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RANDOM ITERATION OF ONE-DIMENSIONAL TRANSFORMATIONS

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1. Introduction

The most familiar one-dimensional dynamical system is given by

$$x_{n+1}=f(x_n) \quad \text{for} \quad n\geqq 0, \tag{1.1}$$

where f is a transformation from an interval into itself. Usually f in (1.1) is assumed to be of piecewise C^2. Moreover if f is uniformly expanding the asymptotic behavior of x_n is investigated in detail (see [7], [8], and [14]). But it will be more natural to consider that f may be changed for each n, by chance. For example, let S be a measurable space, let $\{f_s\}_{s\in S}$ be a family of transformations from the unit interval I into itself and let $\{X_n\}_{n=1}^{\infty}$ be a sequence of S-valued independent and identically distributed random variables on a probability space $(\Omega, \mathcal{F}, P)$. The relation between x_n and x_{n+1} is given by

$$x_{n+1}=f_{X_{n+1}(\omega)}(x_n) \quad n\geqq 0. \tag{1.2}$$

In this paper, we will study the asymptotic behavior of x_n given by (1.2). Following S. Kakutani [5] and T. Ohno [9], we introduce the skew product transformation T on $I\times\Omega$ satisfying

$$\mathrm{proj}_I\circ T^n(x, \omega)=f_{X_n(\omega)}f_{X_{n-1}(\omega)}\cdots f_{X_1(\omega)}\,x, \quad n\geqq 0, \tag{1.3}$$

and deduce our problem to the investigation of the asymptotic behavior of T.

In section 5, we introduce an expanding condition (A.1) for the random transformations f_{X_n}'s. Under this condition, similar to a single piecewise C^2 uniformly expanding transformation, we can obtain the following results:

I. Let m denote the Lebesgue measure on I. Then T has a finite number of $(m\times P)$-absolutely continuous ergodic probability measures such that any $(m\times P)$-absolutely continuous T-invariant σ-additive finite set function can be written as a linear combination of them. These ergodic measures have disjoint supports, each of which is called an ergodic component of T.

II. Each ergodic component of T can be decomposed into finitely many exact components which are permuted cyclically by T.

III. If S is a finite set, then each exact component is also a weak Bernoulli one.

Our results generalize those of [4] (see section 8).

We will prove the above results by investigating the Perron-Frobenius operator $\mathcal{L}$ of T with respect to $m\times P$ and its eigenfunctions corresponding to the eigenvalues of modulus 1. To do this, the results in the deterministic case will play very important roles (for example, Lemma 5.4). From this point of view in section 3 we will summarize the results in [7], [8], and [14] and prove Wagner's theorem [14] by using a different method from the original one. This enable us to apply the results in the deterministic case to our random iteration case.

In section 5, we will prove the main theorem and in section 6 we will give some auxiliary results. Bernoulliness is to be discussed in section 7. In the final section, we will give some examples and remarks.

2. Preliminaries

To begin with, we introduce the so-called Perron-Frobenius operator which plays important roles in this article.

DEFINITION 2.1. Let $(X, \mathcal{B}, m)$ be a probability space and T be an *m-nonsingular transformation on X,* namely it is measurable and for a measurable set A, $m(A)=0$ implies $m(T^{-1}A)=0$. We define *the Perron-Frobenius operator* (*P-F* operator) $\mathcal{L}_{T,m}$ *of T with respect to m* sa follows:

$$\mathcal{L}_{T,m}\phi = \frac{d}{dm}\int_{T^{-1}(\cdot)} \phi\, dm \quad \text{for every} \quad \phi\in L^1(m)\,. \tag{2.1}$$

$\mathcal{L}_{T,m}$ will often be denoted simply by $\mathcal{L}_T$ or $\mathcal{L}_m$.

DEFINITION 2.2. For a measurable transformation T on a measurable space $(X, \mathcal{B})$, we define an operator U_T by

$$U_T\phi = \phi\circ T \quad \text{for each measurable function } \phi\,. \tag{2.2}$$

U_T is called *the operator induced by T.*

Now we summarize some properties of *P-F* operator which are easily verified. As usual, we write n times iteration of T by T^n.

Proposition 2.1. *Let $(X, \mathcal{B}, m)$ be a probability space. Let T be an m-non-singular transformation. Then, we have the following:*

(1) *The Perron-Frobenius operator $\mathcal{L}_m=\mathcal{L}_{T,m}$ is characterized by the identity*

$$\int U_T\psi\cdot\phi\, dm = \int \psi\cdot\mathcal{L}_m\phi\, dm \tag{2.3}$$

for every $\phi\in L^1(m)$ and $\psi\in L^\infty(m)$.

(2) *$\mathcal{L}_m$ is a positive linear operator on $L^1(m)$ with operator norm* 1.

(3) *For every $n \geqq 1$,*

$$\mathcal{L}_m^n = \mathcal{L}_{T^n,m}\,. \tag{2.4}$$

(4) *For every $\phi \in L^1(m)$ and $\psi \in L^\infty(m)$,*

$$\mathcal{L}_m(U_T\psi \cdot \phi) = \psi \cdot \mathcal{L}_m\phi \quad \text{a.e. } (m)\,. \tag{2.5}$$

(5) *For $\phi \in L^1(m)$, $\mathcal{L}_m\phi = \phi$ if and only if ϕm is T-invariant where ϕm is a σ-additive set function defined by $(\phi m)(A) = \int_A \phi\, dm$.*

Furthermore, let μ be an m-absoultely continuous T-invariant probability measure with the density function h.

(6) *For all $\psi \in L^1(\mu)$,*

$$h \cdot \mathcal{L}_\mu\psi = \mathcal{L}_m(h \cdot \psi) \quad \text{a.e. } (m) \tag{2.6}$$

where $\mathcal{L}_\mu = \mathcal{L}_{T,\mu}$.

(7) *For all $\psi \in L^1(\mu)$,*

$$\mathcal{L}_\mu U_T \psi = \psi \text{ a.e. } (\mu)\,. \tag{2.7}$$

(8) *Put $\mathcal{B}_n = T^{-n}\mathcal{B}$, we have*

$$E_\mu[\psi \mid \mathcal{B}_n] = U_T^n \mathcal{L}_\mu^n \psi \quad \text{a.e.} \quad (\mu) \quad \textit{for each} \quad \psi \in L^1(\mu)\,. \tag{2.8}$$

(9) *$\mathcal{L}_\mu$ is the dual operator for the operator U_T on $L^p(\mu)$ $(1 \leqq p < \infty)$.*

Under the same notations in the above we have the following properties about the eigenvalues of $\mathcal{L}_m$ on $L^1(m)$.

Proposition 2.2. (1) *For $\lambda \in \mathbf{C}$ and $\psi \in L^1(\mu)$, the following are equivalent:*

$$\begin{aligned} &\text{(i)} \quad U_T\psi = \lambda\psi \text{ a.e. } (\mu)\,, \\ &\text{(ii)} \quad \mathcal{L}_\mu\psi = \bar{\lambda}\psi \text{ a.e. } (\mu) \textit{ and } |\lambda| = 1\,, \\ &\text{(iii)} \quad \mathcal{L}_m(\psi h) = \bar{\lambda}\psi h \text{ a.e. } (m) \textit{ and } |\lambda| = 1\,. \end{aligned} \tag{2.9}$$

(2) *If μ is maximal, namely any m-absolutely continuous T-invariant probabiilty measure is μ-absolutely continuous, then the set of all eigenvalues of modulus 1 of $\mathcal{L}_\mu$ on $L^1(\mu)$ coincides with that of $\mathcal{L}_m$ on $L^1(m)$.*

Proof. (1) From the formula (2.6), the equivalence of (ii) and (iii) is obvious. So we prove the equivalence of (i) and (ii). If $U_T\psi = \lambda\psi$ for $\lambda \in \mathbf{C}$ and $\psi \in L^1(\mu)$, we have $\psi = \mathcal{L}_\mu U_T \psi = \lambda \mathcal{L}_\mu \psi$ a.e. (μ) by (2.7) and

$$\int |\psi|\, d\mu = \int |U_T\psi|\, d\mu = |\lambda| \int |\psi|\, d\mu\,.$$

Thus $\mathcal{L}_\mu\psi=\bar{\lambda}\psi$ a.e. (μ) and $|\lambda|=1$. Conversely, if $\mathcal{L}_\mu\psi=\bar{\lambda}\psi$ a.e. (μ) and $|\lambda|=1$, we have $E_\mu[\psi|\mathcal{B}_n]=\bar{\lambda}^n\psi\circ T^n$ by (2.8) thus $\bar{\lambda}^n\psi\circ T^n$ converges to some $\tilde{\psi}$ in $L^1(\mu)$. It is easy to see $\tilde{\psi}\circ T=\lambda\tilde{\psi}$ a.e. (μ). We have to show that $\psi=\tilde{\psi}$ .a.e. (μ) but

$$\int |\tilde{\psi}-\psi|\, d\mu = \int |\bar{\lambda}^n\tilde{\psi}\circ T^n-\bar{\lambda}^n\psi\circ T^n|\, d\mu$$
$$= \int |\tilde{\psi}-\bar{\lambda}^n\psi\circ T^n|\, d\mu \to 0 \text{ as } n\to\infty .$$

Hence $U_T\psi=\lambda\psi$ a.e. (μ).

