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AN UPPER BOUND FOR LOEWY LENGTHS OF PROJECTIVE MODULES IN P-SOLVABLE GROUPS

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1. Introduction. Let p be a prime, F a field of characteristic p and let G be a finite group. With every FG-module M there is attached a non-negative integer $L(M)$ called the Loewy length of M. If $J(FG)$ denotes the Jacobson radical of FG, then $L(M)$ is the smallest integer n such that $MJ(FG)^n$ is the zero module. The Loewy length of a module is of some interest since knowledge of it's value or at least of good bounds for it can be very useful determinating of the structure of the module.

In a significant paper [5], Jennings solved the structure problem for the group algebra of a p-group. Unfortunately, little is known about the algebra structure of an arbitrary group algebra, resp. of a p-block. However, in case of p-solvable groups, there are many papers concerning bounds for the Loewy length of a group algebra, resp. of a p-block ([6], [7], [8], [12], [14], [15], [16], [17]).

Recently Ninomiya found a lower bound for the Loewy length of a projective indecomposable module depending on the order of a vertex of it's head ([13]). The aim of this note is to determine an upper bound.

Throughout this paper all groups in questions are finite and p-solvable all modules finitely generated FG-modules where F denotes a field of characteristic p. The notation used in the following is consistent with that in the books of Feit [2] and Huppert/Blackburn [4].

2. Results

Theorem 1. *If P is a projective indecomposable module, then* $L(P) \leqslant \max\{|V| \mid V$ *is a vertex of a composition factor of* $P\}$.

Theorem 2. *Equality holds in Theorem* 1 *if and only if the defect group of the block to which P belongs is cyclic.*

Corollary (Koshitani, Okuyama, Tsushima). *Let B be a p-block with defect group D. Then $L(B) \leqslant |D|$ and equality holds if and only if D is cyclic.*

According to all the examples we know, it seems reasonable to ask the

Question. *Let M be an irreducible module with vertex V. Is it true that $L(P_G(M))\leqslant |V|$ where $P_G(M)$ denotes the projective cover of M ?*

To see that the problem considered here differs from the analogue for p-blocks and defect groups, let us mention the following remarkable example ([10], [11]).

Let p be the prime 3 and let G denote the semidirect product of $SL(2,3)$ with the standard module. Then G possesses an irreducible module M with

$$L(P_G(M)) = |vx(M)|$$

where the vertex $vx(M)$ of M is elementary abelian of order 9.

3. Proofs

In what follows we may always assume that F is algebraically closed. The reduction to such a field is routine since the radical of a group algebra, vertices and defect groups are well behaved by field extensions.

Proof of Theorem 1.

We argue by induction on $(|G|_p, |G|)$ where $|G|_p$ denotes the p-part of the order $|G|$ of G.

Write $P=P_G(M)$ for some irreducible module M.

(1) First assume that $O_p(G)\neq\langle 1\rangle$.

Let E be a normal abelian p-subgroup of G. Put $J=J(FE)$ and let G act by conjugation on the powers J^i of J. By a result of Alperin, Collins and Sibley [1], we have with $\bar{G}=G/E$ an FG-isomorphism

$$P_{\bar{G}}(M)\otimes J^i/J^{i+1}\cong P_G(M)J^i/P_G(M)J^{i+1} .$$

Since $E\subseteq C_G(J^i/J^{i+1})$, the left hand side is an $F\bar{G}$-module, hence a projective $F\bar{G}$-module. As FG-modules, all composition factors of

$$X: = P_{\bar{G}}(M)\otimes J^i/J^{i+1}$$

are composition factors of $P_G(M)$.

Hence by the inductive hypothesis we get

$$\begin{aligned} L(X)&\leqslant\max\{|V| \mid V \text{ is a vertex of a composition factor of } X\}\\ &\leqslant\max\{|\bar{V}| \mid V \text{ is a vertex of a composition factor of } P_G(M)\}\end{aligned}$$

and therefore

$$\begin{aligned} L(P_G(M))&\leqslant\max\{|\bar{V}| \mid V \text{ is a vertex of a composition factor of } P_G(M)\}\cdot L(FE)\\ &\leqslant\max\{|\bar{V}| \mid V \text{ is a vertex of a composition factor of } P_G(M)\}\cdot |E|\\ &=\max\{|V| \mid V \text{ is a vertex of a composition factor of } P_G(M)\}.\end{aligned}$$

Thus it remains to deal with the case

$$H:=O_{p'}(G)\neq\langle 1\rangle\,.$$

Let N be an irreducible constituent of M_H and let I denote the inertial group of N in G, i.e.

$$I=I_G(N)=\{g\,|\,g\in G,\ N\otimes g\cong N\}\,.$$

(2) Next we consider the case $I<G$.

By Clifford's theory, there exists an irreducible FI-module X such that $M\cong X^G$ and $X_H\cong eN$ for some $e\in \boldsymbol{N}$. According to Proposition 2.7 in [18] we have

$$P_G(M)\cong P_I(X)^G\,.$$

We claim that for all composition factors Y of $P_I(X)$ the induced modules Y^G are irreducible. Since $vx(Y)\underset{G}{=}vx(Y^G)$ we are done by the inductive hypothesis. Thus assume that Y is a composition factor of $P_I(X)$ and Z an irreducible submodule of Y^G. Obviously, $Y_H\cong fN$ for some $f\in \boldsymbol{N}$.

