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POLYNOMIAL BOUNDS ON THE NUMBER OF SCATTERING POLES FOR METRIC PERTURBATIONS OF THE LAPLACIAN IN R^n, $n \geq 3$, ODD

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1. Introduction The purpose of this note is to obtain a polynomial bound on the number of the scattering poles associated to the operator

$$G = c(x)^{-1} \sum_{1 \leq i, j \leq n} \partial_{x_i}(g_{ij}(x)\partial_{x_j}) \qquad \text{in} \quad \boldsymbol{R}^n,$$

where $n \geq 3$, odd. We consider this operator under the following assumptions on the coefficients:

(a) $c(x) \in C(\boldsymbol{R}^n; \boldsymbol{R})$ and $c(x) \geq c_0 > 0$ for all $x \in \boldsymbol{R}^n$;

(b) $g_{ij}(x) \in C^1(\boldsymbol{R}^n)$ and the matrix $\{g_{ij}(x)\}$ is a strictly positive hermitian one for all $x \in \boldsymbol{R}^n$, i.e.

(b)$_1$ $g_{ij}(x) = \overline{g_{ji}(x)}, \qquad i, j = 1, \cdots, n, \ \forall x \in \boldsymbol{R}^n$;

(b)$_2$ $\sum_{i \leq j, j \leq n} g_{ij}(x)\xi_i\xi_j \geq C|\xi|^2, \ \forall (x, \xi) \in \boldsymbol{R}^n \times \boldsymbol{R}^n \backslash 0, \ C > 0$;

(c) there exists a $\rho_0 > 0$ so that $c(x) = 1$ and $g_{ij}(x) = \delta_{ij}$ for $|x| \geq \rho_0$, δ_{ij} being the Kronecker's symbol.

It is well known that under the above assumptions the operator G has a self-adjoint realization, which will be again denoted by G, in the Hilbert space $H = L^2(\boldsymbol{R}^n; c(x)dx)$. Note also that it follows from the assumption (b)$_2$ that the operator G is elliptic. By G_0 we shall denote the self-adjoint realization of the Laplacian Δ in the Hilbert space $H_0 = L^2(\boldsymbol{R}^n)$.

It is well known that, under the above assumptions, the scattering matrix corresponding to the pair $\{G, G_0\}$ admits a meromorphic continuation to the entire complex plane $\boldsymbol{C}$. Let $\{\lambda_j\}$ be the poles of this continuation, repeating according to multiplicity, and set

$$N(r) = \#\{\lambda_j : |\lambda_j| \leq r\}.$$

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It is shown in [10] that for any $\varepsilon>0$ there exists a constant $C_\varepsilon>0$ so that

$$M(r)\leq C_\varepsilon r^{n+1+\varepsilon}+C_\varepsilon\,. \tag{1}$$

In fact, (1) is proved there for first order symmetric systems in $\boldsymbol{R}^n$, $n\geq 3$, odd, but the proof easily extends to our case. The aim of this work is to improve (1). More precisely, we have the following

Theorem. *There exists a constant $C>0$ so that the number of the scattering poles associated to the operator G satisfies the bound*

$$N(r)\leq C\,r^{n+1}+C\,. \tag{2}$$

Note that the desired result is to obtain the bound

$$N(r)\leq C\,r^{n}+C\,. \tag{3}$$

In [7] Melrose proved (3) for the Laplacian in exterior domains with Dirichlet or Robin boundary conditions, while in [13] Zworski proved (3) for the Schrödinger operator $\Delta+V(x)$ with a potential $V\in L_0^\infty(\boldsymbol{R}^n)$. Note that a bound of the form (3) is important for obtaining Weyl asymptotics of the phase shift (see [8], where such an asymptotic is obtained for an arbitrary smooth obstacle). To our knowledge, the problem of obtaining a bound of the form (3) on the number of the scattering poles associated to the operator G is still open. Let us also mention Intissar's work [4] where a bound of the form (2) on the number of the scattering poles associated to the operator $(-i\vec{\nabla}+\vec{b}(x))^2+a(x)$ in $\boldsymbol{R}^n$, $n\geq 3$, odd, is obtained, where $a(x)\in C_0^\infty(\boldsymbol{R}^n;\boldsymbol{R})$, $\vec{b}(x)\in C_0^\infty(\boldsymbol{R}^n;\boldsymbol{R})$. In [3] and [4] he has also obtained an analogue of (2) in the case of even n, $n\geq 4$.