(2) Let Λ_m and Λ_μ denote the set of all eigenvalues of modulus 1 of $\mathcal{L}_m$ on $L^1(m)$ and $\mathcal{L}_\mu$ on $L^1(\mu)$ respectively. By (2.4) it is obvious that $\Lambda_m\supset\Lambda_\mu$. Suppose that $\lambda\in\Lambda_m$ and $\phi\in L^1(m)$ is an eigenfunction with $\int|\phi|\, dm=1$. Since $\mathcal{L}_m$ is a positive operator and preserves the value of integral, it is easy to see $\mathcal{L}_m|\phi| =|\phi|$ a.e. (m). Thus $|\phi|m$ is T-invariant probability measure. From our assumption $\{|\phi|=0\}\supset\{h=0\}$, so ϕh^{-1} makes sense and $\phi h^{-1}\in L^1(\mu)$. Therefore we have

$$h\,\mathcal{L}_\mu(\phi h)=\mathcal{L}_m\phi=\lambda\phi=\lambda\phi h^{-1}\,h\,.$$

Hence $\lambda\in\Lambda_\mu$.

Now we consider one-dimensional dynamical systems.

DEFINITION 2.3. Let I be the unit interval. By $\mathcal{D}$ we denote the taotality of transformations f of I satisfying:

(1) There is a partition $0=a_0<a_1<\cdots<a_k=1$ of I such that for each $i=1, 2, \cdots, k$ the restriction $f|_{(a_{i-1},a_i)}$ of f to (a_{i-1}, a_i) is a function of class C^2 and can be extended to the closed interval $[a_{i-1}, a_i]$ as a function of class C^2.

(2) f satisfies the expanding condition

$$d_f=\inf\{|f'(x)|: x\neq a_i\}>0\,. \tag{2.11}$$

REMARK 2.1. In the above definition the partition $0=a_0<a_1<\cdots<a_k=1$ can be chosen to be minimal in the sense of refinement among all partitions satisfying (1). Unless otherwise stated, we always take the minimal partition, so the points $0=a_0<a_1<\cdots<a_k=1$ can uniquely determined by f.

REMARK 2.2. From (2) of Definition 2.3, f is m-nonsingular and $f|_{(a_{i-1},a_i)}$ is strictly monotonic for each i.

DEFINITION 2.4. An element f of $\mathcal{D}$ is said to be *uniformly expanding* if $d_f>1$ and the totality of such transformations is denoted by $\mathcal{D}_e$.

DEFINITION 2.5. For $f\in\mathcal{D}$, set

$$(2.11) \qquad \beta_f = 2d_f^{-1} \max_{1\leq i\leq k} (a_i - a_{i-1})^{-1} + \max_{1\leq i\leq k} \frac{\sup|(f|^{-1}_{(a_{i-1},a_i)})''|}{\inf|(f|^{-1}_{(a_{i-1},a_i)})'|},$$

where $0=a_0<a_1<\cdots<a_k=1$ is the minimal partition satisfying (1) of Definition 2.3.

3. Results in the deterministic case

In this section, we are concerned with a single transformation f in $\mathcal{D}_e$. Unless otherwise stated, $\mathcal{B}$ and m denote the topological Borel field on the unit interval I and the Lebesgue measure respectively.

Theorem 3.1 (Li and Yorke [8]). *Let f be in $\mathcal{D}_e$. Then, there exists a finite collection of sets $L_1, L_2, \cdots, L_l$ and a set of m-absolutely continuous f-invariant probability measures $\{\mu_1, \mu_2, \cdots, \mu_l\}$ such that*

(1) *for each $i=1, 2, \cdots, l$ L_i is a finite union of closed intervals and $fL_i=L_i$;*

(2) *$L_i\cap L_j$ contains at most a finite number of points when $i\neq j$;*

(3) *for each i, $\mu_i(L_i)=1$ and the dynamical system (f, μ_i) is ergodic;*

(4) *if μ is an m-absolutely continuous f-invariant σ-additive finite set function, then it can be written as a linear combination of μ_i's;*

(5) *let $D=I\setminus \bigcup_{i=1}^{l} L_i$, then $D\supset f^{-1}D\supset \cdots$ and $m(f^{-n}D)\to 0$ $(n\to\infty)$.*

Theorem 3.2 (Wagner [14]). *Let f be in $\mathcal{D}_e$, and μ be an m-absolutely continuous ergodic f-invariant probability measure. Put $L=\left\{\frac{d\mu}{dm}>0\right\}$. Then, there is an integer $N>0$ and a collection of disjoint measurable subsets $L_0, L_1, \cdots, L_{N-1}$ of L such that*

(1) *$fL_j=L_{j+1}$ $(0\leq j<N-1)$ and $fL_{N-1}=L_0$;*

(2) *for each $j=0, 1, \cdots, N-1$, the dynamical system (f^N, μ_j) is exact, where $\mu_j=N\,\mu|_{L_j}$.*

REMARK 3.1. Bowen proved that (f^N, μ_j) is weakly Bernoulli in [1].

In the sequel, we will give a poof of Theorem 3.2 by investigating the eigenfunctions of P-F operator $\mathcal{L}_{f,m}$ for the following two reasons. First, Wagner's method is rather complicated; for example, he used the Rohlin criterion (see [12]). The second reason is that our method employed here is also usefull in studying the ergodic behavior of the skew product transformation T as one sees in the later sections.

For a function ϕ from I into $\mathbf{C}$, let $\widetilde{\bigvee}\phi$ denote the total variation of ϕ. For $\phi\in L^1(m)$ we define

$$\bigvee\phi = \inf\{\widetilde{\bigvee}\tilde{\phi}\colon \tilde{\phi} \text{ is any version of } \phi\}$$

and $BV=\{\phi\in L^1(m)\colon \bigvee\phi<\infty\}$.

Lemma 3.1. (1) *Let f belongs to $\mathcal{D}$. Then, for each $\phi \in BV$, we have*

$$\vee \mathcal{L}_{f,m} \phi \leqq 2\alpha_f \vee \phi + \beta_f \|\phi\|_{1,m} . \tag{3.1}$$

where d_f and β_f are those defined in section 2 and $\alpha_f = d_f^{-1}$.

(2) *In addition, if f is in $\mathcal{D}_e$, then there is a positive constant C such that*

$$\limsup_{n \to \infty} \vee \mathcal{L}_{f,m}^n \phi \leqq C \|\phi\|_{1,m} \quad \text{for every } \phi \in BV . \tag{3.2}$$

(3) *If $f \in \mathcal{D}_e$, then $\{\mathcal{L}_{f,m}^n \phi\}_{n=1}^\infty$ is relatively compact in $L^1(m)$ for every $\phi \in L^1(m)$.*

Proof. See [7].

Lemma 3.2. *Let f be in $\mathcal{D}_e$. Let Λ be the set of all eigenvalues of $\mathcal{L}_f = \mathcal{L}_{f,m}$ with modulus 1, and let $E(\lambda)$ denote the eigenspace belonging to $\lambda \in \Lambda$. Then,*

(1) $1 \in \Lambda$.

(2) *if $\lambda \in \Lambda$, then $\phi \in E(\lambda)$ implies that*

$$\vee \phi \leqq C \|\phi\|_{1,m} \tag{3.3}$$

where C is the constant which appeared in the inequality (3.2).

(3) $\dim E(\lambda) < \infty$ *if* $\lambda \in \Lambda$.

(4) $\# \Lambda < \infty$.

Proof. (1) By (3) in Lemma 3.1, we can use the Kakutani-Yosida Theorem [6]. Hence, for each $\phi \in L^1(m)$, the sequence $\left\{\frac{1}{n} \sum_{i=0}^{n-1} \mathcal{L}_f^i \phi\right\}_{n=1}^\infty$ converges in $L^1(m)$. The limit function ϕ^* has the following properties:

$$\mathcal{L}_f \phi^* = \phi^* \text{ and} \int \phi^* \, dm = \int \phi \, dm .$$

Thus $1 \in \Lambda$.

