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ON CERTAIN SEMIGROUPS OF LINEAR OPERATORS AND GENERALIZED CAUCHY PROBLEM

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Introduction

The aim of this paper is to study certain semigroups of linear operators in certain locally convex spaces and to apply them to solve ultraparabolic systems like

$$(1)\qquad \begin{aligned} &\sum_{j=1}^{m} b_j(t)\frac{\partial u}{\partial t_j}-A(x,\partial)u=f(t,x)\,, \qquad t\in O,\ x\in \boldsymbol{R}^n \\ &u|_{\partial O\times \boldsymbol{R}^n}=u_0\,. \end{aligned}$$

Here O is an open subset of $\boldsymbol{R}^m$, ∂O is the boundary of O, u_0 is a function defined on $\partial O\times \boldsymbol{R}^n$ with values in $\boldsymbol{C}^N$, $A(x,\partial)$ is an $N\times N$ elliptic system. The work is arranged as follows: the first paragraph contains the study of the general $N\times N$ system, parabolic in the sense of Petrovskii,

$$(2)\qquad \begin{aligned} &\partial_t u-A(x,\partial)u=f(t,x)\,, \qquad t>0,\ x\in \boldsymbol{R}^n \\ &u(0,x)=u_0(x)\,, \qquad x\in \boldsymbol{R}^n \end{aligned}$$

in certain spaces of continuous functions which are rather natural for this kind of problem, from the point of view of semigroup theory. An analogous study is developed in the second paragraph in the spaces $\mathcal{S}(\boldsymbol{R}^n)^N$ and $\mathcal{S}'(\boldsymbol{R}^n)^N$, with results of generation of semigroups, quasi-equicontinuity, holomorphy. The third and final paragraph is an approach to an abstract version of problem (1) and together with the results of the two first carries easily to applications to concrete equations in spaces of functions or distributions.

1. Parabolic systems in K_r spaces

The aim of this paragraph is to study systems which are parabolic in the sense of Petrovskii in the framework of certain spaces we shall call K_r. It will be proved that a convenient realization of the system $A(x,\partial)$ in a suitable K_r space is the infinitesimal generator of a C_0-semigroup of linear continuous operators. We start by stating the assumptions we shall work with. Let n, N, $p\in\boldsymbol{N}$, $x\in\boldsymbol{R}^n$, for $|\alpha|\leq 2p$ $A_\alpha(x)$ an $N\times N$ matrix. Consider the differential operator $\sum_{|\alpha|\leq 2p} A_\alpha(x)\partial_x^\alpha$. For $r\in\boldsymbol{N}\cup\{0, +\infty\}$ let $B^r(\boldsymbol{R}^n)$ be the space of complex valued functions on $\boldsymbol{R}^n$ which are of class C^r with all the derivatives of order not exceeding r bounded. We shall assume:

(h1) the coefficients of $A_\alpha(x)$ are in $B^{|\alpha|}(\boldsymbol{R}^n)$ and have uniformly Hölder continuous derivatives (of exponent $\mu\in]0, 1[$) of order $|\alpha|$.

(h2) The system $L=\partial_t-\sum_{|\alpha|\leq 2p} A_\alpha(x)\partial_x^\alpha$ is uniformly parabolic in the sense of Petrovskii, that is, (see for example [GS] Ch. III.2), if we put $A^0(x, \xi) =\sum_{|\alpha|=2p}\xi^\alpha A_\alpha(x)$, there exists $\lambda_0>0$, such that, $\forall x\in\boldsymbol{R}^n$, $\forall\xi\in\boldsymbol{R}^n$ with $|\xi|=1$, any eigenvalue of the matrix $A^0(x, i\xi)$ has a real part not exceeding $-\lambda_0$.

Fundamental solutions for parabolic systems are described and studied for example in [FR1], [LSU], [EI], in the case of $t\in[0, T]$, with $T<+\infty$. However, for our purposes, we need some estimates valid for any value of t. Most of these estimates can be obtained with methods analogous to the corresponding in the books we have indicated and so their proofs will be only sketched. In the following "C" and "const" will mean constants (which can be different in each estimate), which are not asked to satisfy any special condition. Instead $C(\xi_1, \cdots, \xi_m)$ will be a constant depending on the specified arguments. Assume that $A(x, \partial)=A(\partial)$, that is, that it does not depend on x. Then

$$Z(x, t) = (2\pi)^{-n}H(t)\int_{\boldsymbol{R}^n}\exp(ix\cdot\xi+tA(i\xi))d\xi \tag{3}$$

is a fundamental solution (here H is the Heaviside function and $\cdot$ is the usual scalar product in $\boldsymbol{R}^n$), with $A(i\xi)=\sum_{|\alpha|\leq 2p}(i\xi)^\alpha A_\alpha$. Let $\|\ \|$ be a fixed norm in the space of $N\times N$ matrixes. Then, we have:

Lemma 1.1. *The coefficients of Z are of class $C^\infty(\boldsymbol{R}^{n+1}\setminus\{0\})$ and there exists $\omega_0\in\boldsymbol{R}$, $C_0>0$, depending on $A=\sup_\alpha\|A_\alpha\|$, λ_0 such that $\forall\beta\in\boldsymbol{N}_0^n$,*

$$\|\partial_x^\beta Z(x, t)\|\leq C(\beta)t^{-(n+|\beta|)/2p}\exp(\omega_0 t-C_0(|x|^{2p}/t)^{1/(2p-1)})\,. \tag{4}$$

Proof. As any eigenvalue of $A^0(i\xi)$ has a real part not exceeding $-\lambda_0|\xi|^{2p}$, one has (see [FR2], Ch. 7, Th. 2)

$$\begin{aligned}&\|\exp(tA^0(i\xi))\|\leq\sum_{j=0}^{N-1}2^j\|tA^0(i\xi)\|^j\exp(-\lambda_0 t|\xi|^{2p})\\&\leq C(A)(1+t^{N-1}|\xi|^{2p(N-1)})\exp(-\lambda_0 t|\xi|^{2p})\\&\leq C(A, \varepsilon_1)\exp(-(\lambda_0-\varepsilon_1)t|\xi|^{2p})\,,\qquad\forall\varepsilon_1>0\,.\end{aligned}$$

For ξ, $\eta\in\boldsymbol{R}^n$, put $B(\xi, \eta)=A(i\xi-\eta)-A^0(i\xi)$. One has

$$\begin{aligned}||A^0(i\xi-\eta)-A^0(i\xi)|| &\leq C(A)(|\eta|\,|\xi|^{2p-1}+|\eta|^{2p}) \\ &\leq C(A)(\varepsilon_2|\xi|^{2p}+C(\varepsilon_2)|\eta|^{2p}) \qquad \forall\varepsilon_2>0,\end{aligned}$$

so that

$$||B(\xi, \eta)|| \leq C(A)(\varepsilon_2|\xi|^{2p}+C(\varepsilon_2)|\eta|^{2p}+|\xi|^{2p-1}+|\eta|^{2p-1}+1)\,.$$

One has

$$\exp(tA(i\xi-\eta)) = \exp(tA^0(i\xi))+\int_0^t \exp((t-s)A(i\xi-\eta))B(\xi, \eta)\exp(sA^0(i\xi))ds\,.$$

Using the previous estimates and Gronwall's inequality, one draws:

$$(5) \quad ||\exp(tA(i\xi-\eta))|| \leq C_1(A, \lambda_0)\exp\{-\lambda_1 t|\xi|^{2p}+C_2(A, \lambda_0)t(|\eta|^{2p}+1)\}$$

for certain constants positive $C_1(A, \lambda_0)$, $C_2(A, \lambda_0)$ and $\lambda_1=\lambda/_0 2$. By Cauchy's theorem, $\forall\beta$, $\forall x\in \boldsymbol{R}^n$, $\forall t>0$, $\forall\eta\in\boldsymbol{R}^n$, one has

$$\partial_x^\beta Z(x, t) = (2\pi)^{-n}\int_{\boldsymbol{R}^n}(i\xi-\eta)^\beta \exp(ix\cdot(\xi+i\eta)+tA(i\xi-\eta))d\xi\,,$$

and so, using (5) and choosing $\eta=(|x|/(2C_2(A, \lambda_0)tp))^{1/(2p-1)}(x/|x|)$, one obtains (4).

Lemma 1.2. *Assume* (h1)—(h2) *are satisfied and let*

$$Z(x, t; \xi) = (2\pi)^{-n}\int_{\boldsymbol{R}^n}\exp(ix\cdot\eta+tA^0(\xi; i\eta))d\eta\,,$$

with

$$A^0(\xi; \eta) = \sum_{|\alpha|=2p}\eta^\alpha A_\alpha(\xi)\,.$$

Let us indicate with ∂_1^β and ∂_3^β the derivatives with respect to x and ξ. Then there exist $\omega_1\in\boldsymbol{R}$ and $\forall\beta$, $\gamma\in\boldsymbol{N}_0^n$ with $|\gamma|\leq 2p$ a constant $C(\beta, \gamma)>0$, such that

$$(6) \quad ||\partial_1^\beta\partial_3^\gamma Z(x, t; \xi)|| \leq C(\beta, \gamma)t^{-(n+|\beta|)/2p}\exp(\omega_1 t-C_1(|x|^{2p}/t))^{1/(2p-1)})\,,$$

with C_1 independent of β and γ.

Proof. We pose, for η, $\theta\in\boldsymbol{R}^n$, $V(t, \xi)=\exp(tA(\xi; i\eta-\theta))$. Then

$$\partial_t V(t, \xi) = A(\xi; i\eta-\theta)V(t, \xi),\ V(0, \xi) = I\,.$$

By well known results concerning O.D.E.s, if $1\leq|\gamma|\leq 2p$,

$$\partial_t\partial_\xi^\gamma V(t, \xi) = A(\xi, i\eta-\theta)\partial_\xi^\gamma V(t, \xi)+\sum_{\gamma<\delta}\binom{\gamma}{\delta}\partial_\xi^{\gamma-\delta}A(\xi; i\eta-\theta)\partial_\xi^\delta V(t, \xi)\,,$$

$$\partial_\xi^\gamma V(0, \xi) = 0\,,$$

so that

$$\partial_\xi^\gamma V(t, \xi) = \int_0^t V(t-s, \xi)[\sum_{\gamma<\delta}\binom{\gamma}{\delta}\partial_\xi^{\gamma-\delta}A(\xi; i\eta-\theta)\partial_\xi^\delta V(s, \xi)]ds\,.$$

By induction, from (5) one has

$$(7)\quad \|\partial_\xi^\gamma V(t,\xi)\| \leq \sum_{j=1}^{|\gamma|} C(\gamma,j)t^j(1+|\theta|^{2p}+|\eta|^{2p})^j \exp\{-\lambda_1 t|\eta|^{2p}+C_2(A,\lambda_0)t(|\theta|^{2p}+1)\}\,.$$

One has

$$\partial_1^\beta\partial_3^\gamma Z(x,t;\xi) = (2\pi)^{-n}\int_{R^n}(i\eta-\theta)^\beta \exp(ix\cdot(\eta+i\theta))\partial_\xi^\gamma \exp(tA^0(\xi;i\eta-\theta))d\eta\,,\quad \forall\theta\in R^n\,.$$

So (6) can be obtained with the same method developed in Lemma 1.1.

In [FR1] a fundamental solution $\Gamma(x,t,\xi)$ of L is built in the following way: define

$$K_1(x,t,\xi) = (A(x,\partial_1)-A^0(\xi,\partial_1))Z(x-\xi,t;\xi)$$

and, by induction,

$$K_{j+1}(x,t,\xi) = \int_0^t\int_{R^n} K_1(x,t-\tau,\eta)K_j(\eta,\tau,\xi)d\eta d\tau\,.$$

Put

$$\Phi(x,t,\xi) = \sum_{j=1}^{\infty} K_j(x,t,\xi)\,.$$

Then

$$\Gamma(x,t,\xi) = Z(x-\xi,t,\xi)+\int_0^t\int_{R^n} Z(x-\eta,t-\tau;\eta)\Phi(\eta,\tau,\xi)d\eta d\tau\,.$$

The following estimates can be established:

Lemma 1.3. *Under the conditions* (h1), (h2),

$$\|\Phi(x,t,\xi)\| \leq C(\mu,A,A_\mu,\lambda_0)t^{(\mu-n-2p)/2p}\exp(\omega_2 t - C_2(|x-\xi|^{2p}/t)^{1/(2p-1)})\,,$$

with

$$A_\mu = \sup_{|\alpha|\leq 2p}\sup_{x\neq\xi}\|A_\alpha(x)-A_\alpha(\xi)\|/|x-\xi|^\mu\,.$$

Moreover, $\forall\rho<\mu$

$$\|\Phi(x,t,\xi)-\Phi(y,t,\xi)\| \leq C(\rho)|x-y|^\rho t^{(\mu-\rho-n-2p)/2p}e^{\omega_3 t}$$
$$\times[\exp(-a(\rho)(|x-\xi|^{2p}/t)^{1/(2p-1)})+\exp(-a(\rho)(|y-\xi|^{2p}/t)^{1/(2p-1)})]\,,$$

with ω_3 *independent of* ρ.