If $\{g_1=1, g_2, \cdots, g_r\}$ denotes a right transversal of I in G, then

$$Z_H\subseteq(Y^G)_H=Y_H\otimes g_1\oplus\cdots\oplus Y_H\otimes g_r$$

where the $Y_H\otimes g_i$ are precisely the homogeneous components of $(Y^G)_H$.

Now choose a homogeneous component of Z_H, say W. We may assume that $W\subseteq Y_H\otimes g_1$, otherwise we consider a suitable G-conjugate of W. Since I acts irreducibly on Y, we get $W_H=Y_H\otimes g_1$ and therefore $Z=Y^G$.

(3) Finally, let $I=G$.

Now, Fong's reduction theorem ([2], Chap. X) asserts that there exists a finite group $\tilde{G}$ and a short exact sequence

$$\langle 1\rangle\to Z\to\tilde{G}\to G\xrightarrow{f}\langle 1\rangle$$

with Z a cyclic p'-group in the center of $\tilde{G}$.

Furthermore,

(i) $\tilde{G}$ contains a normal subgroup $\tilde{H}\cong H$ with $Z\tilde{H}=Z\times\tilde{H}=f^{-1}(H)$.

(ii) There is an $F\tilde{G}$-module $\tilde{N}$ on which $\tilde{H}$ acts irreducibly and an $F\tilde{G}$-module $\tilde{M}$ with $\tilde{H}\subseteq\ker(\tilde{M})$ such that M considered as an $F\tilde{G}$-module is isomorphic to $\tilde{M}\otimes\tilde{N}$.

(iii) $P_{\tilde{G}}(M)\cong P_{\tilde{G}}(\tilde{M})\otimes\tilde{N}$.

Since $\tilde{H}\subseteq O_{p'}(\mathrm{Ker}(\tilde{M}))$, we have $\tilde{H}\subseteq\mathrm{Ker}(\tilde{Y})$ for all composition factors $\tilde{Y}$ of $P_{\tilde{G}}(\tilde{M})$ (see [18], 3.1). Thus by ([4], Chap. VII, 9.12), the tensor product $\tilde{Y}\otimes\tilde{N}$ is irreducible. Because of ([4], Chap. VII, 14.1 and 14.2), there are isomorphisms

$$P_{\tilde{G}}(M)\simeq P_{\tilde{G}/Z}(M)\simeq P_G(M) \text{ and } P_{\tilde{G}}(\tilde{M})\simeq P_{\tilde{G}/\tilde{H}}(\tilde{M}).$$

In particular, each composition factor Y of $P_G(M)$ is of the form $Y\simeq\tilde{Y}\otimes\tilde{N}$ for some composition factor $\tilde{Y}$ of $P_{\tilde{G}}(\tilde{M})$. Since $\dim N$ is prime to p, we get $|vx(Y)|=|vx(\tilde{Y})|$, by ([3], 2.1). Now, $|\tilde{G}/\tilde{H}|_p=|G|_p$ and $O_p(\tilde{G}/\tilde{H})\neq\langle 1\rangle$. Apply part (1) of the proof to $P_{\tilde{G}/\tilde{H}}(\tilde{M})\simeq P_{\tilde{G}}(\tilde{M})$ and the proof is complete.

Proof of Theorem 2.

Assume first that the defect group D of the p-block B to which P belongs is cyclic. In this case it's well-known that

$$B\simeq \mathrm{Mat}(n, FU)$$

where $D\trianglelefteq U\leqslant \mathrm{holomorph}(D)$ and U/D is a cyclic p'-group. From this we deduce quite easily that all the projective indecomposable modules in B are uniserial of length $|D|$. Since D is a vertex for all irreducible modules in B, the assertion follows.

For the other direction ln Theorem 2 assume that $P=P_G(M)$ for some irreducible module M and

$$\begin{aligned} L(P_G(M)) &= \max\{|V| \mid V \text{ is a vertex of a composition factor of } P_G(M)\} \\ &= |V_0|\,. \end{aligned}$$

We claim by induction on $(|G|_p, |G|)$ that V_0 must be cyclic. Then it's well-known that V_0 coincides up to G-conjugation with the defect group D. To do this assume first that $O_{p'}(G)$ is contained in the center $Z(G)$ of G. In this case

$$O_{p',p}(G) = O_{p'}(G)\times O_p(G)$$

and $G/O_{p',p}(G)$ acts faithfully on $O_p(G)$. Let E be an abelian normal p-subgroup of G and put $\bar{G}=G/E$.

Similiar to part (1) of the proof of Theorem 1 we get

$$L(P_{\bar{G}}(M)) = |\bar{V}_0| \text{ and } E \text{ has to be cyclic.}$$

By the inductive hypothesis, $\bar{V}_0$ is cyclic. Now, if G has at least two minimal normal p-subgroups, then V_0 is abelian.

In particular, $O_p(G)$ is abelian and therefore cyclic. Since $G/O_{p',p}(G)$ acts faithfully on $O_pG)$, $O_p(G)$ is a Sylow p-subgroup of G. Hence G has only one minimal normal p-subgroup E. This implies that E is contained in the center of $O_p(G)$. Since $O_p(G)/E$ is cyclic, $O_p(G)$ must be abelian, hence cyclic and therefore a Sylow p-subgroup of G. The proof can now be finished by the same line as we did in (2) and (3) of the proof of Theorem 1.

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