(2) Since BV is dense in $L^1(m)$, we can choose $\phi_p \in BV$ such that $\|\phi_p - \phi\|_{1,m} \leqq \frac{1}{p}$ for every p. Pick a sequence $\{n_i\}_{i=1}^\infty \subset \mathbf{N}$ with $n_i < n_{i+1}$ and $\lambda^{n_i} \to 1$ ($i \to \infty$). By (3.2) and Helley's theorem, we can find a subsequence $\{n'_j\}_{j=1}^\infty \subset \{n_i\}_{i=1}^\infty$ with $n'_j < n'_{j+1}$ and $\tilde{\phi}_p \in BV$ such that

$$\|\tilde{\phi}_p - \mathcal{L}_f^{n'_j} \phi_p\|_{1,m} \to 0 \quad (j \to \infty) \text{ and } \vee \tilde{\phi}_p \leqq C \|\phi_p\|_{1,m} .$$

It is easy to see that $\|\tilde{\phi}_p - \phi\|_{1,m} \leqq \frac{1}{p}$. Again, we can apply Helley's theorem to the sequence $\{\tilde{\phi}_p\}_{p=1}^\infty$. Without loss of generality, we may assume that $\tilde{\phi}_p$ converges to some $\tilde{\phi} \in BV$ with $\vee \tilde{\phi} \leqq C \|\phi\|_{1,m}$ in $L^1(m)$. It is obvious that $\tilde{\phi} = \phi$.

(3) From the fact that just has been proved above, any bounded set in $E(\lambda)$ is relatively compact in $L^1(m)$. Therefore, $\dim E(\lambda) < \infty$.

(4) Suppose that $\{\lambda_n\}_{n=1}^{\infty}$ is an infinite sequence of distinct eigenvalues of modulus 1. Let ϕ_n be an eigenfunction belonging to λ_n for each n. Let F_n be the linear span of $\phi_1, \phi_2, \cdots, \phi_n$ for $n \geqq 1$ and put $F_0=\{0\}$. It is obvious that $\mathcal{L}_f F_n=F_n$ and $F_n \subsetneqq F_{n+1}$ for all $n \geqq 0$. By Riesz' lemma, there exists a $\psi_n \in F_n$ such that $||\psi_n||_{1,m}=1$ and $||\psi_n-\psi||_{1,m} \geqq \frac{1}{2}$ for any $\psi \in F_{n-1}$ for $n=1, 2, \cdots$. Thus we can easily show that

$$\bar{\lambda}_n^N \mathcal{L}_f^N \psi_n - \psi_n \in F_{n-1} \text{ for } n=1, 2, \cdots, \text{ and for any } N \geqq 1 .$$

Therefore, if $p>q$, then

$$\begin{aligned} (3.4) \qquad & ||\bar{\lambda}_p^N \mathcal{L}_f^N \psi_p - \bar{\lambda}_q^M \mathcal{L}_f^M \psi_q||_{1,m} \\ & = ||\bar{\lambda}_p^N \mathcal{L}_f^N \psi_p - \psi_p + \psi_p - \bar{\lambda}_q^M \mathcal{L}_f^M \psi_q||_{1,m} \\ & \geqq \frac{1}{2}, \text{ for all } N, M \in \mathbf{N} . \end{aligned}$$

On the other hand, since $\psi_n \in BV$ so

$$\limsup_{N \to \infty} \bigvee (\bar{\lambda}_p^N \mathcal{L}_f^N \psi_p) \leqq C \text{ for all } p .$$

Hence, we can choose a sequence $\{N_n\}_{n=1}^{\infty}$ such that $N_n<N_{n+1}$ and $N \geqq N_n$ implies that $\bigvee (\bar{\lambda}_n^N \mathcal{L}_j^N \psi_n) \leqq C+1$. By Helley's theorem, $\{\bar{\lambda}_n^{N_n} \mathcal{L}_f^{N_n} \psi_n\}_{u=1}^{\infty}$ is relatively compact in $L^1(m)$. But this contradicts (3.4). The proof of Lemma 3.2 is now complete.

Here we give two general properties of ergodic transformations.

Proposition 3.1. *Let (T, μ) be an ergodic dynamical system such that the set of all eigenvalues of U_T is written as $\{1, \lambda, \cdots, \lambda^{N-1}\}$ where λ is a primitive N-th root of* 1. *Then the eigenfunction ψ corresponding to λ has the following form up to constant multiplication.*

$$(3.5) \qquad \psi(x) = \sum_{i=0}^{N-1} \lambda^i \, 1(x)_{\{\psi=\lambda^i\}} .$$

Proof. Let ψ be an eigenfunction corresponding to λ. Since (T, μ) is ergodic we may assume that $|\psi|=1$. Thus we can write $\psi(x)=e^{2\pi i a(x)}$ where $0 \leqq a(x)<1$ and $a(Tx)=a(x)+\frac{1}{N} \bmod 1$. Put $L_i=\left\{\frac{i}{N} \leqq a(x) < \frac{i+1}{N}\right\}$ for $i=0, 1, \cdots, N-1$. Then $L_0, L_1, \cdots, L_{N-1}$ are disjoint, $T^{-1} L_{i+1}=L_i$ $(0 \leqq i < N-1)$ and $T^{-1} L_0=L_{N-1}$. Define a function $\tilde{\psi}$ by $\tilde{\psi}(x)=\sum_{i+0}^{N-1} \lambda^i \, 1_{L_i}(x)$. Obviously $U_T \tilde{\psi}(x) =\lambda \tilde{\psi}(x)$. By the ergodicity of (T, μ), there is a constant c such that $\tilde{\psi}(x)=c\psi(x)$.

Proposition 3.2. *Let (T, μ) be a dynamical system where μ is T-invariant.*

Assume that $\{\mathcal{L}_{\mu}^{n}\psi\}_{n=1}^{\infty}$ *is relatively compact in* $L^1(\mu)$ *for any* $\psi\in L^1(\mu)$. *Then, if* (T, μ) *is weakly mixing, it is exact and*

$$(3.6)\qquad \|\mathcal{L}_{T,\mu}^{n}\psi-\int\psi\, d\mu\|_{1,\mu}\to 0\ (n\to\infty)\ \textit{for any}\ \psi\in L^1(\mu)\,.$$

Proof. Let (T, μ) be weakly mixing. There is a subset $\mathbf{J}$ of N with density zero such that

$$\int_B \mathcal{L}_{T,\mu}^{n}\psi\, d\mu=\int U_T^n 1_B\,\psi\, d\mu\to\mu(B)\int\psi\, d\mu\quad(n\to\infty,\, n\notin\mathbf{J})\,,$$

for any $\psi\in L^1(\mu)$ and for any $B\in\mathcal{B}$ (see [10, p. 70]). Thus $\{\mathcal{L}_{T,\mu}^{n}\psi\}_{n\in\mathbf{N}\setminus\mathbf{J}}$ converges weakly to $\int\psi\, d\mu$ as $n\to\infty$. By our assumption $\{\mathcal{L}_{\mu}^{n}\psi\}_{n=1}^{\infty}$ is relatively compact in $L^1(\mu)$ so we may assume that

$$\int|\mathcal{L}_{T,\mu}^{n}\psi-\int\psi\, d\mu|\, d\mu\to 0\quad(n\to\infty,\, n\notin\mathbf{J})\,.$$

Put $\mathcal{B}_\infty=\bigcap_{n=1}^{\infty}T^{-n}\mathcal{B}$, we have

$$\begin{aligned}&\int|E_\mu(\psi|\mathcal{B}_\infty)-\int\psi\, d\mu|\, d\mu\\&=\int|E_\mu(\psi|\mathcal{B}_\infty)-E_\mu(\psi|\mathcal{B}_n)|\, d\mu\\&+\int|E_\mu(\psi|\mathcal{B}_n)-\int\psi\, d\mu|\, d\mu\,.\end{aligned}$$

The first term tend to 0 as $n\to\infty$ by Doob's theorem. The second term coincides with $\int|\mathcal{L}_{T,\mu}^{n}\psi-\int\psi\, d\mu|\, d\mu$ by (2.8). This implies that

$$E_\mu(\psi|\mathcal{B}_\infty)=\int\psi\, d\mu\ \text{and}\int|\mathcal{L}_{T,\mu}^{n}\psi-\int\psi\, d\mu|\, d\mu\to 0\quad(n\to\infty)\,.$$

Now we can prove Theorem 3.2.