To prove (2) we exploit the fact that the scattering poles, with multiplicity, coincide with the poles of the meromorphic continuation of the cutoff resolvent $R_\chi(z)=\chi R(z)\chi$ from $\{z\in\boldsymbol{C}\colon \operatorname{Im} z>0\}$ to the entire complex plane $\boldsymbol{C}$, where $R(z)=(G+z^2)^{-1}$ for $\operatorname{Im} z>0$, and $\chi\in C_0^\infty(\boldsymbol{R}^n)$ is such that $\chi=1$ for $|x|\leq\rho_0+1$, $\chi=0$ for $|x|\geq\rho_0+2$ (see [5]). This enables us to characterize the scattering poles as the poles of a meromorphic function of the form $(1-K(z))^{-1}$ where $K(z)$ is an entire family of compact operators on H such that $K(z)^p$, $p=(n+1)/2$, is trace class. Then, following [4], [6], [7] and [13], we deduce that the scattering poles, with multiplicity, are among the zeros of the entire function $h(z)=\det(1-K(z)^p)$, and hence, to prove (2) it suffices to show that the order of $h(z)$ is less than or equal to $n+1$. However, in our case the operator $K(z)$ is much more complicated and therefore it is not so easy to obtain the desired order of $h(z)$. To overcome the difficulties we use precise estimates of the cutoff free resolvent for $\operatorname{Im} z\geq 0$ combained with an application of the Phragmen-Lindelof principle.

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2. Representation of the cutoff resolvent

First, we shall introduce some notations. Given two Hilbert spaces X and Y, $\mathcal{L}(X,Y)$ will denote the space of all linear bounded operators acting from X into Y. Given any $s>0$, H^s will denote the usual Sobolev space $H^s(\boldsymbol{R}^n)$. Finally, given a compact operator $\mathcal{A}$, $\mu_j(\mathcal{A})$ will denote the characteristic values of $\mathcal{A}$, i.e. the eigenvalues of $(\mathcal{A}^*\mathcal{A})^{1/2}$, ordered, with multiplicity, to form a nonincreasing sequence.

Denote by $R_0(z)$ the outgoing resolvent of Δ, i.e. that one with kernel $E(x-y,z)$ where $E(x,z)$ is the outgoing fundamental solution of the operator $\Delta+z^2$. Then we have $R_0(z)=(G_0+z^2)^{-1}\in\mathcal{L}(H_0, H_0)$ for $\operatorname{Im} z>0$. Moreover, it is well known that the kernel of $R_0(z)$ is given in terms of Hankel's functions by

$$R_0(z)(x,y) = -(i/4)(2\pi)^{(n-2)/2}(z/|x-y|)^{(n-2)/2}H^{(1)}_{(n-2)/2}(z|x-y|)\,. \tag{4}$$

It follows easily from this representation that $\chi R_0(z)\chi$ forms an entire family of compact pseudodifferential operators of order -2 in $\mathcal{L}(H_0, H_0)$, χ being the function introduced in the previous section. Using this we shall build the meromorphic continuation of $R_\chi(z)$. Set $Q=G_0-G$ and fix a $z_0\in\boldsymbol{C}$, $\operatorname{Im} z_0>0$. Clearly, for $\operatorname{Im} z>0$, we have