Proof. Using the method developed in [FR1], Ch. 9.4, one verifies that

$$\|K_j(x, t, \xi)\| \leq A_j t^{(j\mu-n-2p)/2p} \exp(\alpha_j t - C_j(|x-\xi|^{2p}/t)^{1/(2p-1)})$$

for certain constants A_j, α_j, C_j. If $m_0 \geq (n+2p)/\mu$, again by induction, one can show that, $\forall r \in \boldsymbol{N}$,

$$\|K_{m_0+r}(x, t, \xi)\| \leq AB^r \Gamma(r\mu/(2p)+1)^{-1} \exp(\omega' t - C'(|x-\xi|^{2p}/t)^{1/(2p-1)})$$

for certain constants A, B, $C'>0$, $\omega' \in \boldsymbol{R}$. From this the first estimate follows. Now we prove the second estimate of the statement. With the same method developed in Ch. 9.4 and Ch. 1.4, Th. 7 in [FR1], one shows that, for $0<t\leq 1$,

$$\|K_1(x, t, \xi)-K_1(y, t, \xi)\| \leq C(\rho)|x-y|^{\rho} t^{(\mu-\rho-n-2p)/2p}[\exp(-a(\rho)(|x-\xi|^{2p}/t)^{1/2p-1)}) + \exp(-a(\rho)(|y-\xi|^{2p}/t)^{1/(2p-1)})] .$$

If $t>1$, one has

$$\|K_1(x, t, \xi)-K_1(y, t, \xi)\| \leq \|A(x, \partial_1)Z(x-\xi, t; \xi)-A(y, \partial_1)Z(y-\xi, t; \xi)\| + \|A^0(\xi, \partial_1)[Z(x-\xi, t; \xi)-Z(y-\xi, t; \xi)]\|$$

Using Lemma 1.1, each addend can be majorized with

$$\text{const } \{\exp(\omega_0 t - C_0(|x-\xi|^{2p}/t)^{1/(2p-1)}) + \exp(\omega_0 t - C_0(|y-\xi|^{2p}/t)^{1/(2p-1)})\}$$

and with const $\exp(\omega_0 t)|x-y|^{\mu}$. So, by interpolation, we have the estimate for $\|K_1(x, t, \xi)-K_1(y, t, \xi)\|$ for any $\omega_3>\omega_0$. The result follows from this, from the first estimate of the lemma and from

$$\Phi(x, t, \xi) = K_1(x, t, \xi) + \int_0^t \int_{\boldsymbol{R}^n} K_1(x-\eta, t-\tau, \eta)\Phi(\eta, \tau, \xi) d\eta d\tau .$$

Definition 1.4. Let $r \in \boldsymbol{R}$. We define

$$K_r = \{u \in C(\boldsymbol{R}^n; \boldsymbol{C}^N) : \forall b>0 \quad \exp(-b|x|^r)u \in L^\infty(\boldsymbol{R}^n; \boldsymbol{C}^N)\} .$$

K_r is a Frechèt space with the family of seminorms

$$p_b(u) = \|\exp(-b|x|^r)u\|_\infty \qquad (b>0) .$$

We remark that the space K_r coincides with the space $K_{r,0}$, in [GS], Ch. III, Th. 2. Now, define $q=2p/(2p-1)$. We have:

Lemma 1.5. *Assume* (h1), (h2) *are satisfied. Let* $f \in C(\boldsymbol{R}^+; K_q)$ *be such that*;

(a) $\forall b>0$ $p_b(f(t)) \leq \chi_b(t)$, *with* $\chi_b \in L^\infty_{\text{loc}}(\boldsymbol{R}^+)$ *and* $\chi_b(t)=O(t^{-\gamma})(t\to 0)$, *for some* $\gamma \in [0, 1[$.

(b) $\exists \nu \in]0, 1[$ *such that*

$$|f(t)(x)-f(t)(y)| \leq \chi_b(t)|x-y|^{\nu}[\exp(b|x|^q)+\exp(b|y|^q)] \quad \forall b>0 .$$

Put

$$V(t)(x)=\int_0^t\int_{\boldsymbol{R}^n} Z(x-\xi,\, t;\, \xi)f(\tau)(\xi)d\xi\, d\tau\,.$$

Then:

(I) $t\to V(t)\in C^1(\boldsymbol{R}^+;\, K_q)$;

(II) $K_q\text{-}\lim_{t\to 0} V(t)=0$;

(III) $\forall t>0,\ V(t)\in C^{2p}(\boldsymbol{R}^n;\, \boldsymbol{C}^N)$ *and* $\forall\beta,\ |\beta|\le 2p,\ t\to\partial_x^\beta V(t)\in C(\boldsymbol{R}^+;\, K_q)$;

(IV) *if* $|\beta|\le 2p,\ K_q\text{-}\lim_{t\to 0} t\partial_x^\beta V(t)=0$.

Proof. Assume $0<t\le T<+\infty$. Then, by Lemma 1.1, $\forall b'>0$,

$$\begin{aligned}|V(t)(x)|&\le C(T,\, b')\int_0^t\int_{\boldsymbol{R}^n}(t-\tau)^{-n/(2p)}\exp\left(-C_0(|x-\xi|^{2p}/(t-\tau))^{1/(2p-1)}\right)\\&\quad\times\exp(b'|\xi|^q)\chi_b'(\tau)d\xi\, d\tau\\&\le C(T,\, b')\int_0^t\chi_b'(\tau)\left(\int_{\boldsymbol{R}^n}\exp(|\xi|^q(-C_0+2^q b' T^{q/(2p)}))d\xi\right)d\tau\exp(b|x|^q),\end{aligned}$$

if $2^q b'\le b$, $2^q b' T^{q/(2p)}<C_0$, for a fixed $b>0$. From this it follows $V(t)\in K_q$ and $K_q\text{-}\lim_{t\to 0} V(t)=0$. The continuity of V in $t_0>0$ can be proved in an analogous way. If $|\beta|\le 2p-1$, one has, for $t>0$,

$$\partial_x^\beta V(t)(x)=\int_0^t\int_{\boldsymbol{R}^n}\partial_1^\beta Z(x-\xi,\, t-\tau;\, \xi)f(\tau)(\xi)d\xi\, d\tau$$

and, for $0<t\le T<+\infty$, $\forall b'>0$,

$$\begin{aligned}&|\partial_x^\beta V(t)(x)|\\&\le C(\beta,\, T,\, b')\int_0^t\int_{\boldsymbol{R}^n}(t-\tau)^{-(n+|\beta|)/(2p)}\chi_{b'}(\tau)\exp\left(-C_0(|x-\xi|^{2p}/(t-\tau))^{1/(2p-1)}\right)\\&\quad\times\exp(b'|\xi|^q)d\xi\,.\end{aligned}$$

For a fixed $b>0$, if $b'<\min\{2^{-q}b,\, C_0 2^{-q}T^{-q/(2p)}\}$, one has

$$|\partial_x^\beta V(t)(x)|\le C(\beta,\, T,\, b')\exp(b|x|^q)\int_0^t(t-\tau)^{-|\beta|/(2p)}\chi_{b'}(\tau)d\tau\,.$$

From this, one draws $K_q\text{-}\lim_{t\to 0} t\partial_x^\beta V(t)=0$, for $|\beta|\le 2p-1$. It is also easily seen that $t\to\partial_x^\beta V(t)\in C(\boldsymbol{R}^+;\, K_q)$. If $|\beta|=2p$, one has (with methods analogous to [FR1], Th. 4)

$$\begin{aligned}\partial_x^\beta V(t)(x)&=\int_0^t\left(\int_{\boldsymbol{R}^n}\partial_1^\beta Z(x-\xi,\, t-\tau;\, \xi)f(\tau)(\xi)d\xi\right)d\tau\\&=\int_0^t\left(\int_{\boldsymbol{R}^n}\partial_1^\beta Z(x-\xi,\, t-\tau;\, \xi)[f(\tau)(\xi)-f(\tau)(x)]d\xi\right)d\tau\\&\quad+\int_0^t\left(\int_{\boldsymbol{R}^n}[\partial_1^\beta Z(x-\xi,\, t-\tau;\, \xi)-\partial_1^\beta Z(x-\xi,\, t-\tau;\, x)]d\xi\right)f(\tau)(x)d\tau\\&\qquad\qquad\left(\text{as}\int_{\boldsymbol{R}^n}\partial_1^\beta Z(y,\, t;\, x)dy=0\right).\end{aligned}$$

From Lemmas 1.1 and 1.2 and assumptions (a) and (b), one draws the existence of $\partial_x^\beta V(t)(x)$ for any $t>0$, $x\in\boldsymbol{R}^n$ and the continuity on $\boldsymbol{R}^+$ of $t\to\partial_x^\beta V(t)$. If $0<t\leq 1$, one has, again using Lemmas 1.1 and 1.2 and assumptions (a) and (b), for any $b>0$, for any b' sufficiently small,

$$|\partial_x^\beta V(t)(x)|\leq \text{const}\,(\int_0^t[(t-\tau)^{1/(2p)-1}+(t-\tau)^{\nu/(2p)-1}]\chi_{b'}(\tau)d\tau)\exp(b|x|^q),$$

so that

$$p_b(\partial_x^\beta V(t))\leq \text{const}\,(\int_0^t[(t-\tau)^{1/(2p)-1}+(t-\tau)^{\nu/(2p)-1}]\chi_{b'}(\tau)d\tau)$$

and

$$p_b(t\partial_x^\beta V(t))\leq \text{const}\, t^{\nu/(2p)-\gamma+1}\underset{t\to 0}{\to} 0\,.$$

Finally, like in [FR1], Ch. 9.3, Lemma 5, for $t>0$,

$$\partial_t V(t)(x)=f(t)(x)+\int_0^t(\int_{\boldsymbol{R}^n}A(\xi,\partial_1)Z(x-\xi,t-\tau;\xi)f(\tau)(\xi)d\xi)d\tau$$

and from the continuity of $t\to\partial_x^\beta V(t)$ $\forall\beta$, $|\beta|\leq 2p$, one has that $t\to\partial_t V(t)\in C(\boldsymbol{R}^+;K_q)$ so that the lemma is completely proved.

Definition 1.6. Let $f\in K_q$. Define $T(0)=I$, and for $t>0$,

$$T(t)f(x)=\int_{\boldsymbol{R}^n}\Gamma(x,t,\xi)f(\xi)d\xi$$

(here Γ is the fundamental solution of L described previously).

Lemma 1.7. $\forall t>0$, $T(t)\in\mathfrak{L}(K_q)$. *Moreover* $\forall f\in K_q$, $t\to T(t)f\in C([0,+\infty[;K_q)$.

Proof. One has, for $t>0$, $T(t)=U(t)+V(t)$, with

$$U(t)f(x)=\int_{\boldsymbol{R}^n}Z(x-\xi,t,\xi)f(\xi)d\xi\,,$$

$$V(t)f(x)=\int_0^t\int_{\boldsymbol{R}^n}Z(x-\eta,t-\tau;\eta)(\int_{\boldsymbol{R}^n}\Phi(\eta,\tau,\xi)f(\xi)d\xi)d\tau d\eta\,.$$

For what concerns $U(t)f$, it is easily seen that $U(t)f\in K_q$ $\forall t>0$, $t\to U(t)f\in C(\boldsymbol{R}^+;K_q)$. Now, we show that $K_q-\lim_{t\to 0}U(t)f=f$. One has

$$\begin{aligned}U(t)f(x)-f(x)&=\int_{\boldsymbol{R}^n}[Z(x-\xi,t;\xi)-Z(x-\xi,t;x)]f(\xi)d\xi+\\&\quad+\int_{\boldsymbol{R}^n}Z(x-\xi,t;x)[f(\xi)-f(x)]d\xi+[\exp(tA^0(x;0))-I]f(x)\\&=U_1(t)f(x)+U_2(t)f(x)+U_3(t)f(x)\,.\end{aligned}$$

We prove that, for $j=1, 2, 3$, $U_j(t)f \underset{t\to 0}{\to} 0$. The fact that $U_1(t)f \underset{t\to 0}{\to} 0$ follows from Lemma 1.2. Even the convergence of $U_3(t)f$ to 0 as $t\to 0$ is easy. Consider $U_2(t)f$. Fix $b, \varepsilon>0$. One has, for $0<t\leq 1$,

$$(8)\quad \exp(-b|x|^q)\left|\int_{R^n} Z(x-\xi, t; \xi)(f(\xi)-f(x))d\xi\right|$$
$$\leq \text{const}\exp(-b|x|^q)\left(\int_{R^n}\exp(-C_1|\xi|^q)|f(x-t^{1/(2p)}\xi)|d\xi+|f(x)|\right).$$

One has $\lim_{|x|\to\infty}\exp(-b|x|^q)f(x)=0$ and there exists $R_1(\varepsilon)>0$ such that if $|x|\geq R_1(\varepsilon)$, $\exp(-b|x|^q)|f(x)|\leq\varepsilon$. Moreover,

$$\exp(-b|x|^q)\int_{R^n}\exp(-C_0|\xi|^q)|f(x-t^{1/(2p)}\xi)|d\xi$$
$$\leq p_{b'}(f)\exp(-b|x|^q)\int_{R^n}\exp(-C_0|\xi|^q+b'|x-t^{1/(2p)}\xi|^q)d\xi \underset{|x|\to+\infty}{\to} 0,$$

if $b'<2^{-q}b$, $2^{-q}b'<C_0$. It follows that, for some $R(\varepsilon)\geq 0$, one has that (8) can be majorized by const ε, if $|x|\geq R(\varepsilon)$, $t\in]0, 1]$. Assume $|x|\leq R(\varepsilon)$. Then, for $0<t\leq 1$, $M\geq 0$,

$$\exp(-b|x|^q)\left|\int_{R^n} Z(x-\xi, t; \xi)(f(\xi)-f(x))d\xi\right|$$
$$\leq \text{const}\left(\int_{|\xi|\leq M}\exp(-C_0|\xi|^q)|f(x-t^{1/(2p)}\xi)-f(x)|d\xi\right.$$
$$\left.+\int_{|\xi|>M}\exp(-C_0|\xi|^q)|f(x-t^{1/(2p)}\xi)-f(x)|d\xi\right).$$

One has

$$\lim_{M\to\infty}\int_{|\xi|>M}\exp(-C_0|\xi|^q)|f(x-t^{1/(2p)}\xi)|d\xi=0,$$

uniformly for $|x|\leq R(\varepsilon)$, $t\in]0, 1]$. For any $M\geq 0$,

$$\int_{|\xi|\leq M}\exp(-C_0|\xi|^q)|f(x-t^{1/(2p)}\xi)-f(x)|d\xi \underset{t\to 0}{\to} 0,$$

uniformly for $|x|\leq R(\varepsilon)$, in force of the uniform continuity of f on compact subsets of R^n. So

$$U_2(t)f \underset{t\to 0}{\to} 0, \quad\text{and}\quad U(t)f \underset{t\to 0}{\to} f.$$

Now we show that $V(t)f \underset{t\to 0}{\to} 0$. Put, for $t>0$, $x\in \boldsymbol{R}^n$,

$$g(t)(x)=\int_{R^n}\Phi(x, t, \xi)f(\xi)d\xi.$$

From Lemma 1.3 one has that g satisfies assumption (a) of Lemma 1.5. Moreover, by [FR1], Ch. IX, estimate 4.17, one has that g satisfies also assumption (b) of the same Lemma. Therefore, $V(t)f \underset{t\to 0}{\to} 0$, and Lemma 1.7 is completely proved.