Proof of Theorem 3.2. It is enough to show that the conclusions are valid if the word "exact" in the statmeent (2) is replaced by "weakly mixing" because of Proposition 3.2. Let G be the set of all eigenvalues of the operator $U_f\colon L^1(\mu)\to L^1(\mu)$. Then from (2.9) and (4) of Lemma 3.2 G is a finite subgroup of the unit circle S^1. Therefore, there is a positive integer N and a primitive N-th root λ of 1 with $G=\{1, \lambda, \cdots, \lambda^{N-1}\}$. Let ψ be a eigenfunction of U_f corresponding to λ. From Proposition 3.1, we may assume that

$$\psi(x)=\sum_{i=0}^{N-1}\lambda^i\, 1_{L_i}(x)\,,$$

where $L_i=\{\psi=\lambda^i\}$ (μ) $i=0, 1, \cdots, N-1$ and $fL_i=L_{i+1}$ $(0\leqq i\leqq N-2)$, $fL_{N-1}=$

L_0. We will prove that (f^N, μ_0) is ergodic where $\mu_0 = N\,\mu|_{L_0}$. If a measurable set $A \subset L_0$ satisfies $f^N A = A$ and $\mu_0(A) > 0$, then $\{f^j A\}_{j=0}^{N-1}$ is a collection of disjoint sets and $\bigcup_{i=0}^{N-1} f^j A$ is f-invariant. Since (f, μ) is ergodic, we have $\mu(\bigcup_{j=0}^{N-1} f^j A) = 1$. Consequently $\mu(A) = \frac{1}{N}$. Thus (f^N, μ_0) is ergodic.

To prove weak mixing property of (f^N, μ_0), it is enough to show that the operator $U_{f^N}: L^1(\mu_0) \to L^1(\mu_0)$ has a unique eigenvalue 1. If not, since f^N also belongs to $\mathcal{D}_e$, we can apply the same argument as above to f^N and μ_0. Therefore, there is an integer $M \geqq 2$ and a measurable set $L_{0,0}$ such that $\{f^{jM} L_{0,0}\}_{j=0}^{M-1}$ are mutually disjoint and $f^{NM} L_{0,0} = L_{0,0}$. So we see that $\{f^j L_{0,0}\}_{j=0}^{MN-1}$ are mutually disjoint and $\mu(\bigcup_{j=0}^{MN-1} f^j L_{0,0}) = 1$. Define a function χ by

$$\chi(x) = \kappa^j \quad \text{if} \quad x \in f^j L_{0,0}\,,$$

where κ is a primitive NM-th root of 1. Then $\chi \in L^2(\mu)$ and $U_f \chi = \kappa \chi$. This contradicts the fact $G = \{1, \lambda, \cdots, \lambda^{N-1}\}$.

4. Random iteration and skew product transformation

From now on, we are concerned with the random iteration of transformations. Our formulation is due to [5]. Let S be a set with Borel structure, $\{f_s\}_{s \in S}$ be a family of transformations from the unit interval I into itself. Let $X_1, X_2, \cdots$ be a sequence of S-valued random variables defined on a probability space $(\Omega, \mathcal{F}, P)$.

DEFINITION 4.1. For each $x \in I$ and $\omega \in \Omega$, set

$$x_0 = x$$

$$x_n = f_{X_n(\omega)}(x_{n-1}) \qquad \text{for} \quad n \geqq 1\,. \tag{4.1}$$

The sequence $\{x_n = x_n(x, \omega)\}_{n=0}^{\infty}$ is called the random orbit of x determined by the random iteration $f_{X_n} f_{X_{n-1}} \cdots f_{X_1}$ (simply random orbit of x).

We will study the ergodic properties of the random orbit x_n under the following

Assumptions. (1) *S is a eomplete separable metric space with the topological Borel field $\mathcal{B}(S)$ and π is a probability measure on $(S, \mathcal{B}(S))$.*

(2) *The sequence of random variables obtained as follows:*

$$\Omega = S^\infty,\ \mathcal{F} = \mathcal{B}(\Omega),\ P = \pi^\infty\,, \quad \text{and}$$

$$X_n(\omega) = \omega_n \qquad for \quad \omega \in \Omega$$

where ω_n is the n-th coordinate of ω, that is, $\{X_n\}_{n=1}^{\infty}$ is a sequence of independent and identically π-distributed rnadom variables.

(3) *The family* $\{f_s\}_{s\in S}$ *is included in* $\mathcal{D}$ *and the map* $(s, x)\to f_s x$ *is* $\mathcal{B}(S\times I)|\mathcal{B}(I)$*-measurable.*

In order to investigate the behavior of the random orbit we consider the following skew product transformation T and deduce our problems to the study of its ergodic properties.

DEFINITION 4.2. Define a transformation T: $I\times\Omega\to I\times\Omega$ by

$$T(x, \omega) = (f_{X_1(\omega)}(x), \sigma\omega) \quad \text{for} \quad (x, \omega)\in I\times\Omega\,, \tag{4.2}$$

where σ: $\Omega\to\Omega$ is the shift transformation, that is, $(\sigma\omega)_n=\omega_{n+1}$.

Since f_s's are all m-nonsingular and P is σ-invariant T becomes $(m\times P)$-nonsingular. This fact enable us to consider the Perron-Frobenius operator of T with respect to $m\times P$.

Let $\mathcal{L}=\mathcal{L}_{T,m\times P}$ be the Perron-Frobenius operator of T, and $\mathcal{L}_s=\mathcal{L}_{f_s,m}$ for every $s\in S$.

Lemma 4.1. (1) *For* $\phi\in L^1(m)$, *we can choose a* $\mathcal{B}(S^n\times I)$*-measurable function* $\psi(s_1, s_2, \cdots, s_n)$ *which is a version of* $(\mathcal{L}_{s_1}\mathcal{L}_{s_2}\cdots\mathcal{L}_{s_n}\phi)(x)$. *Moreover, if* $\phi\in BV$ *we can choose* $\psi(s_1, s_2, \cdots, s_n)$ *so that*

$$\widetilde{\bigvee}\psi(s_1, s_2, \cdots, s_n) = \bigvee\mathcal{L}_{s_1}\mathcal{L}_{s_2}\cdots\mathcal{L}_{s_n}\phi\,. \tag{4.3}$$

(2) *If* $\Phi\in L^1(m\times P)$ *has the form*

$$\Phi(x, \omega) = \phi(x)\,\psi(X_1(\omega), X_2(\omega), \cdots, X_k(\omega))\,,$$

where $\phi\in L^1(m)$ *and* $\psi\in L^\infty(\pi^k)$, *then for* $n\geqq k$, *we have*

$$(\mathcal{L}^n\Phi)(x, \omega) = \int\psi(s_1, \cdots, s_k)\,(\mathcal{L}_{s_n}\cdots\mathcal{L}_{s_1}\phi)(x)\,d\pi^n(s_1, \cdots, s_n) \tag{4.4}$$

a.e. $(m\times P)$, *that is, we can regard* $L^n\Phi$ *as an element in* $L^1(m)$.

Proof. We only prove (2). For any $A\in\mathcal{B}(I)$ and $\Gamma\in\mathcal{F}$, we have

$$\begin{aligned}
&\int_{A\times\Gamma}\mathcal{L}^n\Phi\,dm\,dP\\
&=\int_{T^{-n}(A\times\Gamma)}\Phi\,dm\,dP\\
&=\int\phi(x)\,\psi(X_1(\omega), \cdots, X_k(\omega)\,1_A(f_{X_n(\omega)}\cdots f_{X_1(\omega)}x)\,1_\Gamma(\sigma^n\omega)\,dm\,dP\\
&=P(\Gamma)\int\phi(x)\,\psi(s_1, \cdots, s_k)\,1_A(f_{s_n}\cdots f_{s_1}x)\,dm(x)\,d\pi^n(s_1, \cdots, s_n)\\
&\quad\text{(since } \sigma^{-n}\Gamma \text{ is } (X_{n+1}(\omega), X_{n+2}(\omega), \cdots)\text{-measurable)}\\
&=P(\Gamma)\int\psi(s_1, \cdots, s_k)\,\Big(\int_{f_{s_1}^{-1}\cdots f_{s_n}^{-1}A}\phi(x)\,dm(x)\Big)\,d\pi^n(s_1, \cdots, s_n)
\end{aligned}$$

$$=P(\Gamma)\int \psi(s_1, \cdots, s_k)\left(\int_A (\mathcal{L}_{s_n}\cdots \mathcal{L}_{s_1}\phi)(x)\,dm(x)\right)d\pi^n(s_1, \cdots, s_n)$$
$$=\int_{A\times\Gamma}\left(\int \psi(s_1, \cdots, s_k)(\mathcal{L}_{s_n}\cdots \mathcal{L}_{s_1}\phi)(x)\right)d\pi^n(s_1, \cdots, s_n)\,dm\,dP$$

This proves (4.4).

5. The main theorem

For $\{f_s\}_{s\in S}\subset\mathcal{D}$ put

$$\alpha(s)=d_{f_s}^{-1}$$

and

$$\beta_N(s_1, s_2, \cdots, s_N)=\beta f_{s_1}f_{s_2}\cdots f_{s_N}\,,$$

where d_{f_s} and $\beta f_{s_1}f_{s_2}\cdots j_{s_N}$ are those which were defined in section 2.

We set the following assumptions:

(A.1) $\alpha\in L^1(\pi)$ and $\int \log\alpha\,d\pi<0$.

