases} 1 & (n\equiv 1, 5 \text{ or } 6 \pmod 8 \text{ and } m\equiv 1, 4 \text{ or } 5 \pmod 8) \\ 0 & (\text{otherwise}). \end{cases}$$

Then (2) is obtained by making use of the S-duality (Lemma 2.9). q.e.d.

In order to state the next proposition, we set

$$(2.13)\quad \begin{cases} a_0(m, n) = [m/4]+[(m+7)/8]+[(m+4)/8] \\ \qquad\qquad -[(n+1)/4]-[(n+1)/8]-[(n+6)/8] \\ b_0(m, n) = [m/8]+[(m+6)/8]-[(n+7)/8]-[(n+5)/8]\,. \end{cases}$$

$$(2.14)\quad \begin{cases} a(j, m, n) = \min\,\{\nu+1, a_0(m+j, n+j)\} \\ b(j, m, n) = \min\,\{\nu+1, b_0(m+j, n+j)\}\ ; \end{cases}$$

where ν is the integer defined by

$$\nu = \begin{cases} \nu_2(j) & (j \neq 0) \\ m & (j=0)\,. \end{cases}$$

Let $\mathfrak{m}(s)$ denote the function defined on positive integers as follows (cf. [3]):

$$\nu_p(\mathfrak{m}(s)) = \begin{cases} 0 & (p \neq 2 \text{ and } s\not\equiv 0 \pmod{(p-1)}) \\ 1+\nu_p(s) & (p \neq 2 \text{ and } s\equiv 0 \pmod{(p-1)}) \\ 1 & (p = 2 \text{ and } s\not\equiv 0 \pmod 2) \\ 2+\nu_2(s) & (p = 2 \text{ and } s\equiv 0 \pmod 2)\,. \end{cases}$$

Proposition 2.15 ([15, Theorem 3]). *Let j, m and n be non-negative in-*

tegers with $m>n$ *and* $j\equiv n+1\equiv 0 \pmod 4$. *Then we have*

$$\tilde{J}(S^j(L_4^m/L_4^n)) \cong \begin{cases} \boldsymbol{Z}/\mathfrak{m}((n+j+1)/2)\cdot 2^c \oplus \boldsymbol{Z}/2^{d+i} \oplus \boldsymbol{Z}/2^k & (b(j,m,n)\geqq 0) \\ \boldsymbol{Z}/\mathfrak{m}((n+j+1)/2) & (b(j,m,n)<0), \end{cases}$$

where i, k, c *and* d *are integers defined by*

$$(2.16)\qquad \begin{cases} i=\begin{cases}\min\{\nu_2(n+1)-1,\ a(j,m,n)\} & (n+j\equiv 7 \pmod 8) \\ \min\{\nu_2(n+1),\ a(j,m,n)\} & (n+j\equiv 3 \pmod 8)\end{cases} \\ k=\min\{\nu_2(n+1)-1,\ b(j,m,n)\} \\ c=\max\{a(j,m,n)-i,\ b(j,m,n)-k\} \\ d=\min\{a(j,m,n)-i,\ b(j,m,n)-k\}. \end{cases}$$

In order to state the next proposition, we set

$$(2.17)\qquad \begin{cases} a_3(m,n)=[(m-2)/8]-[(n+5)/8] \\ a_4(m,n)=[m/8]-[(n+7)/8] \\ a_5(m,n)=a_0(m,n)+[m/8]-[(m-4)/8] \\ a_6(m,n)=[(m+4)/8]+[(m-2)/8]-[(n+1)/4]. \end{cases}$$

$$(2.18)\qquad \begin{cases} b_1(j,m,n)=\min\{\nu-a_3(n,n+4),\ a_5(m,n)\} \\ b_2(j,m,n)=\min\{\nu-a_3(n-5,n),\ a_6(m,n)\} \\ b_3(j,m,n)=\min\{\nu+1,\ a_4(m,n)\}, \end{cases}$$

where ν is the integer defined in (2.14).

Proposition 2.19 ([16, Theorem 2]). *Let* j, m *and* n *be non-negative integers with* $m>n$ *and* $j\equiv 0 \pmod 8$.