$$R(z) = R_0(z)+R(z)QR_0(z) \tag{5}$$

and

$$R(z) = R(z_0)+(z_0^2-z^2)R(z)R(z_0)\,. \tag{6}$$

Combining these identities yields

$$R(z)(1-(z_0^2-z^2)QR_0(z)R(z_0)) = R(z_0)+(z_0^2-z^2)R_0(z)R(z_0)$$

for $\operatorname{Im} z>0$. Multiplying the both sides of this identity by χ, since $Q=\chi Q$, we get

$$R_\chi(z)(1-K(z)) = R_\chi(z_0)+K_1(z) \qquad \text{for} \quad \operatorname{Im} z>0\,, \tag{7}$$

where $K(z)=(z_0^2-z^2)QR_0(z)R(z_0)\chi$ and $K_1(z)=(z_0^2-z^2)\chi R_0(z)R(z_0)\chi$. We need now the following

Lemma 1. *The operator-valued functions $K(z)$ and $K_1(z)$ have analytic continuations from $\{z\in\boldsymbol{C}: \operatorname{Im} z>0\}$ to the entire $\boldsymbol{C}$ with values in the compact operators in $\mathcal{L}(H,H)$. Moreover, there exists a constant $C>0$ so that*

$$\mu_j(K(z))\leq C\exp(C|z|),\ \forall z\in\boldsymbol{C},\ \forall j; \tag{8}$$

$$\mu_j(K(z))\leq C(1+|z|)^3(1+|\operatorname{Im} z|)^{-1}j^{-2/n},\ \forall z\in\boldsymbol{C},\ \text{if } j\geq C(1+|z|)^n\,. \tag{9}$$

Assuming that the conclusions of Lemma 1 are fulfilled, we shall complete the proof of (2). Obviously, $1-K(z)$ is invertable in $\mathcal{L}(H,H)$ at $z=z_0$, and since $K(z)$ is an entire family of compact operators, $(1-K(z))^{-1}$ is a meromorphic $\mathcal{L}(H,H)$-valued function on $\boldsymbol{C}$. By (7) we deduce that so is true for $R_\chi(z)$ and the poles of $R_\chi(z)$, with multiplicity, are among the poles of $(1-(K(z))^{-1}$, and hence among the poles of $(1-K(z)^p)^{-1}$ where $p=(n+1)/2$. On the other hand, it follows from (9) and the well known inequality

$$(10) \qquad \mu_{pj}(K(z)^p)\leq\mu_j(K(z))^p \qquad \forall j\,,$$

that $K(z)^p$ is trace class for all $z\in\boldsymbol{C}$. Hence we can introduce the entire function

$$h(z)=\det(1-K(z)^p)$$

and conclude that the poles of $R_\chi(z)$, with multiplicity, are among the zeros of $h(z)$. Hence, (2) will be proved if we show that

$$(11) \qquad |h(z)|\leq C\exp(C|z|^{n+1}) \qquad \forall z\in\boldsymbol{C}\,.$$

By (8), (9) and (10) we obtain with some constant $C'>0$:

$$\mu_j(K(z)^p)\leq C'\exp(C'|z|) \qquad \forall z\in\boldsymbol{C},\ \forall j\,,$$

and

$$\mu_j(K(z)^p)\leq C'(1+|z|)^{3p}(1+|\mathrm{Im}\,z|)^{-p}j^{-(n+1)/n} \qquad \forall z\in\boldsymbol{C}\,,$$

if $j\geq C'(1+|z|)^n$. Now by Weyl's convexity estimate we get

$$\begin{aligned}|h(z)|&\leq\prod_{j=1}^{\infty}(1+\mu_j(K(z)^p))\\&\leq\Big(\prod_{j\leq C'(1+|z|)^n}C'\exp(C'|z|)\Big)\exp\Big(\sum_{j\geq C'(1+|z|)^n}\mu_j(K(z)^p)\Big)\\&\leq\exp(C''(1+|z|)^{n+1})\exp\Big(C'(1+|z|)^{3p}(1+|\mathrm{Im}\,z|)^{-p}\sum_{j=1}^{\infty}j^{-(n+1)/n}\Big).\end{aligned}$$