(A.2) For some $N>(-\log 2)\left(\int\log\alpha\,d\pi\right)^{-1}$ (if $\int\log\alpha\,d\pi=-\infty$ we regard the right hand side as 0), $\beta_N(s_1, s_2, \cdots, s_N)\in L^1(\pi^N)$ and $\beta_1(s)\in L^1(\pi)$.

Then we have:

Theorem 5.1. *Assume* (A.1) *and* (A.2). *Then, there exist finitely many $(m\times P)$-absolutely continuous T-invariant probability measures $Q_1, Q_2, \cdots, Q_n$ with the following properties:*

(i) (1) *For each $i=1, 2, \cdots, n$, the dynamical system (T, Q_i) is ergodic.*

(2) *Any $(m\times P)$-absolutely continuous T-invariant σ-additive finite set function can be represented as a linear combination of Q_i's.*

(3) *For each $i=1, 2, \cdots, n$, the support of Q_i has the form $A_i\times\Omega$ where $A_i\in\mathcal{B}(I)$.*

(ii) (4) *For each $i=1, 2, \cdots, n$, there is an integer N_i and a collection of sets $A_{i,0}, A_{i,2}, \cdots, A_{i,N_i-1}\in\mathcal{B}(I)$ such that setting $L_{i,j}=A_{i,j}\times\Omega$ we have $TL_{i,j}=L_{i,j+1}$ $(0\leqq j<N_i-1)$ and $TL_{i\cdot N_i-1}=L_{i,0}$.*

(5) *The dynamical system $(T^{N_i}, Q_{i,j})$ is exact, where $Q_{i,j}=N_i\,Q_i|_{L_{i,j}}$.*

In the following we give the proof of Theorem 5.1. We put

$$\Omega_n=\{\alpha(X_1)\,\alpha(X_2)\cdots\alpha(X_n)>\tau^n\}\quad\text{for}\quad\tau>0\,. \tag{5.1}$$

Lemma 5.1. *Assume* (A.1). *Then for any $\tau>\exp\left(\int\log\alpha\,d\pi\right)$, there exists a positive constant $C=C(\tau)$ such that*

$$P(\Omega_n)\leqq C\,e^{-\sqrt{n}}\,. \tag{5.2}$$

Proof. In Ibragimov and Linnik [3, Lemma 12.2.1], it is shown that if $Y_1, Y_2, \cdots$ is a sequence of independent and identically distributed random variables with $E[Y_1]=0$, $E[Y_1^2]=\sigma^2$, then for any $0<\gamma<\frac{1}{2}$ and $\rho(n)$ which is strictly increasing and goes to ∞ with n, there exist constants $C_1, C_2>0$ such that

$$P\{Y_1+Y_2+\cdots+Y_n \geqq n^{1/2+\gamma}\rho(n)^{-1}\sigma\} \leqq C_1 \exp\left[-C_2(n^\gamma \rho(n)^{-1})^2\right].$$

We apply this to $(-t)\vee\log\alpha(X_n)-\int(-t)\vee\log\alpha\, d\pi$, where t is positive and chosen to satisfy $\log\tau>\int(-t)\vee\log\alpha\, d\pi$. Then we have

$$\begin{aligned} P(\Omega_n) &\leqq P\{(\alpha(X_1)\vee e^{-t})(\alpha(X_2)\vee e^{-t})\cdots(\alpha(X_n)\vee e^{-t})\geqq \tau^n\} \\ &\leqq P\{\sum_{i=1}^n((-t)\vee\log\alpha(X_i)-\int(-t)\vee\log\alpha\, d\pi) \\ &\geqq n(\log\tau-\int(-t)\vee\log\alpha\, d\pi)\}. \end{aligned}$$

From this we can easily prove the lemma.

By Lemma 4.1, $\mathcal{L}|_{L^1(m)}$ is to be an operator from $L^1(m)$ into $L^1(m)$. For the simplicity we also write it by $\mathcal{L}$.

Lemma 5.2 (basic). *If* (A.1) *and* (A.2) *are satisfied, then for each* $p\in\mathbf{N}$, *there exist two sequences of linear operators* $\{U_n^{(p)}\}_{n=1}^\infty$ *and* $\{S_n^{(p)}\}_{n=1}^\infty$ *from* $L^1(m)$ *into* $L^1(m)$ *such that*

(1) *for every* $n\in\mathbf{N}$,

$$\mathcal{L}^n = U_n^{(p)}+S_n^{(p)}; \tag{5.3}$$

(2) *there is a positive constant* K *which is independent of* p *with*

$$\limsup_{n\to\infty} \vee U_n^{(p)}\phi \leqq K^p\|\phi\|_{1,m} \quad \textit{if} \quad \phi\in BV; \tag{5.4}$$

(3) *there is a positive constant* C *which is independent of* p *and* n *with*

$$\|S_n^{(p)}\|_{1,m}\leqq C\sqrt{p}\, e^{-\sqrt{p}}. \tag{5.5}$$

Proof. Take N in (A.2) and choose $\tau>0$ satisfying $\exp\left[\int\log\alpha\, d\pi\right]<\tau<\sqrt[N]{\frac{1}{2}}$. Put $\tilde{\Omega}_p=\bigcap_{n>p}\Omega_n^c$ where Ω_n is defined by (5.1). Then from (5.2) $\gamma_p=P(\Omega\backslash\tilde{\Omega}_p)\leqq C\sqrt{p}\, e^{-\sqrt{p}}$ for some positive constant independent of p. For fixed $p\in\mathbf{N}$ define

$$U_n^{(p)}\phi=\int_{\tilde{\Omega}_p}\mathcal{L}_{X_1}\mathcal{L}_{X_2}\cdots\mathcal{L}_{X_n}\phi\, dP$$

and

$$S_n^{(p)}\phi=\int_{\Omega\backslash\tilde{\Omega}_p}\mathcal{L}_{X_1}\mathcal{L}_{X_2}\cdots\mathcal{L}_{X_n}\phi\,dP$$

for $\phi\in L^1(m)$.

Then obviously $\mathcal{L}^n\phi=U_n^{(p)}\phi+S_n^{(p)}\phi$ and

$$\begin{aligned}\|S_n^{(p)}\phi\|_{1,m}&=\int dm\left|\int_{\Omega\backslash\tilde{\Omega}_p}\mathcal{L}_{X_1}\mathcal{L}_{X_2}\cdots\mathcal{L}_{X_n}\phi\,dP\right|\\&\leqq\int dm\int_{\Omega\backslash\tilde{\Omega}_p}|\mathcal{L}_{X_1}\mathcal{L}_{X_2}\cdots\mathcal{L}_{X_n}\phi|\,dP\\&\leqq\gamma_p\|\phi\|_{1,m}.\end{aligned}$$

So statement (1) and (3) have been proved.

Then we have to show (2). Without loss of generality we may assume that $\|\phi\|_{1,m}=1$. We can write $p=qN+r$ $(0\leqq r<N)$ and for $n\geqq p+1$, $n=kN+j$ $(0\leqq j<N)$. Applying the inequality (3.1) again and again we have for each $\omega\in\tilde{\Omega}_p$

$$\begin{aligned}\bigvee\mathcal{L}_{\omega_1}\mathcal{L}_{\omega_2}\cdots\mathcal{L}_{\omega_{kN+j}}\phi&\leqq2^k\alpha(\omega_1)\cdots\alpha(\omega_{kN})\bigvee\mathcal{L}_{\omega_{kN+1}}\cdots\mathcal{L}_{\omega_{kN+j}}\phi\\&+\sum_{i=0}^{k-1}2^i\alpha(\omega_1)\cdots\alpha(\omega_{iN})\beta_N(\omega_{iN+1},\cdots,\omega_{(i+1)N}).\end{aligned}$$

From (A.1) and (A.2), $2^i\alpha(\omega_1)\cdots\alpha(\omega_{iN})<(2\tau^N)^i$ if $i>q$. Consequently we have

$$\limsup_{n\to\infty}\bigvee U_n^{(p)}\phi\leqq\left(\frac{1}{1-2\tau^N}+\frac{(2(\int\alpha\,d\pi)^N)^q-1}{2(\int\alpha\,d\pi)^N-1}\right)\int\beta_N\,d\pi^N. \tag{5.6}$$

Lemma 5.3. *If* (A.1) *and* (A.2) *are satisfied, then* $\{\mathcal{L}^n\phi\}_{n=1}^\infty$ *is relatively compact in* $L^1(m\times P)$, *for any* $\Phi\in L^1(m\times P)$. *In addition if the limit* $\Phi^*=\lim_{i\to\infty}\mathcal{L}^{n_i}\Phi$ *exists for some* $\{n_i\}_{i=1}^\infty\subset\mathrm{N}$ *with* $n_i<n_{i+1}$ *in* $L^1(m\times P)$ *then* $\Phi^*(x,\omega)=\phi^*(x)$ *a.e.* (P) *for* $\phi^*\in L^1(m)$.