(1) *If* $n\not\equiv 3 \pmod 4$ *and* $m\geqq 2[(n+6)/8]+2[n/8]+4[(n+15)/8]$, *then we have*

$$\tilde{J}(S^j(L_8^m/L_8^n)) \cong \begin{cases} (\oplus_{i=1}^3 \boldsymbol{Z}/2^{b_i(j,m,n)})\oplus \boldsymbol{Z}/2 & (n\equiv 2 \pmod 8) \\ \oplus_{i=1}^3 \boldsymbol{Z}/2^{b_i(j,m,n)} & (\textit{otherwise}). \end{cases}$$

(2) *If* $n\not\equiv 3 \pmod 4$ *and* $2[(n+6)/8]+2[n/8]+4[(n+15)/8]>m>n$, *then we have*

$$\tilde{J}(S^j(L_8^m/L_8^n)) \cong \begin{cases} \boldsymbol{Z}/2^{b_1(j,m,n)}\oplus \boldsymbol{Z}/4 & (n\equiv 2 \pmod 8) \text{ and } m\geqq n+6) \\ \boldsymbol{Z}/8 & (n\equiv 1 \pmod 8) \text{ and } m\geqq n+3) \\ \widetilde{KO}(S^j(L_8^m/L_8^n)) & (\textit{otherwise}). \end{cases}$$

(3) *If* $n\equiv 3 \pmod 4$ *and* $m\geqq n+5$, *then we have*

$$\tilde{J}(S^j(L_8^m/L_8^n)) \cong \boldsymbol{Z}/\mathfrak{m}((n+j+1)/2)\cdot 2^{c_1}\oplus \boldsymbol{Z}/2^{c_2+i_1}\oplus \boldsymbol{Z}/2^{i_2}\oplus \boldsymbol{Z}/2^{b_3(j,m,n)},$$

where i_1, i_2, c_1 *and* c_2 *are integers defined by*

$$(2.20)\quad \begin{cases} i_1 = \begin{cases} \min\{b_1(j,m,n), \nu_2(n+1)-1\} & (n\equiv 7 \pmod 8) \\ \min\{b_1(j,m,n), \nu_2(n+1)\} & (n\equiv 3 \pmod 8) \end{cases} \\ i_2 = \begin{cases} \min\{b_2(j,m,n), \nu_2(n+1)-2\} & (n\equiv 7 \pmod 8) \\ \min\{b_2(j,m,n), \nu_2(n+1)-1\} & (n\equiv 3 \pmod 8) \end{cases} \\ c_1 = \max\{b_1(j,m,n)-i_1, b_2(j,m,n)-i_2\} \\ c_2 = \min\{b_1(j,m,n)-i_1, b_2(j,m,n)-i_2\}. \end{cases}$$

(4) *If* $n\equiv 3 \pmod 4$ *and* $n+5>m>n$, *then we have*

$$\tilde{J}(S^j(L_8^m/L_8^n)) \cong \begin{cases} \boldsymbol{Z}/\mathfrak{m}((n+j+1)/2)\oplus\widetilde{KO}(S^j(L_8^m/L_8^{n+1})) & (m\geqq n+2) \\ \boldsymbol{Z}/\mathfrak{m}((n+j+1)/2) & (m=n+1). \end{cases}$$

In order to state the next proposition, we set

$$(2.21)\quad (1)\quad \begin{cases} \sigma_{2^i}=\eta^{2^i}-1 & (0\leqq i\leqq 2) \\ \sigma_6=\sigma_4\sigma_2 \\ \sigma_{2i+1}=\sigma_{2i}\sigma_1 & (1\leqq i\leqq 3). \end{cases}$$

(2) Let $F(x)$ denote the free abelian group generated by $x_1, x_2, x_3, x_4, x_5, x_6$ and x_7. Then X_i and $X_i(n)$ $(1\leqq i\leqq 7, n\geqq 0)$ denote the elements of $F(x)$ defined by $X_1=4x_1+2x_3+2x_5+x_7$, $X_2=2x_2+x_6$, $X_3=2x_3+x_7$, $X_6=x_6+x_7$, $X_i=x_i$ $(i=4, 5$ or $7)$, $X_1(n)=2^{[n/2]}X_1$,