Thus we have obtained the estimate

$$(12) \qquad |h(z)|\leq C\exp(C(1+|z|)^{3p}(1+|\mathrm{Im}\,z|)^{-p}) \qquad \forall z\in\boldsymbol{C}\,,$$

with some constant $C>0$. We shall show that this estimate implies (11). Clearly, by (12) we have

$$(13) \qquad |h(z)|\leq C\exp(C|z|^{3p}) \qquad \forall z\in\boldsymbol{C}\,,$$

with possibly a greater constant $C>0$. Introduce the sets $S^\pm=\{z\in\boldsymbol{C}\colon |\mathrm{Im}\,z|<\pm\gamma\,\mathrm{Re}\,z\}$, where $\gamma=\mathrm{tg}(\pi/8p)$, and set $S=S^+\cup S^-$. It is easy to see that

$$(14) \qquad (1+|z|)(1+|\mathrm{Im}\,z|)^{-1}\leq 1+\gamma^{-1} \qquad \forall z\in\boldsymbol{C}\backslash S\,.$$

Thus, by (12) and (14), we obtain

$$|h(z)| \leq C \exp(C|z|^{n+1}) \qquad \forall z \in \boldsymbol{C} \backslash S\,. \tag{15}$$

Now we are going to show that such an estimate holds on S. To this end we need the following fundamental lemma (for the proof, see [9]).

Lemma 2. *Let* $\Lambda = \{z \in \boldsymbol{C} : \theta_1 < \arg z < \theta_2\}$ *and let* $\alpha < 1/2$ *be such that* $\theta_2 - \theta_1 = \pi/\alpha$. *Let the function* $f(z)$ *be holomorphic in a neighbourhood of* Λ *and satisfy the conditions:*

(i) $$|f(z)| \leq M \qquad \textit{for} \quad \arg z = \theta_k,\ k = 1,2\,;$$

(ii) $$|f(z)| \leq M'(\exp(M'|z|^{\beta}) \qquad z\,,$$

with a β *such that* $0 < \beta < \alpha$.
Then

$$|f(z)| \leq M \qquad \forall z \in \Lambda\,.$$

Clearly, $S^+ = \{z \in \boldsymbol{C} : -\pi/8p < \arg z < \pi/8p\}$. Introduce the function $f(z) = h(z)\exp(q\,z^{2p})$ where $q \in \boldsymbol{R}$ is a parameter to be chosen later on. In view of (13) we have

$$|f(z)| \leq C' \exp(C'|z|^{2p}) \qquad \forall z \in \boldsymbol{C}\,. \tag{16}$$

Writing $z = r\,e^{i\varphi}$, $r = |z|$, $\varphi = \arg z$, for $\varphi = \pm\pi/8p$, in view of (15), we have

$$\begin{aligned} |f(z)| &\leq C \exp(C\,r^{2p})\,|\exp(q\,r^{2p}\,e^{i2p}\varphi)| \\ &= C \exp(C\,r^{2p}) \exp(q\,r^{2p} \cos(\pi/4))\,, \end{aligned}$$

and taking $q = -C/\cos(\pi/4)$, we deduce

$$|f(z)| \leq C \qquad \text{for} \quad \arg z = \pm\pi/8p\,. \tag{17}$$

Now, in view of (16) and (17), we can apply Lemma 2 with $\Lambda = S^+$, $\alpha = 4p$, $\beta = 3p$, to conclude that

$$|f(z)| \leq C \qquad \forall z \in S^+\,,$$

which in turn yields

$$|h(z)| \leq C\ |\exp(-q\,z^{2p})| \leq C \exp(C'|z|^{2p}) \qquad \forall z \in S^+\,.$$

Similarly, so is true for all $z \in S^-$, and hence for all $z \in S$. This together with (15) imply (11) since $2p = n+1$.