Now we indicate with $A'(x, \partial)$ the formal adjoint of $A(x, \partial)$:

$$A'(x, \partial) = \sum_{|\alpha| \leq 2p} (-1)^{|\alpha|} \partial^\alpha (A_\alpha(x)^T) \qquad (B^T = \text{transpose of } B) .$$

Owing to (h1), (h2), $A'(x, \partial)$ has properties analogous to $A(x, \partial)$. We remark that $\forall f \in K_q$, $\phi \in \mathcal{D}(\boldsymbol{R}^n)^N$, the mapping

$$\phi \to \int_{\boldsymbol{R}^n} f(x) \cdot A'(x, \partial)\phi \, dx$$

defines an element $\mathcal{A}f$ in $\mathcal{D}(\boldsymbol{R}^n)^N$. We give the following

DEFINITION 1.8. *We define* $D(A) = \{u \in K_q \mid \mathcal{A}u \in K_q\}$. *For* $u \in D(A)$, $Au = \mathcal{A}u$.

It is immediate to verify that A is a closed densely defined operator in K_q.

Lemma 1.9. *Let* $u \in C([0, +\infty[; K_q) \cap C^1(]0, +\infty[; K_q)$. *Assume that*:

(a) $u(t) \in D(A) \forall t > 0$.

(b) $\dfrac{du}{dt}(t) = Au(t)$, $\forall t > 0$.

(c) $u(0) = 0$.

Then $u(t) = 0$, $\forall t \geq 0$.

Proof. $\forall \varepsilon > 0$ take $\omega_\varepsilon \in \mathcal{D}(\boldsymbol{R}^n)$, such that supp $(\omega_\varepsilon) \subseteq \{x \in \boldsymbol{R}^n \mid |x| \leq \varepsilon\}$, $\int_{\boldsymbol{R}^n} \omega_\varepsilon(x) dx = 1$ and define $u_\varepsilon(t) = \omega_\varepsilon * u(t)$. It is clear that $u_\varepsilon(t) \in C^\infty(\boldsymbol{R}^n; \boldsymbol{C}^N)$, $u_\varepsilon(t) \underset{\varepsilon\to 0}{\to} u(t)$ uniformly on bounded subsets of $[0, +\infty[$. Moreover, $\forall t > 0$ $A(x, \partial)u_\varepsilon(t) \underset{\varepsilon\to 0}{\to} Au(t)$ in $(\mathcal{D}^{2p}(\boldsymbol{R}^n)^N)'$. Let $\bar{t} > 0$, $\bar{x} \in \boldsymbol{R}^n$. We shall prove that $u(\bar{t})(\bar{x}) = 0$. For $R > 0$, define $B_R = \{x \in \boldsymbol{R}^n \mid |x - \bar{x}| < R\}$, $B'_R = B_{R+1} \backslash B_R$. For η_1, η_2 positive, $\eta_1 < \bar{t} - \eta_2$, take $v \in C^{2p}([0, \bar{t} - \eta_2] \times \bar{B}_{R+1}; \boldsymbol{C}^N)$ such that v is zero in a neighbourhood of $\partial B_{R+1} \times [0, \bar{t} - \eta_2]$. Pose $\Omega =]\eta_1, \bar{t} - \eta_2[\times B_{R+1}$. One has

$$\int_\Omega \left(\frac{\partial u_\varepsilon}{\partial t} - A(x, \partial)u_\varepsilon\right) \cdot v \, dx \, dt \underset{\varepsilon\to 0}{\to} \int_\Omega \left(\frac{du}{dt} \cdot v - Au \cdot v\right) dx \, dt = 0 . \tag{9}$$

Posing $L' = -\partial_t - A(x, \partial)'$, by Green's formula,

$$\begin{aligned}\int_\Omega (u_\varepsilon \cdot L'v - Lu_\varepsilon \cdot v) dx \, dt &= -\int_\Omega \partial_t (u_\varepsilon \cdot v) dx \, dt \\ &= \int_{B_{R+1}} u_\varepsilon(\eta_1, x) \cdot v(\eta_1, x) \, dx - \int_{B_{R+1}} u_\varepsilon(\bar{t} - \eta_2, x) \cdot v(\bar{t} - \eta_2, x) \, dx ,\end{aligned}$$

so that

$$\int_{\Omega} u\cdot L'v\,dx\,dt = \lim_{\varepsilon\to 0}\int_{\Omega}(u_\varepsilon\cdot L'v - Lu_\varepsilon\cdot v)dx\,dt$$
$$= \int_{B_{R+1}} u(\eta_1)(x)\cdot v(\eta_1, x)dx - \int_{B_{R+1}} u(\bar{t}-\eta_2)(x)\cdot v(\bar{t}-\eta_2, x)dx\,.$$

Letting $\eta_1\to 0$ and using (c),

$$\int_0^{\bar{t}-\eta_2}\Big(\int_{B_{R+1}} u\cdot L'v\,dx\Big)dt = -\int_{B_{R+1}} u(\bar{t}-\eta_2)(x)\cdot v(\bar{t}-\eta_2, x)\,dx\,. \tag{10}$$

For $t>0$, $x\in\boldsymbol{R}^n$, $\xi\to\Gamma(x, t, \xi)\in C^{2p}(\boldsymbol{R}^n; \boldsymbol{C}^N)$ and

$$\partial_t\Gamma(x, t, \xi)^T = A'(\xi, \partial_3)\Gamma(x, t, \xi)^T \qquad \text{(see [FR1], Ch. 9.5.)} \tag{11}$$

Moreover, by [FR1], Ch. 9, (5.6) $\forall\beta$, $|\beta|\leq 2p$.

$$\forall T>0,\ \|\partial_3^\beta\Gamma(x, t, \xi)^T\|\leq \text{const}\, t^{-(n+|\beta|)/(2p)}\exp\big(-k(|x-\xi|^{2p}/t)^{1/(2p-1)}\big)\,, \tag{12}$$

for some $k>0$, $0<t\leq T$. Assume $u(\bar{t})(\bar{x})\neq 0$, $\forall R>0$ take $h_R\in C_0^{2p}(B_{R+1}; \boldsymbol{C}^N)$, such that $h_R(x)=\overline{u(\bar{t})(\bar{x})}/|u(\bar{t})(\bar{x})|$ if $x\in B_R$ and such that $\sup_R \|\partial^\beta h_R\|_\infty<+\infty$, $\forall\beta$, $|\beta|\leq 2p$. Define

$$v_R(t, x) = -\Gamma(\bar{x}, \bar{t}-t, x)^T h_R(x) \qquad (t<\bar{t},\ x\in\boldsymbol{R}^n)\,.$$

Owing to (11), $L'v_R(t, x)=0$, if $t<\bar{t}$, $x\in B_R$. By (10),

$$\int_{B_R}\int_0^{\bar{t}-\eta_2} u\cdot L'v_R\,dx\,dt = \int_{B_{R+1}}\Gamma(\bar{x}, \eta_2, x)u(\bar{t}-\eta_2)(x)\cdot h_R(x)\,dx\,.$$

By estimate (12), $u\cdot L'v_R\in L^1([0, \bar{t}]\times B'_R)$. So, for $\eta_2\to 0$, as Γ is a fundamental solution of L,

$$\int_0^{\bar{t}}\int_{B_R} u\cdot L'v_R\,dx\,dt = |u(\bar{t})(\bar{x})|\,, \qquad \forall R>0\,. \tag{13}$$

Again using (12), one verifies that the left side of (13) tends to 0 as $R\to+\infty$. So, $u(\bar{t})(\bar{x})=0$ and the result is proved.

Therefore we can state the main reuslt of this paragraph:

Theorem 1.10. *Under the assumptions* (h1), (h2), $\{T(t)\,|\,t\geq 0\}$ *is a* C_0-*semigroup in* K_q *with infinitesimal generator* A (*see Definition* 1.8). *Moreover,* $\forall t>0$, $T(t)(K_q)\subseteq D(A)$ *and,* $\forall f\in K_q$, $\lim_{t\to 0} tAT(t)f=0$.

Proof. By Lemma 1.7, $t\to T(t)$ is strongly continuous on $[0, +\infty[$. As usual take $T(t)=U(t)+V(t)$.

$$U(t)f(x) = \int_{\boldsymbol{R}^n} Z(x-\xi,\, t;\, \xi)f(\xi)d\xi\,,$$

$$\partial_x^\beta U(t)f(x) = \int_{\boldsymbol{R}^n} \partial_1^\beta Z(x-\xi,\, t;\, \xi)f(\xi)d\xi \qquad \forall\beta\in\boldsymbol{N}_0^n,\ \forall t>0\,,$$

$$\text{and } t\to\partial_x^\beta U(t)f\in C(]0,\, +\infty[;\, K_q)\,.$$

Moreover, for $0<t\leq 1$, $\forall b'>0$,

$$|\partial_x^\beta U(t)f(x)|\leq C(\beta)p_{b'}(f)t^{-|\beta|/(2p)}\int_{\boldsymbol{R}^n}\exp(-C_0|\xi|^q+b'|x-t^{1/(2p)}\xi|^q)d\xi$$
$$\leq C(\beta,\, b)p_{b'}(f)t^{-|\beta|/(2p)}\exp(b|x|^q)\,,$$

if b' is sufficiently small (b' depends only on b). This estimate implies that $\forall\beta$, $|\beta|\leq 2p-1$, $\lim_{t\to 0} t\partial_x^\beta U(t)f=0$ and $\forall\beta$, $|\beta|=2p$, the operators $\{t\partial_x^\beta U(t)|\, t\in]0,\,1[\}$ are equicontinuous. Owing to the remark at the end of the proof of Lemma 1.7,

$$V(t)f(x) = \int_0^t\int_{\boldsymbol{R}^n} Z(x-\xi,\, t-\tau;\, \xi)g(\tau)(\xi)d\xi\, d\tau$$

with g satisfying the assumptions of Lemma 1.5. So, $\forall t>0$, $x\to T(t)f(x)\in C^{2p}(\boldsymbol{R}^n;\, \boldsymbol{C}^N)$. From [FR1], Ch. 9, Th. 3, one has, for $t>0$, $x\in\boldsymbol{R}^n$,

$$\partial_t T(t)f(x) = A(x,\, \partial_x)T(t)f(x)\,.$$

So, $\forall t>0$, $T(t)f\in D(A)$ and, as $t\to\partial_x^\beta T(t)f\in C(]0;\, +\infty[;\, K_q)\forall\beta$, $|\beta|\leq 2p$, $t\to T(t)f\in C^1(]0,\, +\infty[;\, K_q)$ and $\partial_t T(t)f=AT(t)f$, $\forall t>0$. Now we prove the semigroup property $T(t)T(s)=T(t+s)$. Let $f\in K_q$. Define, for $s\geq 0$, $u(t)=T(t)T(s)f-T(t+s)f$, $t\geq 0$. Then $u\in C([0,\, +\infty[;\, K_q)\cap C^1(]0,\, +\infty[;\, K_q)$, $u(0)=0$, $\partial_t u(t)=Au(t)$. From Lemma 1.9, one has $u(t)=0$ $\forall t\geq 0$, so that $T(t)T(s)=T(t+s)$. We prove that the infinitesimal generator $\bar{A}$ of the semigroup is coincident with A. Let $f\in D(\bar{A})$. For $t>0$, $\partial_t T(t)f=AT(t)f$, so that $\bar{A}T(t)f=AT(t)f$. But $T(t)f\underset{t\to 0}{\to}f$, $\bar{A}T(t)f\underset{t\to 0}{\to}\bar{A}f$; as A is closed, it follows $f\in D(A)$ and $Af=\bar{A}f$. On the other hand, assume $f\in D(A)$. Put

$$u(t) = T(t)f-f-\int_0^t T(s)Af ds \qquad \text{(integral in the Riemann sense).}$$

Then $u\in C([0,\, +\infty[;\, K_q)$, $u(0)=0$. Moreover, $\forall t>0$, $\partial_t u(t)=AT(t)f-T(t)Af$. But (see [KOMU], Prop. 1.2, Cor.) $\int_0^t T(s)Af ds\in D(\bar{A})$ and, as A is an extension of $\bar{A}$,

$$A\int_0^t T(s)Af ds = T(t)Af-Af\,.$$

It follows, $\forall t>0$,

$$Au(t) = AT(t)f - T(t)Af = \partial_t u(t)\,.$$

By Lemma 1.7, $u(t)=0$ $\forall t>0$ and so the result follows from [KOMU], Prop. 1.2. It remains to show that $\lim_{t\to 0} tAT(t)f=0$ $\forall f\in K_q$. One has $tAT(t)f=tAU(t)f+tAV(t)f$. By Lemma 1.5, $tAv(t)f\underset{t\to 0}{\to} 0$. From the first part of the proof one has that $\{tAU(t)\,|\,t\in]0, 1]\}$ is equicontinuous. But, if $f\in D(A)$, $tAT(t)f=tT(t)Af\underset{t\to 0}{\to} 0$. So the result follows, because $D(A)$ is dense in K_q.