$$\begin{aligned} X_2(n) &= 2^{[n/4]}X_2-2^{2[n/4]}X_1, \\ X_3(n) &= 2^{[(n-2)/4]}X_3+(2^{[n/2]}-2^{2[(n-2)/4]+1})X_1, \\ X_4(n) &= 2^{[n/8]}X_4+(2^{[n/4]}-2^{2[n/8]})X_2+2^{[n/4]+2[n/8]}X_1, \\ X_5(n) &= 2^{[(n-2)/8]}X_5+(2^{[(n-2)/4]}-2^{2[(n-2)/8]})X_3 \\ &\quad -2^{[(n+2)/4]+2[(n-2)/8]}X_1, \\ X_6(n) &= 2^{[(n-4)/8]}X_6+(2^{[n/4]}-2^{2[(n-4)/8]+1})X_2 \\ &\quad -2^{[n/4]+2[(n-4)/8]+1}X_1 \end{aligned}$$

and

$$\begin{aligned} X_7(n) &= 2^{[(n-6)/8]}X_7-(2^{[(n-2)/4]}-2^{2[(n-6)/8]+1})(X_3-2X_2) \\ &\quad +2^{2[(n+2)/8]+[(n-2)/4]}X_1. \end{aligned}$$

(3) Let $\varphi\colon F(x)\to\tilde{K}(L_8^m)$ be the homomorphism defined by setting $\varphi(x_i)=\sigma_i$ $(1\leqq i\leqq 7)$.

Proposition 2.22 (Kobayashi and Sugawara [12]). *The homomorphism* φ *is an epimorphism, and the kernel of* φ *coincides with the subgroup of* $F(x)$ *generated by* $\{X_i(m)\mid 1\leqq i\leqq 7\}$.

For each integer n with $0\leqq n<m$, we denote the inclusion map of L_8^n into

L_8^m by i_n^m, and denote the kernel of the homomorphism

$$(i_n^m)^!: \tilde{K}(L_8^m) \to \tilde{K}(L_8^n)$$

by V_n. Set $u=[(n+1)/2]$ and $S_i=\varphi(X_i(2u))$ $(1\leqq i\leqq 7)$. Then V_{2u} is the subgroup of $\tilde{K}(L_8^m)$ generated by S_i $(1\leqq i\leqq 7)$, and we have

$$\tilde{K}(L_8^m/L_8^n) \approx \begin{cases} V_{2u} & (n\equiv 0 \pmod 2) \\ \boldsymbol{Z}\oplus V_{2u} & (n\equiv 1 \pmod 2)\,. \end{cases}$$

According to [1], we have the following lemma.

Lemma 2.23. *The Adams operations are given by the following formulae, where* $s_i=\varphi(X_i)$ $(1\leqq i\leqq 7)$ *and* $k\equiv 1 \pmod 2$.

(1) $\psi^k(s_i)=s_i$ $(i=1, 2$ *or* $4)$.

(2) $$\psi^k(s_3)=\begin{cases} s_3 & (k\equiv 1 \pmod 4) \\ -s_3-2s_2 & (k\equiv 3 \pmod 4)\,. \end{cases}$$

(3) $$\psi^k(s_5)=\begin{cases} s_5 & (k\equiv 1 \pmod 8) \\ s_5+s_6 & (k\equiv 3 \pmod 8) \\ -s_5-2s_4 & (k\equiv 5 \pmod 8) \\ -s_5-2s_4-s_6 & (k\equiv 7 \pmod 8)\,. \end{cases}$$

(4) $$\psi^k(s_6)=\begin{cases} s_6 & (k\equiv \pm 1 \pmod 8) \\ -s_6 & (k\equiv \pm 3 \pmod 8)\,. \end{cases}$$

(5) $$\psi^k(s_7)=\begin{cases} s_7 & (k\equiv 1 \pmod 8) \\ -s_7+2s_4 & (k\equiv 3 \pmod 8) \\ s_7-2s_6 & (k\equiv 5 \pmod 8) \\ -s_7+2s_4+2s_6 & (k\equiv 7 \pmod 8)\,. \end{cases}$$

3. Proofs of Theorems

In this section we prove Theorems 1, 2, 3 and 4.

3.1. Proof of Theorems 1 and 3. According to [8], we have

$$h(q,k)=\begin{cases} 2^{h_1(k)} & (q=4) \\ 2^{h_2(k)} & (q=8)\,. \end{cases}$$

Then Theorems 1 and 3 follows from (2.10) for the case $m\geqq n+3$. Note that we have $L_q^{n+1}/L_q^n \approx S^{n+1}$ and

$$L_q^{n+2}/L_q^n \simeq \begin{cases} S^{n+1}\vee S^{n+2} & (n\equiv 1 \pmod 2) \\ S^n L_q^2 & (n\equiv 0 \pmod 2)\,. \end{cases}$$

This completes the proof of Theorems 1 and 3.