Proof. First we prove that $\{\mathcal{L}^n\phi\}_{n=1}^\infty$ is relatively compact in $L^1(m)$ for $\phi\in BV$. Let $\{n_i\}_{i=1}^\infty$ be a sequence of natural numbers with $n_i<n_{i+1}$. Because of (5.4) and the diagonal method we can choose a subsequence $\{n_j'\}_{j=1}^\infty$ with $n_j'<n_{j+1}'$ such that $\{U_{n_j'}^{(p)}\phi\}_{j=1}^\infty$ is convergent sequence in $L^1(m)$ for fixed p. Then we have

$$\begin{aligned}&\|\mathcal{L}^{n_j'}\phi-\mathcal{L}^{n_k'}\phi\|_{1,m}\\&\leqq\|\mathcal{L}^{n_j'}\phi-U_{n_j'}^{(p)}\phi\|_{1,m}\\&+\|U_{n_j'}^{(p)}\phi-U_{n_k'}^{(p)}\phi\|_{1,m}\\&+\|U_{n_k'}^{(p)}\phi-\mathcal{L}^{n_k'}\phi\|_{1,m}\\&\leqq2\gamma_p+\|U_{n_j'}^{(p)}\phi-U_{n_k'}^{(p)}\phi\|_{1,m}.\end{aligned}$$

Therefore $\{\mathcal{L}^{n_j'}\phi\}_{j=1}^{\infty}$ is a Cauchy sequence in $L^1(m)$. This proves that the statement is valid for $\phi\in BV$. Thus it is valid for $\phi\in L^1(m)$ since $\mathcal{L}$ is contractive. From the formula (4.4) we can easily see that the statements are still valid for the function $\Phi\in L^1(m\times P)$ with the form

$$\Phi(x, \omega) = \phi(x)\, \psi(X_1(\omega), \cdots, X_k(\omega)) \text{ where } \phi\in L^1(m) \text{ and } \psi\in L^\infty(\pi^k)\,.$$

Since the linear combinations of such functions are dense in $L^1(m\times P)$, the statements are valid for any $\Phi\in L^1(m\times P)$.

Proposition 5.1. *Assume* (A.1) *and* (A.2). *Then for each* $\lambda\in\mathbf{C}$ *with* $|\lambda|=1$ *and* $\Phi\in L^1(m\times P)$, *the sequence* $\left\{\frac{1}{n}\sum_{i=0}^{n-1}\bar{\lambda}^i\,\mathcal{L}^i\,\Phi\right\}_{n=1}^{\infty}$ *is convergent in* $L^1(m\times P)$. *And the limit* Φ^* *has the following:*

$$\Phi^*\in L^1(m)\,. \tag{5.7}$$

$$\mathcal{L}\Phi^* = \lambda\,\Phi^*\,. \tag{5.8}$$

In the case of $\lambda=1$, *we have*

$$\int \Phi\, d(m\times P) = \int \Phi^*\, dm\,. \tag{5.9}$$

In particular if $\Phi\geqq 0$ *and* $\int\Phi\, d(m\times P)>0$, $(\Phi^* m)\times P$ *is an* $(m\times P)$*-absolutely continuous T-invariant measure.*

Proof. From Lemma 5.3, we can apply the Kakutani-Yosida Theorem [6, Theorem 1] to $\mathcal{L}$. So we get the above.

REMARK 5.1. If we assume (A.1′) instead of (A.1)

$$\int \alpha\, d\pi < 1\,. \tag{A.1′}$$

Then K^p in (5.4) can be replaced by a constant which is independent of p. Thus there is a constant $C>0$ with

$$\limsup_{n\to\infty} \vee\, \mathcal{L}^n\phi \leqq C\,\|\phi\|_{1\,m} \quad \text{for all} \quad \phi\in BV\,. \tag{5.10}$$

From this one can get the smae results as Lemma 3.1 and Lemma 3.2 when $\mathcal{L}_{f,m}$ is replaced by $\mathcal{L}$.

Put

$$\tilde{h} = \lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\mathcal{L}^i\, 1,\ \tilde{\mu} = \tilde{h}m,\ \tilde{Q} = \tilde{\mu}\times P$$

and

$$\mathcal{B}_{\text{inv}} = \{\Gamma \in \mathcal{B}(I \times \Omega) \colon T^{-1}\Gamma = \Gamma\ (\tilde{Q})\} .$$

To prove the ergodic decomposition and weakly mixing one, we need the following:

Lemma 5.4 (basic). (1) *If $\Gamma \in \mathcal{B}_{\text{inv}}$, there is a measurable set $A \subset \{\tilde{h} > 0\}$ such that $\Gamma = A \times \Omega (m \times P)$.*

(2) *If $A \times \Omega \in \mathcal{B}_{\text{inv}}$, then for almost all s (π). $f_s^{-1} A \supset A(m)$.*

(3) *If $\Psi \in L^1(\tilde{Q})$ and $U_T \Psi = \lambda \Psi$, then for almost all s (π), $\Psi \circ f_s = \lambda \Psi$ $(\tilde{\mu})$.*

(4) *Put $E = \{s \colon f_s \in \mathcal{D}_e\}$, Then $\pi(E) > 0$.*

(5) *If $A \times \Omega \in B_{\text{inv}}$, then A contains at least one ergodic component of f_s for almost all $s \in E$. Hence $\mathcal{B}_{\text{inv}}$ is a finite set.*

Proof. (1) From the formula (2.5), if $\Gamma \in \mathcal{B}_{\text{inv}}$ we have

$$\mathcal{L}(1_\Gamma \tilde{h}) = \mathcal{L}(1_\Gamma \circ T \tilde{h}) = 1_\Gamma \mathcal{L} \tilde{h} = 1_\Gamma \tilde{h} .$$

So $1_\Gamma \tilde{h}$ is independent of ω.

(2) It is easy to see $T^{-1}(A \times \Omega) \supset A \times \Omega (m \times P)$. Thus

$$\begin{aligned} 0 &= \tilde{Q}(T^{-1}(A \times \Omega)) - \tilde{Q}(A \times \Omega) \\ &= \int |1_A(f_{X_1(\omega)}\, x) - 1_A(x)|\, d\tilde{Q} \\ &= \int |1_A(f_s\, x) - 1_A(x)|\, d\tilde{\mu}(x)\, d\pi(s) . \end{aligned}$$

So $f_s^{-1} A = A$ $(\tilde{\mu})$ for almost all s (π). Hence $f_s^{-1} A \supset A(m)$ for almost all s (π).

(3) If $U_T \Psi = \lambda \Psi$, then $\mathcal{L}(\Psi \tilde{h}) = \lambda \Psi \tilde{h}$ from (2.9). Thus $\Psi \tilde{h}$ is independent of ω, so we have (3).

(4) Since $\int \log \alpha\, d\pi < 0$, $\pi(E) > 0$.

(5) Let S_0 be the measurable set such that $\pi(S_0) = 1$ and $s \in S_0$ implies that $f_s^{-1} A \supset A(m)$ for $A \subset \{\tilde{h} > 0\}$ with $A \times \Omega \in \mathcal{B}_{\text{inv}}$. If $s \in E \cap S_0$, f_s has the ergodic components $L_1^{(s)}, L_2^{(s)}, \cdots, L_n^{(s)}$ from Theorem 3.1. From (5) in Theorem 3.1 $m(A \cap L_j^{(s)}) > 0$ for some $j = 1, 2, \cdots, n$. $f_s^{-1} A \supset A(m)$ implies $f_s^{-1}(A \cap L_j^{(s)}) \supset A \cap L_j^{(s)}(\mu_j)$. Thus $\mu_j(A \cap L_j^{(s)}) = 1$ from the ergodicity of f_s. Hence $A \supset L_j^{(s)}(m)$.

Proof of Theorem 5.1. We first prove the assertions (1) and (3) of (i). By Lemma 5.4, we may asssume

$\mathcal{B}_{\text{inv}} = \sigma(\{A_i \times \Omega \colon i = 1, 2, \cdots, n\})$, where $\bigcup_{i=1}^{n} A_i = \{\tilde{h} > 0\}(m)$ and $m(A_i \cap A_i) = 0$ if $i \neq j$. Set $\tilde{h}_i = \tilde{\mu}(A_i)^{-1} \tilde{h} 1_{A_i}$, $\tilde{\mu}_i = \tilde{h}_i\, m$ and $Q_i = \tilde{\mu}_i \times P$. Then it is clear that $\mathcal{L} \tilde{h}_i = \tilde{h}_i$ and (T, Q_i) is ergodic.