2. Parabolic systems in the spaces $\mathcal{S}(\boldsymbol{R}^n)^N$ and $\mathcal{S}'(\boldsymbol{R}^n)^N$

Now we want to study parabolic systems via semigroup theory in the spaces $\mathcal{S}(\boldsymbol{R}^n)^N$ and $\mathcal{S}'(\boldsymbol{R}^n)^N$ (which we shall abbreviate respectively with $\mathcal{S}$ and $\mathcal{S}'$). An analogous study in the spaces $H^\infty(\boldsymbol{R}^n)$ was developed in [MI] and [BA] (for the case of $\mathcal{S}'(\boldsymbol{R}^n)$ and $A(x, \partial)=\Delta$ see also [SZ], Ch. V). We assume the following:

(h3) $\forall\alpha$, $|\alpha|\leq 2p$, the coefficients of A_α belong to $B^\infty(\boldsymbol{R}^n)$.

Let

$$\mathcal{S} = \mathcal{S}(\boldsymbol{R}^n)^N = \{\phi\in C^\infty(\boldsymbol{R}^n; \boldsymbol{C}^N)\,|\,\forall\alpha,\ \beta\in\boldsymbol{N}_0^n,\ x\to x^\alpha\partial^\beta\phi\in L^\infty(\boldsymbol{R}^n)^N\}\,.$$

We pose, for $m\geq 0$, $\beta\in\boldsymbol{N}_0^n$, $\mu\in]0, 1]$, $\phi\in\mathcal{S}$,

$$p_{m,\beta}(\phi) = \|\,|x|^m|\partial^\beta\phi|\,\|_\infty,\quad \|\phi\|_{m,r} = \sum_{|\beta|\leq r}\|(1+|x|^m)|\partial^\beta\phi|\,\|_\infty\,,$$

$$[\phi]_\mu = \sup_{x\neq y}|\phi(x)-\phi(y)|/|x-y|^\mu,\quad \|\phi\|_{m,r,\mu} = \|\phi\|_{m,r}+\sum_{|\beta|=r}[\partial^\beta\phi]_\mu\,.$$

The norms $\|\ \|_{m,r}$ or, alternatively, $\|\ \|_{m,r,\mu}$ form a calibration in $\mathcal{S}$. We continue to call $T(t)$ the restriction of the operator defined in K_q to $\mathcal{S}$. One has:

Theorem 2.1. *Assume* (h2), (h3). *Then,*
(I) $\forall t\geq 0$, $T(t)\in\mathfrak{L}(\mathcal{S})$.
(II) $t\to T(t)\phi\in C([0, +\infty[; \mathcal{S})$ $\forall\phi\in\mathcal{S}$.
(III) $\{T(t)\,|\,t\geq 0\}$ *is a C_0-semigroup of linear operators in $\mathcal{S}$, with infinitesimal generator $A_{\mathcal{S}}$ defined as follows*: $D(A_{\mathcal{S}})=\mathcal{S}$, $A_{\mathcal{S}}\phi=A(x, \partial)\phi=A\phi$.

Proof. By theorem (8) of [FR1], Ch. 9, $T(t)\phi\in C^\infty(\boldsymbol{R}^n; \boldsymbol{C}^N)\,\forall\, t\geq 0$ and $\forall\beta$

$$\partial_x^\beta T(t)\phi(x) = \int_{\boldsymbol{R}^n}\partial_1^\beta\Gamma(x, t, \xi)\phi(\xi)d\xi\,.$$

For $m\geq 0$, $0<t\leq T$, $(T>0)$, by [FR1], Ch. 9, 6 Th. 7,

$$|x|^m\left|\int_{\boldsymbol{R}^n}\partial_1^\beta\Gamma(x, t, \xi)\phi(\xi)d\xi\right|$$
$$\leq C(T, \beta)t^{-|\beta|/(2p)}\int_{\boldsymbol{R}^n}\exp(-C|\xi|^q)|x|^m|\phi(x-t^{1/(2p)}\xi)|\,d\xi\,.$$

One has

$$|x|^m|\phi(x-t^{1/(2p)}\xi)|\leq C(m)(p_{m,0}(\phi)+t^{m/(2p)}|\xi|^m p_{0,0}(\phi)) .$$

It follows, for $0<t\leq T$,

$$p_{m,\beta}(T(t)\phi)\leq C(T,\beta,m)(t^{-|\beta|/(2p)}p_{m,0}(\phi)+t^{(m-|\beta|)/(2p)}p_{0,0}(\phi)) .$$

So, (I) is proved.

Because of the semigroup property, to have (II) it is sufficient to show that $\forall\phi\in\mathcal{S}\ \lim_{t\to 0} T(t)\phi=\phi$. As usual, $T(t)=U(t)+V(t)$. We start by showing that $U(t)\phi\to\phi$. One has

$$U(t)\phi(x)=\int_{R^n}Z(x-\xi;\, t,\, \xi)\phi(\xi)d\xi=\int_{R^n}Z(\xi,\, t;\, x-\xi)\phi(x-\xi)d\xi .$$

Owing to [FR1], Ch. 9, Lemma 4,

$$\begin{aligned}\partial_x^\beta U(t)\phi(x)&=\sum_{\gamma\leq\beta}\binom{\beta}{\gamma}\int_{R^n}\partial_3^{\beta-\gamma}Z(\xi,\, t;\, x-\xi)\partial^\gamma\phi(x-\xi)d\xi\\ &=\int_{R^n}Z(\xi,\, t;\, x-\xi)\partial^\beta\phi(x-\xi)d\xi+\sum_{\gamma<\beta}\binom{\beta}{\gamma}\int_{R^n}\partial_3^{\beta-\gamma}Z(\xi,\, t;\, x-\xi)\partial^\gamma\phi(x-\xi)d\xi .\end{aligned}$$

For $m\geq 0$,

$$\begin{aligned}&|x|^m\left|\int_{R^n}Z(\xi,\, t;\, x-\xi)\partial^\beta\phi(x-\xi)d\xi-\partial^\beta\phi(x)\right|\\ &\leq|x|^m\left|\int_{R^n}\|Z(\xi,\, t;\, x-\xi)-Z(\xi,\, t;\, x)\|\,|\partial^\beta\phi(x-\xi)|d\xi+\right.\\ &\quad+|x|^m\left|\int_{R^n}\|Z(\xi,\, t;\, x)\|\,|\partial^\beta\phi(x-\xi)-\partial^\beta\phi(x)|d\xi+\right.\\ &\quad+|x|^m\|\exp(tA^0(x;\, 0))-I\|\,|\partial^\beta\phi(x)|=I_1(x)+I_2(x)+I_3(x) .\end{aligned}$$

For $0<t\leq 1$, using Lemma 1.2 one draws

$$I_1(x)\leq C(m,\, \beta)(t^{1/(2p)}p_{m,\beta}(\phi)+t^{m+1/(2p)}p_{0,\beta}(\phi))\underset{t\to 0}{\to}0 .$$

Again for $0<t\leq 1$, $m\geq 0$,

$$I_2(x)\leq C(0)\int_{R^n}|x|^m\exp(-C_0|\xi|^q)|\partial^\beta\phi(x-t^{1/(2p)}\xi)-\partial^\beta\phi(x)|d\xi .$$

Let $R\geq 0$, $|x|\leq R$. Then the expression is majorized by $C(R)p_{0,|\beta|+1}(\phi)t^{1/(2p)}\underset{t\to 0}{\to}0$. On the other hand, for any value of x,

$$\begin{aligned}I_2(x)&\leq C(0)|x|^m\int_{R^n}\exp(-C_0|\xi|^q)|\partial^\beta\phi(x-t^{1/(2p)})\xi|d\xi\\ &\quad+|x|^m\int_{R^n}\exp(-C_0|\xi|^q)d\xi|\partial^\beta\phi(x)| .\end{aligned}$$

The first addend is dominated by

$$C(m)\Big[\int_{|\xi|\leq R'} \exp(-C_0|\xi|^q)|x-t^{1/(2p)}\xi|^m|\partial^\beta\phi(x-t^{1/(2p)}\xi|)d\xi+$$
$$+\int_{|\xi|>R'} \exp(-C_0|\xi|^q)|x-t^{1/(2p)}\xi|^m|\partial^\beta\phi(x-t^{1/(2p)}\xi)|d\xi+t^{m/(2p)}p_{0,\beta}(\phi)\Big].$$

If $|x|\geq R$, $|\xi|\leq R'$, $t\leq 1$, $|x-t^{1/(2p)}\xi|\geq R-R'$ and as $\lim_{|x|\to\infty}|x|^m\partial^\beta\phi(x)=0$, one has that the first integral tends to 0 as $|x|\to+\infty$ $\forall R'\geq 0$, uniformly with respect to $t\in]0, 1]$. The second integral is majorized by

$$\int_{|\xi|>R'} \exp(-C_0|\xi|^q)d\xi p_{m,\beta}(\phi) \underset{R'\to+\infty}{\to} 0.$$

So, $I_2(x)\underset{t\to 0}{\to} 0$, uniformly for $x\in\boldsymbol{R}^n$. Finally,

$$|x|^m\|\exp(tA^0(x;0))-I\||\partial^\beta\phi(x)|\leq \text{const}\, tp_{m,\beta}(\phi)\underset{t\to 0}{\to} 0.$$

If $\gamma<\beta$, $0<t\leq 1$,

$$\begin{aligned}
&|x|^m\Big|\int_{\boldsymbol{R}^n}\partial_3^{\beta-\gamma}Z(\xi, t; x-\xi)\partial^\gamma\phi(x-\xi)d\xi\Big|\\
&\leq|x|^m\Big|\int_{\boldsymbol{R}^n}[\partial_3^{\beta-\gamma}Z(\xi, t; x-\xi)-\partial_3^{\beta-\gamma}Z(\xi, t; x)]\partial^\gamma\phi(x-\xi)d\xi\Big|\\
&+|x|^m\Big|\int_{\boldsymbol{R}^n}\partial_3^{\beta-\gamma}Z(\xi, t; x)[\partial^\gamma\phi(x-\xi)-\partial^\gamma\phi(x)]d\xi\Big|\\
&+|x|^m\Big\|\int_{\boldsymbol{R}^n}\partial_3^{\beta-\gamma}Z(\xi, t; x)d\xi\Big\||\partial^\gamma\phi(x)| = I_1(x)+I_2(x)+I_3(x).
\end{aligned}$$

I_1 and I_2 can be treated using methods analogous to the previous and [FR1] Ch. 9, Lemma 4 to show that they converge to 0 as $t\to 0$ uniformly in x. $I_3(x)=|x|^m\|\partial_x^{\beta-\gamma}\exp(t(A(x;0)))\||\partial^\gamma\phi(x)|\underset{t\to 0}{\to} 0$ uniformly for $x\in\boldsymbol{R}^n$. With this we have proved that $U(t)\phi\underset{t\to 0}{\to}\phi$.

It remains to show that $V(t)\phi\underset{t\to 0}{\to} 0$. We start by remarking that, for $0<t\leq 1$, $\forall j\in\boldsymbol{N}_0$,

$$|x|^m\Big|\partial_x^\beta\int_{\boldsymbol{R}^n}K_j(x, t, \xi)\phi(\xi)d\xi\Big|\leq C(j)t^{j/(2p)-1}\|\phi\|_{m,|\beta|}.$$

This can be easily proved by induction, using the estimates of [FR1] Ch. 9.6, Th. 7 and observing that

$$\begin{aligned}
\partial_x^\beta\int_{\boldsymbol{R}^n}K_1(x, t, \xi)\phi(\xi)d\xi &= \sum_{\beta_1+\beta_2+\beta_3=\beta}C(\beta_1, \beta_2, \beta_3)\int_{\boldsymbol{R}^n}[A^{(\beta_1)}(x, \partial_1)\\
&-A^{0(\beta_1)}(x-\xi, \partial_1)]\partial_3^{\beta_2}Z(\xi, t; x-\xi)\partial_x^{\beta_3}\phi(x-\xi)d\xi
\end{aligned}$$

and

$$\partial_x^\beta \int_{R^n} K_{j+1}(x, t, \xi)\phi(\xi)d\xi = \sum_{\beta_1+\beta_2+\beta_3=\beta} C(\beta_1, \beta_2, \beta_3) \int_0^t \int_{R^n} [A^{(\beta_1)}(x, \partial_1)$$
$$-A^{0(\beta_1)}(x-\xi, \partial_1)]\partial_3^{\beta_2} Z(\eta, t-\tau; x-\eta)\partial_x^{\beta_3}\Big(\int_{R^n} K_j(x-\eta, \tau, \xi)\phi(\xi)d\xi\Big)d\eta\, d\tau\,.$$
$$(A^{(\beta)}(x, \partial) = \sum_{|\alpha|\le 2p} \partial_x^\beta A(x)\partial^\alpha)\,.$$

So,

$$|x|^m |\partial_x^\beta \int_0^t \int_{R^n} Z(x-\eta, t-\tau; \xi)\Big(\int_{R^n} K_j(\xi, \tau, \eta)\phi(\eta)d\eta\Big)d\xi\, d\tau|$$
$$= |x|^m |\sum_{\gamma\le\beta}\binom{\beta}{\gamma}\int_0^t\int_{R^n} \partial_3^{\beta-\gamma} Z(\xi, t-\tau; x-\xi)\partial_x^\gamma\Big(\int_{R^n} K_j(x-\xi, \tau, \eta)\phi(\eta)d\eta\Big)d\xi\, d\tau|$$
$$\le \text{const}\,(t^{j/(2p)}\|\phi\|_{m,|\beta|} + t^{(m+j)/(2p)}\|\phi\|_{0,|\beta|}) \underset{t\to 0}{\to} 0\,.$$