3. Proof of Lemma 1

Since $R(z_0) = R_0(z_0) + R_0(z_0)QR(z_0)$, we have

$$K(z)=(z_0^2-z^2)QR_0(z)R_0(z_0)\chi K_2 \quad \text{for} \quad \operatorname{Im} z>0, \tag{18}$$

where $K_2=1+QR(z_0)\chi$. Since G is an elliptic second order differential operator, we have $R(z_0)\in\mathcal{L}(H,H^2)$, and hence $K_2\in\mathcal{L}(H,H)$. Choose functions χ_1, $\chi_2\in C_0^\infty(\boldsymbol{R}^n)$ such that $\chi_1=1$ on supp Q, $\chi_2=1$ on supp χ_1 and $\chi=1$ on supp χ_2. Now, using (5) with $R_0(z)$, after an easy computation, we obtain from (18):

$$K_3(z)=(K+(z_0^2-z^2)K_4)\chi R_0(z)\chi K_2+K_5 \quad \text{for} \quad \operatorname{Im} z>0, \tag{19}$$

where

$$K_3=QR_0(z_0)[G_0,\chi_1]R_0(z_0)[G_0,\chi_2],$$

$$K_4=QR_0(z_0)\chi_1+QR_0(z_0)[G_0,\chi_1]R_0(z_0)\chi_2,$$

$$K_5=-K_3R_0(z_0)\chi K_2.$$

Here $[\ ,\ ]$ stands for the commutator. Clearly, $K_3, K_4\in\mathcal{L}(H,H)$ and $K_5\in\mathcal{L}(H,H^2)$. Hence K_5 is a compact operator in $\mathcal{L}(H,H)$. Furthermore, as mentioned above, $\chi R_0(z)\chi$ is an entire family of compact operators in $\mathcal{L}(H_0,H_0)$, and hence, by (19), $K(z)$ can be continued analytically to the entire $\boldsymbol{C}$ with values in the compact operators in $\mathcal{L}(H,H)$. Clearly, so is true for $K_1(z)$. To prove (8) and (9) we need the following

Lemma 3. *There exists a constant* $C>0$ *so that*

$$\|\chi R_0(z)\chi\|_{\mathcal{L}(H_0,H_0)}\le C\exp(C|z|) \qquad \forall z\in\boldsymbol{C}; \tag{20}$$

$$\|\chi R_0(z)\chi\|_{\mathcal{L}(H_0,H_0)}\le C(1+|z|)^{-1}(1+\operatorname{Im} z)^{-1} \quad \textit{for} \quad \operatorname{Im} z\ge 0; \tag{21}$$

$$\|\chi R_0(z)\chi\|_{\mathcal{L}(H_0,H^2)}\le C(1+|z|)(1+\operatorname{Im} z)^{-1} \quad \textit{for} \quad \operatorname{Im} z\ge 0. \tag{22}$$

Assume for a moment that the conclusions of Lemma 3 are fulfilled. Now (8) immediately follows from (19), (20) and the well known inequality $\mu_j(\mathcal{A})\le\|\mathcal{A}\|$, $\forall j$. Turn to the proof of (9). Set $B=\{x\in\boldsymbol{R}^n\colon |x|\le\rho_0+3\}$ and denote by Δ_B the self-adjoint realization of the Laplacian Δ with domain $D(\Delta)=C_0^\infty(B)$ in the Hilbert space $L^2(B)$. It is well known that

$$\mu_j((1-\Delta_B)^{-m})\le C_B^m\cdot j^{-2m/n},\ \forall j,\ \forall \text{integer } m\ge 1. \tag{23}$$