The property (2) of (i) is proved as follows. Let Q be any $(m \times P)$-absolutely continuous T-invariant σ-additive set function. Then it is easy to check that Q is $\tilde{Q}$-absolutely continuous and $Q|_{A_i \times \Omega}$ is also T-invariant for every

$i=1, 2, \cdots, n$. Since (T, Q_i)'s are ergodic, we have $Q|_{A_i\times\Omega}=Q(A_i\times\Omega)\, Q_i$ $i=1, 2, \cdots, n$. Hence

$$Q=\sum_{i=1}^{n} Q(A_i\times\Omega)\, Q_i\,.$$

Now we prove the statement (ii). For the sake of simplicity we assume that $(T, \tilde{Q})$ is ergodic. By (3) in Lemma 5.4, there is an s such that any eigenvalue of U_T is that of U_{f_s}. Thus U_T has only a finite number of eigenvalues and so the totality of all eigenvalues is $\{1, \lambda, \cdots, \lambda^{N-1}\}$, where λ is a primitive N-th root of 1. Let Ψ be an eigenfunction of U_T on $L^1(\tilde{Q})$ belonging to λ. From Proposition 3.1, we can write $\Psi=\sum_{i=0}^{N-1}\lambda^i\, 1_{L_i}$, where $L_i=\{\Psi=\lambda^i\}$. So we can see that $TL_i=L_{i+1}$, $0\leqq i<N-1$, $TL_{N-1}=L_0$ and for each i, $(T^N, \tilde{Q}_i)$ is weakly mixing in the same manner as in the proof of Theorem 3.2. From Proposition 3.2, $(T^N, \tilde{Q}_i)$ is exact.

6. Auxiliary results

In this section we always assume (A.1) and (A.2) and use the same notations as before.

Analogously to Theorem 3.1, (5) we have

Proposition 6.1. *Let $L=\{\tilde{h}>0\}$ and $D=I\times\Omega\backslash L$. Then $T^{-1}D\subset D(m\times P)$ and $(m\times P)\,(T^{-n}D)\to 0$ $(n\to\infty)$.*

Proof. It is obvious that $T^{-1}D\subset D$. And we have

$$\begin{aligned}
&\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}(m\times P)\,(T^{-i}D)\\
&=\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\int_D 1_D\circ T^i\, d(m\times P)\\
&=\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\int_D \mathcal{L}^i\, 1_D\, d(m\times P)\\
&\leqq\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\int_D \mathcal{L}^i\, 1\; d(m\times P)\\
&=\int_D \tilde{h}\; d(m\times P)\\
&=0
\end{aligned}$$

From the above Proposition one can see that $I\times\Omega=L\cup D$ is the Hopf decomposition for the Markov operator U_T (see [2]). So from the Chacon-Ornstein Theorem we have

Corollary. *For each $\Phi\in L^1(m\times P)$, $\frac{1}{n}\sum_{i=0}^{n-1}\mathcal{L}^i\,\Phi$ converges almost everywhere*

$(m\times P)$.

The operator $\mathcal{L}$ has the following spectral decomposition.

Proposition 6.2. *Let Λ be the set of all eigenvalues of $\mathcal{L}$ on $L^1(m\times P)$ with modulus* 1. *Let $E(\lambda)$ be the eigenspace belonging to λ. Then*:

(1) $1\in\Lambda$.

(2) $\#\Lambda<\infty$.

(3) *Put $\Lambda=\{1=\lambda_1, \lambda_2, \cdots, \lambda_l\}$. Then $E(\lambda_i)\subset L^1(m)$ and* dim $E(\lambda_i)<\infty$ *for everp $i=1, 2, \cdots, l$.*

(4) *For every $n\in\mathbf{N}$, we have*

$$\mathcal{L}^n = \sum_{i=1}^{l} \lambda_i^n \mathcal{P}_i+\mathcal{N}^n , \tag{6.1}$$

where P_i is projection onto the eigenspace, namely, $\mathcal{P}_i \mathcal{P}_j=\mathcal{P}_j \mathcal{P}_i=O$ $(i\neq j)$, $\mathcal{P}_i \mathcal{P}_i=\mathcal{P}_i$ and $\mathcal{P}_i \mathcal{N}=\mathcal{N}\mathcal{P}_i=O$.

(5) *For each $i=1, 2, \cdots, l$, λ_i is a root of* 1.

(6) *For each $\Phi\in L^1(m\times P)$.*

$$||\mathcal{N}^n \Phi||_{1,m\times p}\to 0 \quad (n\to\infty) . \tag{6.2}$$

Proof. (1), (2), and (4) are easy consequences of the Kakutani-Yosida Theorem and Proposition 5.1. For (3) we have only to show that dim $E(\lambda_i)<\infty$. But if we consider T^{N_i} instead of T where $\lambda_i^{N_i}=1$, we can deduce the problem to the case $\lambda_i=1$. It remains to prove (6.2). Without loss of generality, we may assume that $(T, \tilde{Q})$ is ergodic. Let $L_0, L_1, \cdots, L_{N-1}$ be the exact components of T^N and let L and D be the sets stated in Proposition 6.1. For $\Phi\in L^1(m\times P)$ we have

$$\mathcal{L}^{nN}\Phi = 1_D \mathcal{L}^{nN}\Phi+1_L \mathcal{L}^{nN}\Phi .$$

The first term tends to 0 by Proposition 6.1. Next using (2.5) and (2.6) we have

$$\begin{aligned} 1_L \mathcal{L}^{nN} \Phi &= \mathcal{L}^{nN}(1_L\circ T^{nN} \Phi) \\ &= \sum_{j=0}^{N-1} h_j \mathcal{L}_{Q_j}^{nN}(1_{L_j}\circ T^{nN} \Phi h_j^{-1}) \\ &= \sum_{j=0}^{N-1} h_j (\mathcal{L}_{Q_j}^{nN}(\Phi h_j^{-1})-\int_{L_j} \Phi h_j^{-1}\, dQ_j)+ \sum_{j=0}^{N-1} h_j \int_{L_j} \Phi h_j^{-1}\, dQ_j \end{aligned}$$

where $h_j=N \tilde{h}|_{L_j}$ and $Q_j=N \tilde{Q}|_{L_j}$. The last term corresponds to the projections. From Proposition 3.2,

$$h_j(\mathcal{L}_{Q_j}^{nN}(\Phi h_j^{-1})-\int \Phi h_j^{-1}\, dQ_j)\to 0 \text{ in } L^1(m\times P) \text{ as } n\to\infty .$$

This completes the proof.

7. Bernoulliness

Assume that the S is a finite set and assumption (A.1) is satisfied. We have proved the existence of $(m\times P)$-absolutely continuous T-invariant probability measures. Let Q be one of such measures. Here we have:

Theorem 7.1. *If the dynamical system (T,Q) is weakly maixing, then it is weakly Bernoulli.*

To prove the above we need some lemmas.

Lemma 7.1. *Let ϕ be a nonnegative function of bounded variation. If* (A.1) *and* (A.2) *are satisfied, any limit point ϕ^* of the sequence $\{\mathcal{L}^n\phi\}_{n=1}^{\infty}$ has the following.*

For any τ with $0<\tau<1$, there are positive cosntants K_0, K_1, K_2 and ρ with $0<\rho<1$ such that

$$\int_B \phi^*\, dm \leqq K_0\, \rho^n + K_1\, e^{-K_2\sqrt{n}}$$

for every $B\in\mathcal{B}(I)$ with $m(B)<\tau^n$.

Proof. Let ϕ^* be a limit point of $\{\mathcal{L}^n\phi\}_{n=1}^{\infty}$. From Lemma 5.2, we have

$$\phi^* = \phi_p^{*1}+\phi_p^{*2} \quad \text{for each } p\in\mathbf{N},$$

where ϕ_p^{*1} and ϕ_p^{*2} are functions with

$$\bigvee \phi_p^{*1} \leqq K^p\|\phi\|_{1,m} \quad \text{for some } K \text{ and}$$
$$\|\phi_p^{*2}\|_{1,m} \leqq C\sqrt{p}\; e^{-\sqrt{p}}\; \|\phi\|_{1,m}\,.$$

Moreover, we may assume that

$$\|\phi_p^{*1}\|_{\infty,m} \leqq C'\, K^p \quad \text{for some constant } C'\,.$$

Indeed, let ψ_p be a version of ϕ_p^{*1} with

$$\widetilde{\bigvee}\psi_p = \bigvee \phi_p^{*1}\,.$$

We may assume that ψ_p converges to ϕ^* almost everywhere as p goes to ∞. So there is an $x\in I$ such that $\{\psi_p(x)\}_{p=1}^{\infty}$ is a bounded sequence. Therefore

$$\|\phi_p^{*1}\|_{\infty,m} \leqq \sup_p\, |\psi_p(x)| + K^p\|\phi\|_{1,m}\,.$$

Hence we have

$$\|\phi_p^{*1}\|_{\infty,m} \leqq C'K^p\,.$$

For a given τ with $0<\tau<1$ choose a natural number L such that $\tau^L<K^{-1}$

and put $\rho=\tau^L\sqrt{K}$. Then we have

$$\int_B \phi^* \, dm = \int_B \phi^{*1}_{[n/L]} \, dm + \int_B \phi^{*2}_{[n/L]} \, dm$$
$$\leqq C' \rho^n + C\sqrt{\frac{n}{L}}\, e^{-[n/L]}$$

whenever $m(B)<\tau^n$.