3.2. Proof of Theorem 2. The part (1) is obtained by (2.11) (2).

Suppose that the spaces L_4^m/L_4^n and L_4^{m+t}/L_4^{n+t} have the same stable homotopy type, where $\nu_2(n+1)\geqq i$ and $h_1(m-2[(n+1)/2])\geqq i\geqq 3$. Then $m\geqq n+5$ and there exists a homotopy equivalence

$$f\colon S^j(L_4^m/L_4^n)\to S^{j-t}(L_4^{m+t}/L_4^{n+t}),$$

which induces an isomorphism

$$J(f^!)\colon \tilde{J}(S^{j-t}(L_4^{m+t}/L_4^{n+t}))\to\tilde{J}(S^j(L_4^m/L_4^n)). \tag{3.1}$$

We can assume that $\nu_2(j)\geqq h_1(m-2[(n+1)/2])=h_1(m-n-1)$. By (2.11) (3), $t\equiv 0 \pmod 8$. It follows from (2.11) (1) and Proposition 2.15 that we have $\nu_2(t)\geqq[(m-n-1)/2]$,

$$\tilde{J}(S^j(L_4^m/L_4^n))\cong \boldsymbol{Z}/\mathfrak{m}((n+j+1)/2)\cdot 2^{c_1}\oplus\boldsymbol{Z}/2^{d_1+i_1}\oplus\boldsymbol{Z}/2^{k_1}$$

and

$$\tilde{J}(S^{j-t}(L_4^{m+t}/L_4^{n+t}))\cong \boldsymbol{Z}/\mathfrak{m}((n+j+1)/2)\cdot 2^{c_2}\oplus\boldsymbol{Z}/2^{d_2+i_2}\oplus\boldsymbol{Z}/2^{k_2},$$

where $i_1=\min\{\nu_2(n+1)-1,\ h_1(m-n-1)\}$,

$$\begin{aligned}
i_2&=\min\{\nu_2(n+t+1)-1,\ h_1(m-n-1)\},\\
k_1&=\min\{\nu_2(n+1)-1,\ [(m-n-1)/8]+[(m-n+5)/8]\},\\
k_2&=\min\{\nu_2(n+t+1)-1,\ [(m-n-1)/8]+[(m-n+5)/8]\},\\
c_l&=h_1(m-n-1)-i_l \quad (l=1 \text{ or } 2)
\end{aligned}$$

and $d_l=[(m-n-1)/8]+[(m-n+5)/8]-k_l$ ($l=1$ or 2). Since $c_l\geqq d_l\geqq 0$ and $\nu_2(\mathfrak{m}((n+j+1)/2))\geqq i_l\geqq k_l$ ($l=1$ or 2), the isomorphism (3.1) implies that $c_1=c_2$, and hence

$$\min\{\nu_2(n+1)-1,\ h_1(m-n-1)\}=\min\{\nu_2(n+t+1)-1,\ h_1(m-n-1)\}.$$

Since $\nu_2(n+1)\geqq i$, this implies that we have $\nu_2(n+t+1)>i$ if $\nu_2(n+1)>i$ and $\nu_2(n+t+1)=i$ if $\nu_2(n+1)=i$. Thus we have $\nu_2(t)\geqq i+1$. The proof of the part (2) of Theorem 2 is completed by making use of the S-duality (Lemma 2.9).

The part (3) is obtained by (2.11) (3) and the parts (1) and (2) of Theorem 2. This completes the proof of Theorem 2.

3.3. Proof of Theorem 4. Suppose that the spaces L_8^m/L_8^n and L_8^{m+t}/L_8^{n+t} have the same stable homotopy type. Then there exists a homotopy equivalence

$$f\colon S^j(L_8^m/L_8^n)\to S^{j-t}(L_8^{m+t}/L_8^{n+t}),$$

which induces isomorphisms

(3.2) $$f^*\colon \tilde{K}(S^{j-t}(L_8^{m+t}/L_8^{n+t})) \to \tilde{K}(S^j(L_8^m/L_8^n))$$

and

(3.3) $$J(f^!)\colon \tilde{J}(S^{j-t}(L_8^{m+t}/L_8^{n+t})) \to \tilde{J}(S^j(L_8^m/L_8^n)).$$