First, we shall prove (9) for $\operatorname{Im} z\ge 0$. Using the well known inequalities

$$\mu_{2j+1}(\mathcal{A}+\mathcal{B})\le\mu_j(\mathcal{A})+\mu_j(\mathcal{B}) \qquad \forall j, \tag{24}$$

and

$$\mu_j(\mathcal{A}\mathcal{B})\le\begin{cases}\|\mathcal{A}\|\ \mu_j(\mathcal{B}),\ \forall j,\\ \|\mathcal{B}\|\ \mu_j(\mathcal{A}),\ \forall j,\end{cases} \tag{25}$$

by (19), we obtain

$$\begin{aligned}\mu_{2j+1}((K(z))&\leq \mu_j(K_5)+\|K_3+(z_0^2-z^2)K_4\|\mu_j(\chi R_0(z)\chi)\|K_2\|\\&\leq C\mu_j(\chi R_0(z_0)\chi)+C(1+|z|^2)(\mu_j(\chi R_0(z)\chi)\,,\end{aligned}$$

where $\|\cdot\|$ is the norm in $\mathcal{L}(H_0,H_0)$. On the other hand, by (22), (23) and (25) we have

$$\begin{aligned}\mu_j(\chi R_0(z)\chi)&\leq \mu_j((1-\Delta_B)^{-1})\|(1-\Delta)\chi R_0(z)\chi\|_{\mathcal{L}(H_0,H_0)}\\&\leq Cj^{-2/n}(1+|z|)(1+\mathrm{Im}\,z)^{-1}\qquad\text{for}\quad \mathrm{Im}\,z\geq 0,\ \forall j\,.\end{aligned}$$

Now, in this case, (9) follows from the above estimates at once.

Turn to the proof of (9) for $\mathrm{Im}\,z\leq 0$. By (24) we have

$$\mu_{2j+1}(K(z))\leq \mu_j(K(-z))+\mu_j(\tilde{K}(z))\,, \tag{26}$$

where $\tilde{K}(z)=K(z)-K(-z)$. We have already seen above that, when $\mathrm{Im}\,z\leq 0$, $\mu_j(K(-z))$ has the desired bound. To estimate the other term we shall proceed as in [13]. Set $\tilde{R}_0(z)=R_0(z)-R_0(-z)$. It follows from (4) that the kernel of $\tilde{R}_0(z)$ is given by

$$\tilde{R}_0(z)(x,y)=(i/2)(2\pi)^{-n+1}z^{n-2}\int_{S^{n-1}}e^{iz\langle x-y,w\rangle}dw\,,$$

where S^{n-1} denotes the unit sphere in $\boldsymbol{R}^n$. Now it is easy to see that for any multiindex α we have

$$\sup_{\substack{|x|\leq\rho_0+3\\|y|\leq\rho_0+3}}|\partial_x^\alpha \tilde{R}_0(z)(x,y)|\leq C^{|\alpha|+1}|z|^{|\alpha|}e^{C|z|} \tag{27}$$

with some constant $C>0$. By Theorem 1.4.2 of [2], for any integer $m\geq 1$ there exists a function $\chi_m\in C_0^\infty(\boldsymbol{R}^n)$ such that $\chi_m=1$ on supp χ, $\chi_m=0$ for $|x|\geq\rho_0+3$ and $|\partial_x^\alpha\chi_m|\leq C^{|\alpha|+1}|\alpha|!$ for $|\alpha|\leq 2m$ with some constant $C>0$. Using this together with (19), (23), (25) and (27), we get

$$\begin{aligned}\mu_j(\tilde{K}(z))&\leq\|(K_3+(z_0^2-z^2)K_4)\chi\|\mu_j(\chi_m\tilde{R}_0(z)\chi)\|K_2\|\\&\leq C'(1+|z|^2)\mu_j((1-\Delta_B)^{-m})\|(1-\Delta)^m\chi_m\tilde{R}_0(z)\chi\|\\&\leq C''(1+|z|^2)\mu_j((1-\Delta_B)^{-m})\sup_{x,y}|(1-\Delta_x)^m(\chi_m(x)\tilde{R}_0(z)(x,y)\chi(y))|\\&\leq C^{2m+1}(|z|^{2m}+(2m)^{2m})e^{C|z|}j^{-2m/n},\ \forall z\in\boldsymbol{C},\ \forall j,\ \forall m\geq 1\,,\end{aligned}$$

with some constant $C>0$. Here $\|\cdot\|$ again denotes the norm in $\mathcal{L}(H_0,H_0)$. Now, taking $2m=|z|$ we can easily arrange for $j\geq q|z|^n$, $|z|\gg 1$, with large q depending only on C, that

$$\mu_j(\tilde{K}(z))\leq Cj^{-2/n}$$

with possibly another constant $C>0$. Now, in this case, (9) follows from this estimate, (26) and the estimate of $\mu_j(K(-z))$ obtained above.