On the other hand, if $t\le 1$, by the estimates of the derivatives of the K_{j-s} in [FR1] Ch. 9.6, one has

$$\|\partial_x\Big(\sum_{j=\nu}^\infty K_j(x, t, \xi)\Big)\| \le \text{const}\; t^{(\nu-n-2p-|\beta|)/(2p)} \exp\,(-C(|x-\xi|^{2p}/t)^{1/(p-1)})$$

so that

$$|x|^m |\partial_x^\beta \int_{R^n} \sum_{j=\nu}^\infty K_j(x, t, \xi)\phi(\xi)d\xi| \le \text{const}\; t^{(\nu-2p-|\beta|)/(2p)}\|\phi\|_{m,0}\,.$$

If $\nu > |\beta|$, one has

$$|x|^m |\partial_x^\beta \int_0^t \int_{R^n} Z(x-\eta, t-\tau; \eta)\Big(\int_{R^n}\sum_{j=\nu}^\infty K_j(\eta, \tau, \xi)\phi(\xi)d\xi\Big)d\eta\, d\tau|$$
$$= |x|^m |\sum_{\gamma\le\beta}\binom{\beta}{\gamma}\int_0^t\int_{R^n} \partial_3^{\beta-\gamma} Z(\eta, t-\tau; x-\eta)\partial_x^\gamma\Big(\int_{R^n}\sum_{j=\nu}^\infty K_j(x-\eta, \tau, \xi)\phi(\xi)$$
$$\cdot d\xi\, d\eta\, d\tau| \le \text{const}\; t^{(\nu-|\beta|)/(2p)}\|\phi\|_{m,0} \underset{t\to 0}{\to} 0\,.$$

As

$$V(t)\phi(x) = \sum_{j=1}^{\nu-1}\int_0^t\int_{R^n} Z(x-\xi, t-\tau; \xi)\Big(\int_{R^n} K_j(\xi, \tau, \eta)\phi(\eta)d\eta\Big)d\tau\, d\xi$$
$$+\int_0^t\int_{R^n} Z(x-\xi, t-\tau; \xi)\Big(\int_{R^n}\sum_{j=\nu}^\infty K_j(\xi, \tau, \eta)\phi(\eta)d\eta\Big)d\tau\, d\xi\,,$$

we have shown that, $\forall m\ge 0$, $\forall\beta\in N_0^n$, $\forall\phi\in\mathcal{S}$, $p_{m\,\beta}(V(t)\phi)\underset{t\to 0}{\to} 0$. So we have proved that $t\to T(t)\phi\in C([0, +\infty[; \mathcal{S})\forall\phi\in\mathcal{S}$. To verify that the infinitesimal generator of the semigroup $\hat{A}_{\mathcal{S}}$ is $A_{\mathcal{S}}$, remark that, certainly, $\forall\phi\in D(\hat{A}_{\mathcal{S}})$, $\hat{A}_{\mathcal{S}}\phi = A\phi$ (see Def. 1.8) and so, as $\phi\in C^\infty(R^n; C^N)$, $\hat{A}_{\mathcal{S}}\phi = A_{\mathcal{S}}\phi$. As $A_{\mathcal{S}}$ is continuous, it follows that $\hat{A}_{\mathcal{S}}$ is continuous and, as it is closed and densely defined, it is necessarily $\hat{A}_{\mathcal{S}} = A_{\mathcal{S}}$. So the theorem is completely proved.

We recall the following definition (see [HC], page 294):

DEFINITION 2.2. Let $T=\{T(t)\,|\,t\geq 0\}$ be a C_0-semigroup in a locally convex space X. T is said quasi-equicontinuous if there exists $\omega\geq 0$, such that $\{\exp(-\omega t)T(t)\,|\,t\geq 0\}$ is equicontinuous.

We want to emphasize the following important difference between the semigroups we have considered in K_q and in $\mathcal{S}$: in the first case they are not quasi-equicontinuous, in the second they are.

Now we shall give an example to show that quasi-equicontinuity is not satisfied in K_q. Consider the simplest case, namely $n=N=1$, $A(x, \partial)=\partial_x^2$, $2p=q=2$. For $\delta\in R$, put $f_\delta(x)=\exp(\delta x)$; $f_\delta\in K_2\,\forall\delta\in R$. If the semigroup $\{T(t)\,|\,t\geq 0\}$ were quasi-equicontinuous, there would exist $\omega\geq 0$, such that $\exp(-\omega t)T(t)f_\delta(x)\underset{t\to\infty}{\to} 0$, $\forall x\in R^n$, $\forall\delta\in R$. But $T(t)f_\delta(x)=\exp(\delta^2 t+\delta x)$ and so such an ω cannot exist.

Now we want to prove the following

Theorem 2.3. *The semigroup $\{T(t)\,|\,t\geq 0\}$ is quasi-equicontinuous in $\mathcal{S}$.*

To prove this result, we need the following lemma:

Lemma 2.4. *There exists $\rho\geq 0$, such that $\forall m, m'\geq 0$, with $m<m'$, $\forall\mu$, $\nu\in]0, 1[$, with $\mu<\nu$, there exist $C>0$, $\alpha\in]-1, 0]$, $\alpha'>\alpha$, dependent on m, m', μ, ν, such that*

$$\|T(t)\phi\|_{m,2p,\mu}\leq C(t^\alpha+t^{\alpha'})\exp(\rho t)\|\phi\|_{m',0,\nu}\,,$$

$\forall t>0$, $\forall\phi\in\mathcal{S}$.

Proof. Put $\rho=\max\{\omega_j\,|\,j=0, 1, 2, 3\}$. Let $m, m'\geq 0$, with $m<m'$, $\mu, \nu\in]0, 1[$ with $\mu<\nu$. We have to prove that

$$\begin{aligned}\|T(t)\|_{m,2p,\mu}&=\sum_{|\beta|\leq 2p}\|(1+|x|^m)\partial_x^\beta T(t)\phi\|_\infty+\sum_{|\beta|=2p}[\partial_x^\beta T(t)\phi]_\mu\\&\leq C(t^\alpha+t^{\alpha'})\exp(\rho t)(\|(1+|x|^{m'})\phi\|_\infty+[\phi]_\nu)\,,\end{aligned}$$

for certain C, α, α' with the declared properties. As usual, $T(t)=U(t)+V(t)$. If $|\beta|\leq 2p-1$, using Lemma 1.1, one proves that

$$(1+|x|^m)\,|\partial_x^\beta U(t)\phi(x)|\leq C(\beta, m)\exp(\omega_0 t)(t^{-|\beta|/(2p)}+t^{(m-|\beta|)/(2p)})\|\phi\|_{m,0}\,.$$

If $|\beta|=2p$,

$$\begin{aligned}&(1+|x|)^m\,|\partial_x^\beta U(t)\phi(x)|\\&\leq(1+|x|^m)\left|\int_{R^n}[\partial_1^\beta Z(x-\xi, t; \xi)-\partial_1^\beta Z(x-\xi, t; x)]\phi(\xi)d\xi\right|\end{aligned}$$

$$+(1+|x|^m)\int_{R^n}\|\partial_x^\beta Z(x-\xi,\ t;\ x)\|\,|\phi(\xi)-\phi(x)|\,d\xi = I_1+I_2\,.$$

Using Lemma 1.2, one has,

$$I_1\leq C(m,\ \beta)\exp(\omega_1 t)(t^{1/(2p)-1}+t^{(m+1)/(2p)-1})\|\phi\|_{m,0}\,.$$

Moreover,

$$I_2\leq C(\beta)(1+|x|^m)\exp(\omega_0 t)\,t^{-n/(2p)-1}\int_{R^n}\exp(-C_0(|x-\xi|^{2p}/t)^{1/(2p-1)})$$
$$\cdot[\phi]_\nu^\varepsilon|x-\xi|^{\nu\varepsilon}|\phi(\xi)-\phi(x)|^{1-\varepsilon}d\xi\leq(\forall\varepsilon\in]0,\ 1[)$$
$$C(\beta)[\phi]_\nu^\varepsilon\exp(\omega_0 t)t^{(\nu\varepsilon)/(2p)-1}(1+|x|^m)\int_{R^n}\exp(-C_0|\xi|^q/2)\,|\phi(x-t^{1/(2p)}\xi)$$
$$-\phi(x)|^{1-\varepsilon}d\xi\,.$$

One has

$$(1+|x|^m)\int_{R^n}\exp(-C_0(|\xi|^q/2)\,|\phi(x-t^{1/(2p)})\xi-\phi(x)|^{1-\varepsilon}d\xi$$
$$\leq C(\beta,\ \varepsilon)\{(1+|x|^m)\int_{R^n}\exp(-C_0(|\xi|^q/2)\,|\phi(x-t^{1/(2p)}\xi)|^{1-\varepsilon}d\xi+\|\phi\|_{m/(1-\varepsilon),0}^{1-\varepsilon}\}$$
$$\leq C(\beta,\ \varepsilon)\|\phi\|_{m/(1-\varepsilon),0}^{1-\varepsilon}+t^{m/(2p)}p_{0,0}(\phi)^{1-\varepsilon}\}\ ,$$

so that

$$I_2\leq C(\beta,\ \varepsilon)\exp(\omega_0 t)(t^{\nu\varepsilon/(2p)-1}+t^{(m+\nu\varepsilon)/(2p)-1})\,[\phi]_\nu^\varepsilon\|\phi\|_{m/(1-\varepsilon),0}^{1-\varepsilon}\,,$$

$\forall\varepsilon]\in 0,\ 1[$. If $\varepsilon=1-m/m'$, one has

$$I_2\leq C(\beta,\ m,\ m')\exp(\omega_0 t)(t^{\nu(1-m/m')/(2p)-1}+t^{m/(2p)+\nu(1-m/m')/(2p)-1})([\phi]_\nu+\|\phi\|_{m',0})\,.$$

Further, again with $|\beta|=2p$, we have

$$\partial_x^\beta U(t)\phi(x)=\int_{R^n}[\partial_1^\beta Z(x-\xi,\ t;\ \xi)-\partial_1^\beta Z(x-\xi,\ t;\ x)]\phi(\xi)d\xi$$
$$+\int_{R^n}\partial_1^\beta Z(x-\xi,\ t;\ x)[\phi(\xi)-\phi(x)]d\xi\ ,$$

so that

$$\partial_1^\beta U(t)\phi(x)-\partial_1^\beta U(t)\phi(y)=\int_{R^n}[\partial_1^\beta Z(x-\xi,\ t;\ \xi)-\partial_1^\beta Z(x-\xi,\ t;\ x)$$
$$-\partial_1^\beta Z(y-\xi,\ t;\ \xi)+\partial_1^\beta Z(y-\xi,\ t;\ y)]\phi(\xi)d\xi$$
$$+\int_{R^n}[\partial_1^\beta Z(x-\xi,\ t;\ x)-\partial_1^\beta Z(y-\xi,\ t;\ x)]\,[\phi(\xi)-\phi(x)]\,d\xi$$
$$+\int_{R^n}[\partial_1^\beta Z(y-\xi,\ t;\ x)-\partial_1^\beta Z(y-\xi,\ t;\ y)]\,[\phi(\xi)-\phi(x)]d\xi$$

$$+\int_{R^n}\partial_1^\beta Z(y-\xi,\ t;\ y)d\xi[\phi(y)-\phi(x)]=J_1+J_2+J_3+J_4\,.$$

One has

$$J_1=\int_{R^n}[\partial_1^\beta Z(\xi,\ t;\ x-\xi)-\partial_1^\beta Z(\xi,\ t;\ x)]\phi(x-\xi)d\xi$$
$$-\int_{R^n}[\partial_1^\beta Z(\xi,\ t;\ y-\xi)-\partial_1^\beta Z(\xi,\ t;\ y)]\,\phi(y-\xi)d\xi$$
$$=\int_{R^n}[\partial_1^\beta Z(\xi,\ t;\ x-\xi)-\partial_1^\beta Z(\xi,\ t;\ x)-\partial_1^\beta Z(\xi,\ t;\ y-\xi)+\partial_1^\beta Z(\xi,\ t;\ y)]\phi(x-\xi)d\xi$$
$$+\int_{R^n}[\partial_1^\beta Z(\xi,\ t;\ y-\xi)-\partial_1^\beta Z(\xi,\ t;\ y)]\,[\phi(x-\xi)-\phi(y-\xi)]d\xi=J_{11}+J_{12}\,.$$

One has, by Lemma 1.2,

$$\begin{aligned}K(\xi,\ t,\ x,\ y)&=\|\partial_1^\beta Z(\xi,\ t;\ x-\xi)-\partial_1^\beta Z(\xi,\ t;\ x)-\partial_1^\beta Z(\xi,\ t;\ y-\xi)+\partial_1^\beta Z(\xi,\ t;\ y)\|\\&\le C(\beta)t^{-(n+2p)/(2p)}\exp(\omega_1 t)\,|\xi|\exp(-C_1(|\xi|^{2p}/t)^{1/(2p-1)})\\&\le C(\beta)t^{(1-n-2p)/(2p)}\exp(\omega_1 t)\exp\left(-\frac{C_1}{2}(|\xi|^{2p}/t)^{1/(2p-1)}\right).\end{aligned}$$

On the other hand,

$$\begin{aligned}K(\xi,\ t,\ x,\ y)&\le\|\partial_1^\beta Z(\xi,\ t;\ x-\xi)-\partial_1^\beta Z(\xi,\ t;\ y-\xi)\|+\|\partial_1^\beta Z(\xi,\ t;\ x)-\partial_1^\beta Z(\xi,\ t;\ y)\|\\&\le(\text{by Lemma 1.2})\ C(\beta)t^{-(n+2p)/(2p)}|x-y|\ \exp(\omega_1 t)\exp(-C_1(|\xi|^{2p}/t)^{1/(2p-1)})\,,\end{aligned}$$

so that

$$\begin{aligned}(14)\quad &K(\xi,\ t,\ x,\ y)\\&\le C(\beta,\ \mu)|x-y|^\mu t^{(1-\mu-n-2p)/(2p)}\exp(\omega_1 t)\exp(-(C_1/2)(|\xi|^{2p}/t)^{1/(2p-1)})\,.\end{aligned}$$