We can assume that $\nu_2(j) \geqq a_0(m, n)+1$. Suppose $m \geqq n+5$. Then, by (2.11) (3), $t \equiv 0 \pmod 8$. If $n \not\equiv 3 \pmod 4$, then Proposition 2.19 asserts that the exponent of the group $\tilde{J}(S^j(L_8^m/L_8^n))$ is equal to $2^{b_1(j,m,n)}$ and the exponent of the group $\tilde{J}(S^{j-t}(L_8^{m+t}/L_8^{n+t}))$ is equal to $2^{b_1(j-t,m+t,n+t)}$, and the isomorphism (3.3) implies that $b_1(j, m, n)=b_1(j-t, m+t, n+t)$. In the case $n \equiv 3 \pmod 4$, f induces a homotopy equivalence

$$\bar{f}\colon S^j(L_8^m/L_8^{n+1}) \to S^{j-t}(L_8^{m+t}/L_8^{n+t+1}),$$

which induces an isomorphism

$$J(\bar{f}^!)\colon \tilde{J}(S^{j-t}(L_8^{m+t}/L_8^{n+t+1})) \to \tilde{J}(S^j(L_8^m/L_8^{n+1})).$$

Since $b_1(j, m, n)=b_1(j, m, n+1)$ and $b_1(j-t, m+t, n+t)=b_1(j-t, m+t, n+t+1)$, the isomorphism $J(\bar{f}^!)$ implies that $b_1(j, m, n)=b_1(j-t, m+t, n+t)$. In either case, we have $b_1(j, m, n)=b_1(j-t, m+t, n+t)$. Hence $\nu_2(j-t)-a_3(n, n+4) \geqq a_5(m, n)$, and

(3.4) $$\nu_2(t) \geqq \begin{cases} a_5(m, n)-2 & (n \equiv 0,\ 1 \text{ or } 7 \pmod 8) \\ a_5(m, n)-1 & (\textit{otherwise}). \end{cases}$$

By Lemma 2.9, (2.11) (2) and (3.4), the part (1) of Theorem 4 is obtained except for the case $n \equiv m-n \equiv 2 \pmod 8$. So, assume that $n \equiv m-2 \equiv 2 \pmod 8$. Let Y_i be the element of $\tilde{K}(S^j(L_8^m/L_8^n))$, which corresponds to $I^{j/2}(S_i)$ by the isomorphism

$$(p_n^m)^!\colon \tilde{K}(S^j(L_8^m/L_8^n)) \xrightarrow{\cong} I^{j/2}(V_n),$$

where I denotes the Bott periodicity isomorphism $(1 \leqq i \leqq 7)$. Set $u_1 = Y_6$,

$$\begin{aligned}
u_2 &= Y_4+Y_2+2^{(m-n-10)/8}(2Y_2+Y_1)+2^{(3m-3n-6)/8}\,Y_1,\\
u_3 &= Y_7-Y_2-Y_1+2^{(m-n-10)/8}(2Y_2+Y_1-Y_3)+2^{(3m-3n-14)/8}\,Y_1\\
&\quad -2^{(2m-2n-12)/8}\,Y_1,\\
u_4 &= 2Y_4-Y_1+2^{(m-n-2)/8}(2Y_2+Y_1)+2^{(3m-3n+2)/8}\,Y_1,\\
u_5 &= Y_5+Y_1-2^{(3m-3n-6)/8}\,Y_1,\\
u_6 &= 2Y_6-Y_2
\end{aligned}$$

and $u_7=2Y_7+Y_3-2Y_2-2Y_1+2^{(m-n-2)/8}(2Y_2+Y_1-Y_3)+2^{(3m-3n-6)/8}\,Y_1$. Then, by Proposition 2.22, we have

$$\tilde{K}(S^j(L_8^m/L_8^n))\cong\langle\{u_i\,|\,1\leqq i\leqq 7\}\rangle/\langle\{2^{a(i)}\,u_i\,|\,1\leqq i\leqq 7\}\rangle,$$

where
$$a(1)=(m-n+4)/2,$$
$$a(2)=a(3)=(m-n+2)/4,$$
$$a(5)=a(6)=(m-n-2)/8$$
and $a(4)=a(7)=(m-n-10)/8$. According to Lemma 2.23, the Adams operation ψ^5 is given by the following formulae.