4. Proof of Lemma 3

First, we shall derive (22) from (21). Choose functions $\chi_1, \chi_2 \in C_0^\infty(\boldsymbol{R}^n)$ such that $\chi_1 = 1$ on supp χ, $\chi_2 = 1$ on supp χ_1. As above, we have

$$\chi R_0(z)\chi = R_0(z_0)A(z) \tag{28}$$

where

$$A(z) = ((z_0^2 - z^2)(\chi + [G_0, \chi]R_0(z_0)\chi_1) + \\ + [G_0, \chi]R_0(z_0)[G_0, \chi_1]\chi_2 R_0(z)\chi + [G_0, \chi]R_0(z_0)\chi_1 + \chi^2 .$$

Since $R_0(z_0) \in \mathcal{L}(H_0, H^2)$, clearly $A(z) \in \mathcal{L}(H_0, H_0)$ and by (28) we get

$$\begin{aligned} \|\chi R_0(z)\chi\|_{\mathcal{L}(H_0, H^2)} &\leq C\|A(z)\|_{\mathcal{L}(H_0, H_0)} \\ &\leq C'(1+|z|^2)\|\chi_2 R_0(z)\chi_2\|_{\mathcal{L}(H_0, H_0)} + C' \\ &\leq C''(1+|z|)(1+\mathrm{Im}\, z)^{-1} \qquad \text{for} \quad \mathrm{Im}\, z \geq 0 , \end{aligned}$$

provided (21) is fulfilled with χ replaced by χ_2.

To prove (20) and (21) we shall exploit the following resolvent formula

$$R_0(z)f = \int_0^\infty e^{itz} U_0(t) f\, dt \qquad \text{for} \quad \mathrm{Im}\, z > 0, f \in H_0 , \tag{29}$$

where $U_0(t)$ is the propagator of the Cauchy problem

$$\begin{cases} (\partial_t^2 - \Delta)U_0(t)f(x) = 0 & \text{in} \quad \boldsymbol{R}_t \times \boldsymbol{R}_x^n , \\ U_0(0)f(x) = 0, \quad \partial_t U_0(0)f(x) = f(x) . \end{cases}$$

It is easy to see that

$$\|\partial_t U_0(t) f\|_{H_0} \leq \|f\|_{H_0} \qquad \forall f \in H_0, \quad \forall t . \tag{30}$$

Now, integrating by parts in (29) and using that by Huygens' principle there exists a $T>0$ so that $\chi U_0(t)\chi = 0$ for $t \geq T$, we get

$$z\chi R_0(z)\chi f = i\int_0^T e^{itz}\chi\partial_t U_0(t)\chi f\, dt \qquad \text{for} \quad \mathrm{Im}\, z > 0, \quad f \in H_0 , \tag{31}$$

which clearly extends analytically to the entire complex plane $\boldsymbol{C}$. By (30) and (31) we have

$$\|z\chi R_0(z)\chi\|_{\mathcal{L}(H_0, H_0)} \leq C\int_0^T e^{-t\,\mathrm{Im}\, z} dt \leq \begin{cases} C\,T e^{T|z|}, & \forall z \in \boldsymbol{C} , \\ C\,T & \text{for} \quad \mathrm{Im}\, z \geq 0 , \\ C(\mathrm{Im}\, z)^{-1} & \text{for} \quad \mathrm{Im}\, z \geq 1 . \end{cases}$$

Now (20) and (21) follow from these estimates at once.

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