It follows

$$J_{11}\le C(\beta,\ \mu)\exp(\omega_1 t)\,|x-y|^\mu t^{(1-\mu)/(2p)-1}\|\phi\|_{0,0}\,.$$

Further,

$$J_{12}\le C(\beta,\ \mu)t^{1/(2p)-1}\exp(\omega_1 t)\,[\phi]_\mu\,|x-y|^\mu\,.$$

It follows

$$J_1\le C(\beta,\ \mu)\,|x-y|^\mu\exp(\omega_1 t)(t^{(1-\mu)/(2p)-1}+t^{1/(2p)-1})(\|\phi\|_{0,0}+[\phi]_\mu)\,.$$

One has

$$J_2=\int_{R^n}\partial_1^\beta Z(\xi,\ t;\ x)[\phi(x-\xi)-\phi(x)]\,d\xi-\int_{R^n}\partial_1^\beta Z(\xi,\ t;\ x)[\phi(y-\xi)-\phi(x)]d\xi\,.$$

Note that the second addend is equal to $\int_{R^n}\partial_1^\beta Z(\xi,\ t;\ x)\,[\phi(y-\xi)-\phi(y)]\,d\xi$, so that

$$|J_2| \leq C(\beta, \mu) \exp(\omega_0 t) t^{\nu/(2p)-1} [\phi]_\nu .$$

On the other hand,

$$J_2 = \int_{R^n} \partial_1^\beta Z(\xi, t; x) [\phi(x-\xi) - \phi(y-\xi)] d\xi ,$$

and so

$$|J_2| \leq C(\beta, \mu) \exp(\omega_0 t) t^{-1} |x-y| [\phi]_\nu .$$

Putting together the two estimates,

$$|J_2| \leq C(\beta, \mu, \nu) \exp(\omega_0 t) t^{(\nu-\mu)/(2p)-1} |x-y|^\mu [\phi]_\nu .$$

Then we remark that

$$J_3 = \int_{R^n} [\partial_1^\beta Z(y-\xi, t; x) - \partial_1^\beta Z(y-\xi, t; y)] [\phi(\xi) - \phi(y)] d\xi ,$$

so that, by Lemma 1.2,

$$|J_3| \leq C(\mu, \beta, \nu) |x-y|^\mu \exp(\rho t) t^{\nu/(2p)-1} [\phi]_\nu .$$

Finally, $J_4=0$.

So, for $|\beta|=2p$,

$$[\partial_x^\beta U(t)\phi]_\mu \leq C(\mu, \beta, \nu) \exp(\rho t) (t^{(\nu-\mu)/(2p)-1} + t^{1/(2p)-1}) (\|\phi\|_{0,0} + [\phi]_\nu) .$$

Putting together the two estimates we have obtained, we have that

$$\|U(t)\phi\|_{m,2p,\mu} \leq C(m, m', \mu, \nu) \exp(\rho t) (t^{\alpha_1} + t^{\alpha_1'}) \|\phi\|_{m',0,\nu} ,$$

with $-1 < \alpha_1 < \alpha_1'$ and α_1, α_1' dependent on m, m', μ, ν. Now we consider $V(t)\phi$. If $|\beta| \leq 2p-1$, $m \geq 0$,

$$\begin{aligned} &(1+|x|^m) |\partial_x^\beta V(t)\phi(x)| \\ &= (1+|x|^m) \left| \int_0^t \int_{R^n} \partial_1^\beta Z(x-\eta, t-\tau; \eta) \left(\int_{R^n} \Phi(\eta, \tau, \xi) \phi(\xi) d\xi \right) d\eta d\tau \right| \\ &\leq (\text{by Lemmas 1.1 and 1.3})\ C(m, \beta) \exp(\rho t) (t^{(1-|\beta|)/(2p)} + t^{(m+1-|\beta|)/(2p)}) \|\phi\|_{m,0} . \end{aligned}$$

Now let $|\beta|=2p$.

$$\begin{aligned} (1+|x|^m) |\partial_x^\beta V(t)\phi(x)| &\leq (1+|x|^m) \left| \int_0^t \int_{R^n} \partial_1^\beta Z(x-\eta, t-\tau; \eta) \left(\int_{R^n} [\Phi(\eta, \tau, \xi) \right.\right. \\ &\quad \left.\left. - \Phi(x, \tau, \xi)] \phi(\xi) d\xi \right) d\eta d\tau \right| + (1+|x|^m) \left| \int_0^t \int_{R^n} [\partial_1^\beta Z(x-\eta, t-\tau; \eta) \right. \\ &\quad \left. - \partial_1^\beta Z(x-\eta, t-\tau; x)] \left(\int_{R^n} \Phi(x, \tau, \xi) \phi(\xi) d\xi \right) d\eta d\tau \right| = I_1 + I_2 . \end{aligned}$$

From Lemmas 1.1 and 1.3, one has

$$I_1 \le \text{const} \exp(\rho t)(t^{1/(2p)} + t^{(m+1)/(2p)-1}) \|\phi\|_{m,0}\,.$$

Further,

$$I_2 \le (\text{by Lemmas 1.2 and 1.3}) \text{ const} \exp(\rho t)(t^{1/(2p)-1} + t^{(m-2)/(2p)-1}) \|\phi\|_{m,0}\,.$$

So, if $|\beta| = 2p$,

$$\|\partial_x^\alpha V(t)\phi\|_{m,0} \le C(m, \beta) \exp(\rho t)(t^{1/(2p)-1} + t^{(m+2)/(2p)-1}) \|\phi\|_{m,0}\,.$$

It remains to estimate $[\partial_1^\alpha V(t)\phi]_\mu$, for $|\beta| = 2p$. Posing

$$g(t)(x) = \int_{R^n} \Phi(x, \tau, \xi)\phi(\xi)d\xi\,,$$

by Lemma 1.3,

$$|g(t)(x) - g(t)(y)| \le C(\nu) \exp(\omega_3 t) t^{(1-\nu)/(2p)-1} |x-y|^\nu \|\phi\|_{0,0}$$

and

$$|g(t)(x)| \le \text{const} \exp(\omega_2 t) t^{1/(2p)-1} \|\phi\|_{0,0}$$

So,

$$\begin{aligned}
&|\partial_x^\beta V(t)\phi(x) - \partial_x^\beta V(t)\phi(y)| \\
&\le \left|\int_0^t \int_{R^n} [\partial_1^\beta Z(\xi, t-\tau; x-\xi) - \partial_1^\alpha Z(\xi, t-\tau; y-\xi)] g(\tau)(x-\xi) d\xi d\tau\right| \\
&+ \left|\int_0^t \int_{R^n} \partial_1^\beta Z(\xi, t-\tau; y-\xi)[g(\tau)(x-\xi) - g(\tau)(y-\xi)] d\xi d\tau\right| = I_1 + I_2\,.
\end{aligned}$$

$$\begin{aligned}
I_1 &\le \left|\int_0^t \int_{R^n} [\partial_1^\beta Z(\xi, t-\tau; x-\xi) - \partial_1^\beta Z(\xi, t-\tau; x) - \partial_1^\beta Z(\xi, t-\tau; y-\xi)\right. \\
&\left. + \partial_1^\beta Z(\xi, t-\tau; y)] g(\tau)(x-\xi d\xi) d\tau\right| \\
&+ \left|\int_0^t \int_{R^n} [\partial_1^\beta Z(\xi, t-\tau; x) - \partial_1^\beta Z(\xi, t-\tau; y)][g(\tau)(x-\xi) - g(\tau)(x)] d\xi d\tau\right| \\
&= I_{11} + I_{12}\,.
\end{aligned}$$

One has, by (14),

$$\begin{aligned}
I_{11} &\le C(\beta, \mu) \int_0^t \int_{R^n} |x-y|^\mu (t-\tau)^{(1-\mu-n-2p)/(2p)} \exp(\omega_1(t-\tau)) \\
&\times \exp(-(C_1/2)(|\xi|^{2p}/(t-\tau))^{1/(2p-1)}) \exp(\omega_2 \tau)\tau^{1/(2p)-1} \|\phi\|_{0,0} d\tau d\xi \\
&\le \text{const} \exp(\rho t)|x-y|^\mu t^{(2-\mu)/(2p)-1} \|\phi\|_{0,0}\,.
\end{aligned}$$

By Lemma 1.2,

$$I_{12}\leq \text{const}\int_0^t\int_{R^n}|x-y|\exp(\omega_1(t-\tau))(t-\tau)^{-n/(2p)-1}\exp(-C_1(|\xi|^{2p}/(t-\tau))^{1/(2p-1)})$$
$$\times\exp(\omega_3\tau)\tau^{(1-\mu)/(2p)-1}|\xi|^{\mu}\|\phi\|_{0,0}d\xi d\tau\leq \text{const}\ |x-y|\exp(\rho t)t^{1/(2p)-1}\|\phi\|_{0,0}$$

On the other hand,

$$I_{12}\leq \text{const}\exp(\omega_1 t)t^{1/(2p)-1}\|\phi\|_{0,0},$$

so that

$$I_{12}\leq \text{const}\exp(\rho t)t^{1/(2p)-1}\|\phi\|_{0,0}|x-y|^{\mu}.$$

Then,

$$I_2\leq\left|\int_0^t\int_{R^n}[\partial_1^\beta Z(\xi,t-\tau;y-\xi)-\partial_1^\beta Z(\xi,t-\tau;y)][g(\tau)(x-\xi)-g(\tau)(y-\xi)]d\xi\,d\tau\right|$$
$$+\left|\int_0^t\int_{R^n}\partial_1^\beta Z(\xi,t-\tau;y)[g(\tau)(x-\xi)-g(\tau)(y-\xi)-g(\tau)(x)+g(\tau)(y)]d\xi\,d\tau\right|$$
$$=I_{21}+I_{22}.$$

$$I_{21}\leq \text{const}\exp(\rho t)t^{(2-\mu)/(2p)-1}|x-y|^{\mu}\|\phi\|_{0,0}.$$

Further,

$$|g(\tau)(x-\xi)-g(\tau)(x)-g(\tau)(y-\xi)+g(\tau)(y)|$$
$$\leq C(\mu,\beta)\exp(\rho t)\tau^{(1-\nu)/(2p)-1}|x-y|^{\mu}|\xi|^{\nu-\mu}\|\phi\|_{0,0},$$

so that

$$I_{22}\leq \text{const}\exp(\rho t)t^{(1-\mu)/(2p)-1}|x-y|^{\mu}\|\phi\|_{0,0}.$$

It follows

$$[\partial_x^\beta V(t)\phi]_\mu\leq C(\mu,\beta)(t^{(1-\mu)/(2p)-1}+t^{(2-\mu)/(2p)-1})\|\phi\|_{0,0}\exp(\rho t).$$

So the lemma is completely established.

Lemma 2.5. *Let $\beta\in N_0^n$. Then $\forall t\geq 0$*

$$\partial_x^\beta T(t)\phi=T(t)(\partial_x^\beta\phi)+\int_0^t T(t-s)g_\beta(s)ds,$$

with $g_\beta(t)=\sum_{\gamma<\beta}\binom{\beta}{\gamma}A^{(\beta-\gamma)}(x,\partial)\partial_x^\gamma T(t)\phi$ (the integral is intended in the Riemann sense).

Proof. $\forall\beta,\gamma,\ \phi\to A^{(\beta-\gamma)}(x,\partial)\phi$ is continuous in $\mathcal{S}$, so that $g_\beta\in C([0,+\infty[;\mathcal{S})$. By [FR1], Th. 10, $(t,x)\to T(t)\phi(x)\in C^\infty(\boldsymbol{R}^+\times\boldsymbol{R}^n;\boldsymbol{C}^N)$, so that, $\forall t>0$, $\forall x\in\boldsymbol{R}^n$,

$$\partial_t \partial_x^\beta T(t)\phi(x) = A_S \partial_x^\beta T(t)\phi(x) + g_\beta(t)(x).$$

If $\Psi \in \mathcal{S}'(\boldsymbol{R}^n)^N$, $\forall t>0$,

$$\begin{aligned}\partial_t \langle \partial_x^\beta T(t)\phi, \psi\rangle &= \langle A_S \partial_x^\beta T(t)\phi, \psi\rangle + \langle g_\beta(t), \psi\rangle \\ &= \langle \partial_x^\beta T(t)\phi, A_{S'}\psi\rangle + \langle g_\beta(t), \psi\rangle\end{aligned}$$

and this function is continuous on $[0, +\infty[$. From [GU] Th. 3.4 the result follows.