$$(3.5)\quad\begin{cases}(1)\quad \psi^5(u_1)=5^{j/2}(u_1-u_6)\,.\\(2)\quad \psi^5(u_2)=5^{j/2}\,u_2\,.\\(3)\quad \psi^5(u_3)=5^{j/2}(u_3-u_6)\,.\\(4)\quad \psi^5(u_4)=5^{j/2}\,u_4\,.\\(5)\quad \psi^5(u_5)=5^{j/2}(2^{(m-n-2)/8}(2u_2-u_4)-u_4-u_5)\,.\\(6)\quad \psi^5(u_6)=5^{j/2}(-u_6)\,.\\(7)\quad \psi^5(u_7)=5^{j/2}(u_7-2u_6)\,.\end{cases}$$

Choose $v_i\in\tilde{K}(S^{j-t}(L_8^{m+t}/L_8^{n+t}))$ similarly as $u_i\in\tilde{K}(S^j(L_8^m/L_8^n))$ $(1\leqq i\leqq 7)$, and set

$$f^*(v_i)=\sum_{k=1}^7 a_{ik}\,u_k\quad(1\leqq i\leqq 7)\,.$$

By the equality $\psi^5\circ f^*=f^*\circ\psi^5$, we have

$$(3.6)\qquad 5^{j/2}\,a_{11}\equiv 5^{(j-t)/2}(a_{11}-a_{61})\pmod{2^{a(1)}}\,.$$

$$(3.7)\qquad 5^{j/2}\,a_{31}\equiv 5^{(j-t)/2}(a_{31}-a_{61})\pmod{2^{a(1)}}\,.$$

Since $\nu_2(a_{31})\geqq a(1)-a(3)=(m-n+6)/4$, (3.7) implies that $a_{61}\equiv 0\pmod{2^{a(1)}}$. It follows from (3.6) that we have $(5^{t/2}-1)\,a_{11}\equiv 0\pmod{2^{a(1)}}$. Note that $a_{11}\equiv 1\pmod 2$. According to [15, Lemma 3.1], we see that $\nu_2(t)+1\geqq a(1)=(m-n+4)/2$. Hence

$$\nu_2(t)\geqq(m-n+2)/2=[m/2]-[(n+1)/2]+1\,.$$

This completes the proof of the part (1) of Theorem 4.

Supppse that $\nu_2(n+1)\geqq i$ and $h_2(m-2\,[(n+1)/2])\geqq i\geqq 3$. Then $m\geqq n+5$ and $t\equiv 0\pmod 8$ by (2.11) (3). It follows from (2.11) (1) and Proposition 2.19 that we have $\nu_2(t)\geqq[(m-n-1)/2]$,

$$\tilde{J}(S^j(L_8^m/L_8^n))\cong \boldsymbol{Z}/\mathrm{m}((n+j+1)/2)\cdot 2^{c_1}\oplus\boldsymbol{Z}/2^{c_2+i_1}\oplus\boldsymbol{Z}/2^{i_2}\oplus\boldsymbol{Z}/2^{i_3}$$

and

$$\tilde{J}(S^{j-t}(L_8^{m+t}/L_8^{n+t}))\cong \boldsymbol{Z}/\mathrm{m}((n+j+1)/2)\cdot 2^{d_1}\oplus\boldsymbol{Z}/2^{d_2+k_1}\oplus\boldsymbol{Z}/2^{k_2}\oplus\boldsymbol{Z}/2^{k_3}\,,$$

where $i_1=\min\,\{\nu_2(n+1)-1,\,h_2(m-m-1)\}$,

$$k_1=\min\,\{\nu_2(n+t+1)-1,\,h_2(m-n-1)\}\;,$$

$$i_2 = \min\{\nu_2(n+1)-2,\ [(m-n+3)/8]+[(m-n-3)/8]\}\ ,$$
$$k_2 = \min\{\nu_2(n+t+1)-2,\ [(m-n+3)/8]+[(m-n-3)/8]\}\ ,$$
$$i_3 = k_3 = [(m-n-1)/8]\ ,$$
$$c_1 = h_2(m-n-1)-i_1\ ,$$
$$d_1 = h_2(m-n-1)-k_1\ ,$$
$$c_2 = [(m-n+3)/8]+[(m-n-3)/8]-i_2\ ,$$

and $d_2=[(m-n+3)/8]+[(m-n-3)/8]-k_2$. Since $c_1\geqq c_2\geqq 0$, $c_2+i_1\geqq i_3$,