Proof of Theorem 2.3. We show that, $\forall m, m' \geq 0$, with $m<m'$, $\forall r \in \boldsymbol{N}_0$, $r \geq 2p$, $\forall \mu, \nu \in]0, 1[$ with $\mu < \nu$ there exist α, α', $C-1<\alpha<\alpha'$, (dependent on m, m', q, μ, ν) such that

$$\|T(t)\phi\|_{m,r,\mu} \leq C(t^\alpha + t^{\alpha'}) \exp(\rho t)\|\phi\|_{m',r-2p,\nu}, \tag{15}$$

where ρ is the number appearing in the statement of Lemma 2.4. The proof is by induction on $r \geq 2p$. If $r=2p$, it is true by Lemma 2.4. Assume (15) for a certain $r \geq 2p$. It is clear it suffices to show that,

$$\forall \beta,\ |\beta| = r+1-2p,\ \|\partial_x^\beta T(t)\phi\|_{m,2p,\mu} \leq C \exp(\rho t)(t^\alpha + t^{\alpha'})\|\phi\|_{m',|\beta|,\nu},$$

for certain $C>0$, $-1<\alpha<\alpha'<+\infty$. By Lemma 2.4,

$$\partial_x^\beta T(t)\phi = T(t)\partial_x^\beta \phi + \int_0^t T(t-s) g_\beta(s)\,ds.$$

So,

$$\begin{aligned}
&\|\partial_x^\beta T(t)\phi\|_{m,2p,\mu} \leq \|T(t)\partial_x^\beta \phi\|_{m,2p,\mu} + \left\|\int_0^t T(t-s)g_\beta(s)ds\right\|_{m,2p,\mu}. \\
&\|T(t)\partial_x^\beta \phi\|_{m,2p,\mu} \leq C_1 \exp(\rho t)(t^{\alpha_1} + t^{\alpha_2'})\|\partial_x^\beta \phi\|_{m',0,\nu} \\
&\quad \leq C_1 \exp(\rho t)(t^{\alpha_1} + t^{\alpha_1'})\|\phi\|_{m',|\beta|,\nu}. \\
&\left\|\int_0^t T(t-s)g_\beta(s)\,ds\right\|_{m,2p,\mu} \leq \int_0^t \|T(t-s)g_\beta(s)\|_{m,2p,\mu}\,ds, \\
&\|T(t-s)g_\beta(s)\|_{m,2p,\mu} \leq C_2 \exp(\rho(t-s))((t-s)^{\alpha_2} + (t-s)^{\alpha_2'})\|g_\beta(s)\|_{m'',0,\mu'},
\end{aligned}$$

with $m<m''<m'$, $\mu<\mu'<\nu$, by virtue of Lemma 2.3. But

$$\begin{aligned}&\|g_\beta(s)\|_{m'',0,\mu'} \leq \text{const}\,\|T(s)\phi\|_{m'',r,\mu'} \leq (\text{by the inductive assumption}) \\ &\quad \text{const}\exp(\rho s)(s^{\alpha_3} + s^{\alpha_3'})\|\phi\|_{m',r-2p,\nu}.\end{aligned}$$

So,

$$\begin{aligned}\|\partial_x^\beta T(t)\phi\|_{m,2p,\mu} &\leq C_1 \exp(\rho t)(t^\alpha + t^{\alpha'})\|\phi\|_{m',|\beta|,\nu} \\ &\quad + \text{const}\exp(\rho t)\int_0^t [(t-s)^{\alpha_2} + (t-s)^{\alpha_2'}](s^{\alpha_3} + s^{\alpha_3'})ds\,\|\phi\|_{m',|\beta|-1,\nu}.\end{aligned}$$

From this, (15) follows.

Now, let $\omega>\rho$. One has, taking into account that the norms $\|\ \|_{m,r,s}$ are a calibration for $\mathcal{S}$ and inequality (15), that $\{\exp(-\omega t)T(t)\,|\,t\geq 1\}$ is equicontinuous. On the other hand, in a barrelled space every C_0-semigroup is locally equicontinuous (see [KOMU], Prop. 1.1) and this gives the result.

We recall the following definition (see [YO], Ch. IX, 10):

DEFINITION 2.6. Let X be a locally convex space and $T=\{T(t)\,|\,t\geq 0\}$ a linear equicontinuous semigroup in X. T is said holomorphic if it admits a weakly holomorphic extension to some sector $\{|\operatorname{Arg}\lambda|\leq\theta_0\}$, for some θ_0 positive, and the extension is equicontinuous.

We have the following result:

Theorem 2.7. *Assume* (h2), (h3) *are satisfied; then there exists* $\omega\geq 0$ *such that the semigroup* $\{\exp(-\omega t)T(t)\,|\,t\geq 0\}$ *is holomorphic in* $\mathcal{S}$.

Proof. Owint to [YO], ch. IX, 10, it is sufficient to show that there exist $\omega\geq 0$, $\theta_1>\pi/2$, such that $|\operatorname{Arg}\lambda|\leq\theta_1$ implies $\omega+\lambda\in\rho(A_{\mathcal{S}})$ and $\{\lambda^k(\lambda+\omega-A_{\mathcal{S}})^{-k}\,|\,k\in\boldsymbol{N}_0,\ |\operatorname{Arg}\lambda|\leq\theta_1\}$ is equicontinuous. As the coefficients of $A_\alpha(x)$ are uniformly bounded, there exists $M>0$ such that $\forall x\in\boldsymbol{R}^n$, $\forall\xi\in\boldsymbol{R}^n$, with $|\xi|=1$, any eigenvalue of $A^0(x;i\xi)$ satisfies $|\lambda|\leq M$. So, by (h2), there exists $\theta_0>\pi/2$ such that, $\forall x\in\boldsymbol{R}^n$, with $|\xi|=1$, any eigenvalue of $A^0(x;i\xi)$ is of the form $\rho e^{i\theta}$, with $|\theta|\geq\theta_0$. Therefore, for every θ, $|\theta|<\theta_0-\pi/2$, the operators $e^{i\theta}A(x,\partial)$ satisfy (h2), (h3).

Now, fix θ_1, $\pi/2<\theta_1<\theta_0$. If $\theta=\theta_1-\pi/2$, by Theorem 2.3 there exists $\omega_0\geq 0$, such that the semigroups generated by $-\omega+A_{\mathcal{S}}$ and $e^{\pm i\theta}(-\omega+A_{\mathcal{S}})$ are equicontinuous. As $(\lambda e^{-i\theta}+\omega-A_{\mathcal{S}})^{-1}=e^{i\theta}(\lambda+e^{i\theta}\omega-e^{i\theta}A_{\mathcal{S}})^{-1}$, we shall have the result if we prove that, if B is the infinitesimal generator of an equicontinuous semigroup, $\{\lambda\in\boldsymbol{C}\,|\,\operatorname{Re}\lambda>0\}\subseteq\rho(B)$ and the operators $\{\lambda^k(\lambda-B)^{-k}\,|\,|\operatorname{Arg}\lambda|\leq\phi,\ k\in\boldsymbol{N}\}$ are equicontinuous, for any $\phi<\pi/2$. In fact, by the Hille-Yosida theorem (see [YO], Ch. IX, 7) the result is true if $\phi=0$. Let $\lambda'>0$, p a continuous seminorm. For $\lambda\in\boldsymbol{C}$, $k\in\boldsymbol{N}$,

$$p((\lambda-\lambda')^k(\lambda'-B)^{-k-1}x)\leq\lambda'^{(-k-1)}|\lambda-\lambda'|^k q(x)$$

with q continuous seminorm dependent only on p. Therefore, if $|\lambda-\lambda'|<\lambda'$, as $\mathcal{S}$ is complete, $(\lambda-B)^{-1}$ exists and

$$(\lambda-B)^{-1}x=\sum_{k=0}^{\infty}(-1)^k(\lambda-\lambda')^k(\lambda'-B)^{-k-1}x\,.$$

If $|\operatorname{Arg}\lambda|\leq\phi<\pi/2$, there exists $\lambda'>0$, such that $|\lambda-\lambda'|\leq\lambda'\sin\phi$ and $|\lambda|\leq$

λ'. So, for every continuous seminorm p, there exists a continuous seminorm q, such that, $\forall n\in \boldsymbol{N}$,

$$\begin{aligned}
p(\lambda^m(\lambda-B)^{-m}x) &= p(\lambda^m \sum_{k_1=0}^{\infty}\cdots\sum_{k_m=0}^{\infty}(-1)^{k_1+\cdots+k_m}(\lambda'-B)^{-k_1-\cdots-k_m-m}x) \\
&\leq |\lambda|^m \sum_{k_1=0}^{\infty}\cdots\sum_{k_m=0}^{\infty}|\lambda-\lambda'|^{k_1+\cdots+k_m}\lambda'^{(-k_1-\cdots-k_m-m)}q(x) \\
&\leq (1-\sin\phi)^{-1}q(x)\,.
\end{aligned}$$

With this the result is proved.

For what concerns $\mathcal{S}'=\mathcal{S}'(\boldsymbol{R}^n)^N$, we have:

Theorem 2.8. *Assume* $A(x, \partial)$ *satisfies* (h2), (h3). *Define* $D(A_{\mathcal{S}'})=\mathcal{S}'$, $A_{\mathcal{S}'}\phi=A(x, \partial)\phi$. *Then,* $A_{\mathcal{S}'}$ *is the infinitesimal generator of a quasi-equicontinuous semigroup in* $\mathcal{S}$. *Further, there exists* $\omega\geq 0$, *such that* $-\omega+A_{\mathcal{S}'}$ *is the infinitesimal generator of a holomorphic semigroup in* $\mathcal{S}$.

Proof. The formal adjoint $A'(x, \partial)$ satisfies (h2), (h3). Therefore the result follows from Theorems 2.3 and 2.7 and from the results of [YO] Ch. 9, 13, concerning dual semigroups in reflexive spaces.

REMARK 2.9. A consequence of Theorem 2.8 is that the equation $(\lambda-A(x, \partial))u=f$ has a unique solution in $\mathcal{S}'\,\forall f\in\mathcal{S}'$, for any λ sufficiently large.

REMARK 2.10. It is easily seen that, if $\phi\in\mathcal{S}'$, $\forall t>0$ $T(t)\phi(x)=\langle\Gamma(x, t, \xi), \phi(\xi)\rangle$, so that, owing to the estimates of [FRI1] Ch. 9, $(t, x)\rightarrow T(t)\phi(x)\in C^\infty(\boldsymbol{R}^+\times\boldsymbol{R}^n; \boldsymbol{C}^N)$.

3. Generalized Cauchy problem for certain ultraparabolic systems

In this section we shall consider generalized Cauchy problems for certain systems we shall call ultraparabolic, extending some results of [VD]. In the case of equations, papers on this subject (in spaces of Sobolev type) are (under even more general conditions) [GE], [SA], [GI], [VG]. Besides, abstract equations leading to ultraparabolic problems in Banach and Hilbert spaces were considered also by A. Favini [FA] and J.L. Lions [LI].

Now we introduce the general assumptions we shall work with. Let O be an open connected subset of $\boldsymbol{R}^m (m\geq 1)$, Γ the boundary of O. Assume that Γ is a C^1 manifold in $\boldsymbol{R}^m$ of dimension $m-1$, O is on one side of Γ, b is a C^1 real vector field on $\bar{O}$ such that:

(m1) $b(t)\neq 0\,\forall t\in\bar{O}$.

(m2) $b(t)\cdot\nu(t)>0\,\forall t\in\Gamma$, with $\nu(t)$ normal vector to Γ at t, inward to O.

(m3) Let $s\rightarrow S(s, t)(s\in I(t), t\in\bar{O})$ the maximal solution of

$$\frac{dv}{ds}(s) = b(v(s))$$
$$v(0) = t\,.$$

We assume that $\forall t \in \bar{O} \exists t' \in \Gamma$, such that $t = S(s, t')$ for some $s \in I(t')$. Now let X be a sequentially complete, barrelled space, A the infinitesimal generator of a strongly continuous semigroup in X, $c \in C^1(\bar{O}; \boldsymbol{C})$. We consider the problem

$$\begin{aligned} &\sum_{j=1}^{m} b_j(t)\frac{\partial u}{\partial t}(t) + c(t)u(t) - Au(t) = f(t) \qquad \text{in } O; \\ &u(t) = g(t)\,, \qquad t \in \Gamma \end{aligned} \tag{16}$$

with $g \in C(\Gamma; X)$, $f \in C(\bar{O}; X)$. A solution u of (16) is a function $u \in C^1(O; X) \cap C(\bar{O}; X)$ such that, $\forall t \in O$, $u(t) \in D(A)$, the first condition of (16) is satisfied $\forall t \in O$, the second for every $t \in \Gamma$ (for the definition of $C^1(O; X)$ see [GWS], III. III). To study this problem, we need some results concerning the flow S generated by b. First of all we remark that, thanks to (m1)—(m3), for a fixed $t \in \bar{O}$ there exists a unique $t' \in \Gamma$ and a unique $s \in I(t')$, such that $t = S(s, t')$. Moreover, $s \geq 0$ and $s = 0$ if and only if $t \in \Gamma$. So, we can define $\Phi: \bar{O} \to \bar{R}^+ \times \Gamma$, $\Phi(t) = (\Phi_1(t), \Phi_2(t))$, such that $S(\Phi_1(t), \Phi_2(t)) = t$.

One has:

Lemma 3.1 *The mapping Φ is of cass C^1 from $\bar{O}$ to $\boldsymbol{R} \times \boldsymbol{R}^m$. More generally if b is of class C^k and Γ is a C^k manifold in R^m, Φ is a calss C^k ($k \geq 1$).*

Proof. Assume b and Γ of class $C^k (k \geq 1)$. Let $t \in \Gamma$, Ψ a C^k-diffeomorphism between a neighbourhood U of t in $\bar{O}$ and a neighbourhood $\Psi(U)$ of O in $\boldsymbol{R}^m_+ = \{y \in \boldsymbol{R}^m \mid y_m > 0\}$, such that $\Psi(U \cap \Gamma) = \Psi(U) \cap \{(y', 0) \mid y' \in \boldsymbol{R}^{m-1}\}$. For $(y', 0) \in \Psi(U)$, define $\hat{S}(s, y') = S(s, \Psi^{-1}(y', 0))$. $\hat{S}$ is defined on an open subset of $\boldsymbol{R} \times \boldsymbol{R}^{m-1}$, it is of class C^k (because Ψ^{-1} is of class C^k and by well known results of regular dependence of the solution on the initial datum) and $d\hat{S}(0, 0)(\sigma, z) = \sigma b(t) + d_{y'}\Psi^{-1}(y', 0)z\,((\sigma, z) \in \boldsymbol{R} \times \boldsymbol{R}^{m-1})$. One has that $d_{y'}\Psi^{-1}(y', 0)z \in T_t(\Gamma)$ (the tangent space to Γ in t).