$$\nu_2(\mathfrak{m}((n+j+1)/2)) \geqq i_1 \geqq i_2\ ,$$
$$\nu_2(\mathfrak{m}((n+j+1)/2))+c_1 \geqq h_2(m-n-1)+1$$

and $\max\{d_2+k_1, k_2, k_3\}\leqq h_2(m-n-1)$, the isomorphism (3.3) implies that $c_1=d_1$, and hence

$$\min\{\nu_2(n+1)-1,\ h_2(m-n-1)\} = \min\{\nu_2(n+t+1)-1,\ h_2(m-n-1)\}\ .$$

Since $\nu_2(n+1)\geqq i$, this implies that we have $\nu_2(n+t+1)>i$ if $\nu_2(n+1)>i$ and $\nu_2(n+t+1)=i$ if $\nu_2(n+1)=i$. Thus we have $\nu_2(t)\geqq i+1$. The proof of the part (2) of Theorem 4 is completed by making use of the S-duality (Lemma 2.9).

By (2.11) (3) and the parts (1) and (2) of Theorem 4, the part (3) of Theorem 4 is obtained except for the case $n\equiv 0 \pmod 8$ and $m=n+6$. So, assume that $n\equiv 0 \pmod 8$ and $m=n+6$. Let Y_i be the element of $\tilde{K}(S^j(L_8^m/L_8^n))$, which corresponds to $I^{j/2}(S_i)$ by the isomorphism

$$(p_n^m)^!:\ \tilde{K}(S^j(L_8^m/L_8^n)) \xrightarrow{\cong} I^{j/2}(V_n)$$

$(1\leqq i\leqq 7)$. Set $u_1=Y_5$, $u_2=Y_6+2Y_5$ and $u_3=Y_7+2Y_5$. Then, by Proposition 2.22, we have

$$\tilde{K}(S^j(L_8^m/L_8^n)) \cong \langle\{u_i\,|\,1\leqq i\leqq 3\}\rangle/\langle\{2^{a(i)}u_i\,|\,1\leqq i\leqq 3\}\rangle\ ,$$

where $a(1)=32$ and $a(2)=a(3)=4$. According to Lemma 2.23, we have the following formulae.

$$(3.8)\qquad \begin{cases} (1) & \psi^{-1}(u_1) = 5u_1+u_2\ . \\ (2) & \psi^5(u_1) = 5^{j/2}(17u_1+2u_2)\ . \\ (3) & \psi^{-1}(u_2) = 8u_1-u_2\ . \\ (4) & \psi^5(u_2) = 5^{j/2}(16u_1+u_2)\ . \\ (5) & \psi^{-1}(u_3) = -8u_1-u_3\ . \\ (6) & \psi^5(u_3) = 5^{j/2}(16u_1+u_3)\ . \end{cases}$$

Choose $v_i\in\tilde{K}(S^{j-t}(L_8^{m+t}/L_8^{n+t}))$ similarly as $u_i\in\tilde{K}(S^j(L_8^m/L_8^n))$ $(1\leqq i\leqq 3)$, and set

$$f^*(v_i)=\sum_{k=1}^{3} a_{ik}u_k \quad (1\leqq i\leqq 3)\,.$$

By the equality $\psi^{-1}\circ f^*=f^*\circ\psi^{-1}$, we have

$$8a_{12}-8a_{13}\equiv a_{21} \pmod{32}. \tag{3.9}$$

By the equality $\psi^5\circ f^*=f^*\circ\psi^5$, we have

$$5^{j/2}(17a_{11}+16a_{12}+16a_{13})\equiv 5^{(j-t)/2}(17a_{11}+2a_{21}) \pmod{32}. \tag{3.10}$$

By (3.9), (3.10) and the fact $5^{t/2}\equiv 1 \pmod{2^{\nu_2(t)+1}}$, we have

$$(5^{t/2}-1)\,a_{11}\equiv 0 \pmod{32}.$$

Note that $a_{11}\equiv 1 \pmod 2$. According to [15, Lemma 3.1], we see that $\nu_2(t)+1\geqq 5$. Hence

$$\nu_2(t)\geqq 4=[m/2]-[(n+1)/2]+1\,.$$

This completes the proof of Theorem 4.

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Department of Mathematics
Osaka University
Toyonaka, Osaka 560, Japan