So, by condition (m2), $d\hat{S}(0, 0)$ is a linear isomorphism. This implies that $\hat{S}$ is a C^k diffeomorphism between a neighbourhood $\tilde{V}$ of $(0, 0)$ in $[0, +\infty[\times \boldsymbol{R}^{m-1}$ and a neighbourhood $\hat{U} = \hat{S}(V)$ of t in $\bar{O}$. If $\hat{t} \in \hat{U}$, necessarily

$$\Phi(\hat{t}) = ((\hat{S}^{-1})_1(\hat{t}), \Psi^{-1}((\hat{S}^{-1})_2(\hat{t}), 0)$$

so that Φ is of class C^k in a neighbourhood Ω of Γ in $\bar{O}$. Now, assume $\hat{t} \in O$. There exists $s > 0$, such that $S(-s, \hat{t}) \in \Omega$ and, for this fixed s, there exists a neighbourhood V of $\hat{t}$ in O such that $S(-s, t)$ is defined and $S(-s, t) \in \Omega$. For $t \in V$, $\Phi_1(t) = \Phi_1(S(-s,t)) + s$, $\Phi_2(t) = \Phi_2(S(-s, t))$. As Φ is of class C^k in Ω, s

is fixed, $t\to S(-s,t)$ is of class C^k, Φ is of class C^k in V and so it is of class C^k in $\bar{O}$.

Proposition 3.2. *Assume u is a solution of* (1.6). *Then, $\forall t\in\bar{O}$, if* $W(s,t)=\exp\left(\int_0^t c(S(\sigma-\Phi_1(t),t))d\sigma\right)$,

$$
\begin{aligned}
(17)\quad u(t) = {} & W(\Phi_1(t),t)^{-1}\exp(\Phi_1(t)A)g(\Phi_2(t)) \\
& +\int_0^{\Phi_1(t)}\exp((\Phi_1(t)-\sigma)A)W(\Phi_1(t),t)^{-1}W(\sigma,t)f(S(\sigma-\Phi_1(t),t))d\sigma\,.
\end{aligned}
$$

Proof. Put $v(s)=u(S(s-\Phi_1(t),t))(s\in]0,\delta[$, for some $\delta>0)$. Then, for $s\in]0,\delta[$,

$$
\begin{aligned}
v'(s) &= \sum_{j=1}^{m} b_j(S(s-\Phi_1(t),t))\frac{\partial u}{\partial t_j}(S(s-\Phi_1(t),t)) \\
&= -c(S(s-\Phi_1(t),t))v(s)+Av(s)+f(S(s-\Phi_1(t),t))\,.
\end{aligned}
$$

From this, for $s\in]0,\delta[$,

$$\frac{d}{ds}(W(s,t)v(s)) = A(W(s,t)v(s))+W(s,t)f(S(s-\Phi_1(t),t))\,.$$

One has that $\forall x'\in D(A')$ $s\to\langle W(s,t)v(s),x'\rangle\in C^1(]0,\delta[)\cap C([0,\delta[)$ and, for $s>0$,

$$\frac{d}{ds}\langle W(s,t)v(s),x'\rangle = \langle W(s,t)v(s),A'x'\rangle+\langle W(s,t)f(S(s-\Phi_1(t),t)),x'\rangle\,.$$

It follows from [GU] Th. 3.4 that

$$
\begin{aligned}
W(s,t)v(s) &= \exp(sA)v(0)+\int_0^s\exp((s-\sigma)A)W(\sigma,t)f(S(\sigma-\Phi_1(t),t))d\sigma\,, \\
v(s) &= W(s,t)^{-1}\exp(sA)v(0)+\int_0^s\exp((s-\sigma)A)W(s,t)^{-1}W(\sigma,t) \\
&\qquad\qquad\times f(S(\sigma-\Phi_1(t),t))d\sigma\,.
\end{aligned}
$$

From this the result follows.

Proposition 3.3 *Assume that $f=0$. Let $\{\exp(tA)\,|\,t\geq 0\}$ be the semigroup generated by A. Assume that $T(t)(X)\subseteq D(A)\forall t>0$. Then, if $g\in C^1(\Gamma;X)$ (that is, g is the restriction to Γ of a C^1 function defined from $\boldsymbol{R}^m$ to X), $u(t)=W(\Phi_1(t),t)^{-1}\exp(\Phi_1(t)A)g(\Phi_2(t))$ is the only solution of* (17). *If $A\in\mathcal{L}(X)$, u is of class C^1 on $\bar{O}$.*

Proof. One has that $t\to W(\Phi_1(t),t)^{-1}\in C^1(\bar{O};\boldsymbol{C})$, $s\to\exp(sA)x$ belongs to $C([0,+\infty[;X)\cap C^1(]0,+\infty[;X)\forall x\in X$ and to $C^1([0,+\infty[;X)$ if $A\in\mathcal{L}(X)$

and $\dfrac{d}{ds}(\exp(sA))=A\exp(sA)$. As X is a barrelled space and $t\to\exp(\Phi_1(t)A))\cdot g(\Phi_2(t))\in C^1(O;X)\cap C(\bar{O};X)(\in C^1(\bar{O};X)$ if $A\in\mathcal{L}(X))$. From this the result follows by computation.

Proposition 3.4. *Consider* (16) *with* $g=0$. *Assume* $f\in C^1(\bar{O};X)$. *Then*

$$u(t)=W(\Phi_1(t),t)^{-1}\int_0^{\Phi_1(t)}\exp((\Phi_1(t)-\sigma)A)W(\sigma,t)f(S(\sigma-\Phi_1(t),t))d\sigma$$

solves (16).

Proof.
$$\int_0^{\Phi_1(t)}\exp((\Phi_1(t)-\sigma)A)W(\sigma,t)f(S(\sigma-\Phi_1(t),t))d\sigma = \int_0^{\Phi_1(t)}\exp(\sigma A)W(\Phi_1(t)-\sigma,t)f(S(-\sigma,t))d\sigma\,.$$

Put $g(\sigma,t)=W(\Phi_1(t)-\sigma,t)f(S(-\sigma,t))$. g is of class C^1 on its domain. We have, for $t>0$, $|\rho|$ sufficiently small,

$$\rho^{-1}\Big(\int_0^{\Phi_1(t+\rho e^j)}\exp(sA)g(s,t+\rho e^j)ds-\int_0^{\Phi_1(t)}\exp(sA)g(s,t)ds\Big)$$
$$=\rho^{-1}\int_{\Phi_1(t)}^{\Phi_1(t+\rho e^j)}\exp(sA)g(s,t+\rho e^j)ds+\rho^{-1}\int_0^{\Phi_1(t)}\exp(sA)[g(s,t+\rho e^j)-g(s,t)]ds\,.$$

As X is barrelled, $\{\exp(sA)\,|\,s\geq 0\}$ is locally equicontinuous. So,

$$\rho^{-1}\int_0^{\Phi_1(t)}\exp(sA)[g(s,t+\rho e^j)-g(s,t)]ds\underset{\rho\to 0}{\longrightarrow}\int_0^{\Phi_1(t)}\exp(sA)\frac{\partial g}{\partial t_j}(s,t)ds\,.$$

$$\rho^{-1}\int_{\Phi_1(t)}^{\Phi_1(t+\rho e^j)}\exp(sA)g(s,t+\rho e^j)ds=\rho^{-1}\int_{\Phi_1(t)}^{\Phi_1(t)+\rho(\partial\Phi_1/\partial t_j)(t)}\exp(sA)g(s,t+\rho e^j)ds$$
$$+\rho^{-1}\int_{\Phi_1(t)+\rho(\partial\Phi_1/\partial t_j)(t)}^{\Phi_1(t+\rho e^j)}\exp(sA)g(s,t+\rho e^j)ds\underset{\rho\to 0}{\longrightarrow}\frac{\partial\Phi_1}{\partial t_j}(t)\exp\Phi_1((t)A)g(\Phi_1(t),t)\,.$$

Besides, for $h>0$,

$$h^{-1}(\exp(tA)-1)\Big(\int_0^{\Phi_1(t)}\exp(sA)g(s,t)ds\Big)=h^{-1}\int_h^{\Phi_1(t)}\exp(sA)[g(s-h,t)-g(s,t)]ds$$
$$+h^{-1}\int_{\Phi_1(t)}^{\Phi_1(t)+h}\exp(sA)g(s-h,t)ds-h^{-1}\int_0^h\exp(sA)g(s,t)ds$$
$$\to-\int_0^{\Phi_1(t)}\exp(sA)\frac{\partial g}{\partial s}(s,t)ds+\exp(\Phi_1(t)A)(g(\Phi_1(t),t)-g(0,t))\,.$$

From these identities the result follows by computation.

Assume $m=1$, $O=\boldsymbol{R}^+$, $b_1(t)=1$, $c(t)=0$. Then we have the usual system

$$
(18) \qquad \begin{aligned} &\frac{du}{dt}(t) - Au(t) = f(t)\,, \qquad t > 0 \\ &u(0) = g\,. \end{aligned}
$$

Proposition 3.6. *Assume* $\exp(tA)(X) \subseteq D(A)\ \forall t > 0$, $tA \exp(tA)x \underset{t\to 0}{\longrightarrow} 0$ $\forall x \in X$, *f is Hölder continuous from* $[0, +\infty[$ *to* X (*that is,* $\forall p$ *continuous seminorm on* X *there exist* $\alpha \in]0, 1]$, $C \geq 0$, *such that* $p(f(t)-f(s)) \leq C|t-s|^{\alpha}$, $\forall s, t \geq 0$). *Then*

$$
(19) \qquad u(t) = \exp(tA)g + \int_0^t \exp((t-s)A)g(s)ds
$$

solves (18). *If* $A \in \mathcal{L}(X)$, *it suffices to assume that g is continuous.*

Proof. It is sufficient to show that $v(t) = \int_0^t \exp((t-s)A)g(s)ds$ solves (18) with $g=0$. for $\varepsilon > 0$, $t \geq 0$, define $v_\varepsilon(t) = \int_0^{t-\varepsilon} \exp((t-s)A)g(s)ds$. It is easily seen that, $\forall \varepsilon > 0$, for $t > \varepsilon$,

$$
v'_\varepsilon(t) = \exp(\varepsilon A)g(t-\varepsilon) + \int_0^{t-\varepsilon} A\exp((t-s)A)g(s)ds\,.
$$

One has $\exp(\varepsilon A)g(t-\varepsilon) \underset{\varepsilon\to 0}{\longrightarrow} g(t)$ uniformly on compact subsets on R^+ (because the semigroup is locally equicontinuous) and

$$
\begin{aligned}
\int_0^{t-\varepsilon} A\exp((t-s)A)g(s)ds &= \int_0^{t-\varepsilon} A\exp((t-s)A)(g(s)-g(t))ds \\
&\quad + \exp(tA)g(t) - \exp(\varepsilon A)g(t)\,.
\end{aligned}
$$

$$
\int_0^{t-\varepsilon} A\exp((t-s)A)(g(s)-g(t))\,ds \underset{\varepsilon\to 0}{\longrightarrow} \int_0^t A\exp((t-s)A)(g(s)-g(t))\,ds
$$

owing to the assumptions, uniformly on compact subsets of R^+ and $\exp(\varepsilon A)g(t) \underset{\varepsilon\to 0}{\longrightarrow} g(t)$, uniformly on compact subsets of R^+. It follows (see [GWS] III. III) that

$$
v \in C(R^+; X),\ v'(t) = \int_0^t AT(t-s)(g(s)-g(t))\,ds + \exp(tA)g(t)\,.
$$

On the other hand, for $t \geq \varepsilon$,

$$
Av_\varepsilon(t) = \int_0^{t-\varepsilon} AT(t-s)g(s)ds = \int_0^{t-\varepsilon} AT(t-s)(g(s)) - g(t))\,ds + \exp(tA)g(t) - g(t)\,.
$$

As A is closed, it follows, $\forall t > 0$,

$$Av(t)=\int_0^t AT(t-s)(g(s)-g(t))ds+\exp(tA)g(t)-g(t)$$

and the first result is proved. The second statement is easy.

So, we have:

Proposition 3.7. *Assume* (m1)–(m3) *are satisfied. Consider the problem* (16) *in* $X=K_q$, *with* A *defined in definition* 1.8. *Then,* $\forall f\in C^1(\bar{O};K_q)$ *there exists a unique solution of* (16) *given by* (17).

If $X=\mathcal{S}'$ *and we substitute in* (16) A *with* $A'_{\mathcal{S}}$, *an analogous result is true and the solution* $u\in C^1(\bar{O};\mathcal{S}')$. *In the case of problem* (18), *if* $X=K_q$ *and* f *is Hölder continuous,* (19) *furnishes the only solution of the problem. When* $X=\mathcal{S}'$, *it is sufficient to assume that* $f\in C(\overline{R^+};X)$.

Proof. It follows from Theorem 1.10, Theorem 3.1, Propositions 3.4, 3.6, 3.5.

REMARK 3.8. The conditions

$$\exp(tA)(X)\subseteq D(A)\,\forall t>0,\ \lim_{t\to 0} tAT(t)x=0\,\forall x\in X \tag{20}$$

characterize holomorphic semigroups in Banach spaces (see [BB], Prop. 1.1.11). Even in Frechèt spaces this is no more true. For example, take $X=\mathcal{E}(R)$, $A=\dfrac{d}{dt}$. A generates the group of translations and is continuous, so that it satisfies (20), but there exist elements $f\in X$ such that $t\to T(t)f$ does not admit a holomorphic extension to any neighbourhood of R^+ in $\boldsymbol{C}$ (it is sufficient to consider a C^∞ function which is not analytic).

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