From now on, put

$$S = \{1, 2, \cdots, q\}$$
$$\phi_s = \frac{1}{|f'_s|}$$

and

$$p_s = \pi(\{s\})>0 \quad \text{for each} \quad s\in S\,.$$

Let $\{I_i\}_{i=1}^r$ be the partition of I into intervals which is determined by $\{a_{i=0}^{(s)}\}_{i=0}^{k_s}$ $s=1, 2, \cdots, q$.

Put
$$\xi = \{I_i\times[s]\colon 1\leqq i\leqq r,\ 1\leqq s\leqq q\}$$

and

$$\xi_0^N = \bigvee_{i=0}^{N} T^{-i}\,\xi\,.$$

If B is an atom in ξ_0^N, then we can write $B=A\times\Gamma$, where A has the form

$$I_{i_0} \cap f_{j_0}^{-1} I_{i_1} \cap \cdots \cap f_{j_0}^{-1} f_{j_1}^{-1} \cdots f_{j_{N-1}}^{-1} I_{i_N}$$

and Γ is a cylinder set having the form

$$\Gamma = [j_0, j_1, \cdots, j_N]\,.$$

Hence we have the following:

Lemma 7.2. *ξ is a generator for (T,Q).*

The next lemma can be proved in the same way as Lemma 1 in [1].

Lemma 7.3. *Let $A\times\Gamma$ be an atom in ξ_0^{N+1}. If $\bar{A}\cap\{a_j^{(s)}\}_{s=1}^{k_s}{}_{s=1}^{q}=\phi$, then $T(A\times\Gamma)\in\xi_0^N$.*

Lemma 7.4. *Let $h\in L^1(m)$ be the density function of Q. Given $\varepsilon>0$ there exists $g_\varepsilon\in BV$ with*

$$0\leqq g_\varepsilon\leqq h$$

and

$$\int (h-g_\varepsilon)\, dm<\varepsilon\,.$$

Proof. We may assume that $h=\lim_{n\to\infty} \mathcal{L}^n 1$. Taking $\phi=1$ in (5.3) and (5.4) we have

$$\mathcal{L}^n 1 = U_n^{(p)} 1+S_n^{(p)} 1$$

and

$$\limsup_{n\to\infty} \vee\, U_n^{(p)} 1\leqq K^p\,.$$

By Helley's theorem we may assume that $\lim_{n\to\infty} U_n^{(p)} 1=g_p$ exists. Then we see

$$0\leqq g_p\leqq h$$

and

$$\|h-g_p\|_{1,m}\leqq C\sqrt{p}\ e^{-\sqrt{p}} \quad \text{by (5.5)}\,.$$

Lemma 7.5 and Lemma 7.6 correspond to Lemma 3.2 and Lemma 3.3 in [11] respectively.

Lemma 7.5. *Given $\varepsilon>0$, there is an integer $N_0=N_0(\varepsilon)$ and a positive constant H_0 satisfying:*

If $N\geqq N_0$ then there exist collections of atoms $\alpha_N\subset\xi_0^N$ and $\alpha'_N\subset\xi_0^N$ such that

(1) $Q(\cup\,\alpha_N)>1-(H_0\dfrac{1}{N^2}+\varepsilon)$

and

$$Q(\cup\,\alpha'_N)>1-H_0\frac{1}{N^2},$$

where $Q(\cup\,\alpha)$ denotes the total Q-measure of all atoms contained in the collection α.

(2) *$A\times\Gamma\in\alpha_N\cup\alpha'_N$ implies $\Gamma\subset\Omega_N$, where Ω_N is the set defined by* (5.1) *for some fixed $\sqrt[N]{\dfrac{1}{2}}>\tau>\exp\left(\int\log\alpha\, d\pi\right)$.*

(3) $\left|\dfrac{g_\varepsilon(x)}{g_\varepsilon(y)}-1\right|\leqq\dfrac{2}{N^2}$ *for every $x,y\in A$, whenever $A\times\Gamma\in\alpha_N$ and*

$\left|\dfrac{\phi_{i_0}(x)}{\phi_{i_0}(y)}-1\right|\leqq\dfrac{2}{N^2}$ *for every $x,y\in A$, whenever $A\times\Gamma\in\alpha'_N$ and $\Gamma\subset[i_0]$,*

where g_ε is the function which appeared in Lemma 7.4.

Proof. Write g for g_ε. Let $0<\delta<1$. For a cylinder $\Gamma=[i_0, i_1, \cdots, i_N]\subset\Omega_N$ consider the following exhaustive list of possibility for an atom $A\times\Gamma\in\xi_0^N$.

(i) $g(x)\geqq\frac{\delta}{2}$ for all $x\in A$ and $g(y)>e^{\delta}g(z)$ for some $y, z\in A$.

(ii) $g(x)<\frac{\delta}{2}$ and $g(y)\geqq\frac{3}{4}\delta$ for some $x, y\in A$.

(iii) $g(x)<\frac{3}{4}\delta$ for all $x\in A$.

(iv) $g(x)\geqq\frac{\delta}{2}$ for all $x\in A$ and $g(y)\leqq e^{\delta}g(z)$ for all $y, z\in A$.

The variation of g over an $A\times\Gamma$ satisfying (i) and (ii) is at least $\frac{\delta^2}{4}$. The total number of such atoms $A\times\Gamma$ is at most $\frac{4}{\delta^2}\bigvee g$. Thus the total Q-measure of all atoms satisfying (i) and (ii) is at most

$$\sum_{I\subset\Omega_N} P(\Gamma)\sum_{A}\int_A h\, dm\leqq\frac{4}{\delta^2}\bigvee g\, H_1\, e^{-H_2\sqrt{N}}$$

for some constants H_1 and H_2 by Lemma 7.1.

The total Q-measure of all atoms satisfying (iii) is at most $\frac{3}{4}\delta+\varepsilon$. Therefore the total Q-measure of all atoms $A\times\Gamma$ with $\Gamma\subset\Omega_N$ satisfying (i), (ii) or (iii) is at most

$$\frac{4}{\delta^2}\bigvee g\, H_1\, e^{-H_2\sqrt{N}}+\frac{3}{4}\delta+\varepsilon\,.$$

Next put $C_s=\int|f_s'|\, d\mu$, where $\mu=hm$. For each s and $\Gamma_s\subset[s]\cap\Omega_N$ consider the following exhaustive list again:

(i)′ $\phi_s(x)\geqq\frac{\delta}{2}$ for all $x\in A$ and $\phi_s(y)>e^{\delta}\phi_s(z)$ for some $y, z\in A$.

(ii)′ $\phi_s(x)<\frac{\delta}{2}$ and $\phi_s(y)\geqq\frac{3}{4}\delta$ for some $x, y\in A$.

(iii)′ $\phi_s(x)<\frac{3}{4}\delta$ for all $x\in A$.

(iv)′ $\phi_s(x)\geqq\frac{\delta}{2}$ for all $x\in A$ and $\phi_s(y)\leqq e^{\delta}\phi_s(z)$ for all $y, z\in A$.

The variation of ϕ_s over an $A\times\Gamma_s$ satisfying (i)′ and (ii)′ is at least $\frac{\delta^2}{4}$. The total number of such atoms $A\times\Gamma_s$ is at most $\frac{4}{\delta^2}\bigvee\phi_s$. Thus the total Q-measure of all atoms satisfying (i)′ and (ii)′ is at most

$$\frac{4}{\delta^2}\bigvee\phi_s\, H_1\, e^{-H_2\sqrt{N}}\, p_s\,.$$

The total Q-measure of all atoms satisfying (iii)′ is at most $\frac{3}{4}\delta\, C_s\, p_s$. Therefore the total Q-measure of all atoms $A\times\Gamma$ with $\Gamma\subset[s]\cap\Omega_N$ and satisfying (i)′, (ii)′ or (iii)′ for some s is at most

$$\frac{4}{\delta^2}H_1\sqrt{N}\,e^{-H_2\sqrt{N}}\cdot\sum_{s=1}^{l}\vee\phi_s\,p_s+\frac{3}{4}\delta\sum_{s=1}^{l}C_s\,p_s\,.$$

Put

$$\delta=\frac{1}{N^2}$$

$$\alpha_N=\{A\times\Gamma\in\xi_0^N\colon\Gamma\subset\Omega_N\text{ and (iv) holds}\}\quad\text{and}$$

$$\alpha'_N=\{A\times\Gamma\in\xi_0^N\colon\Gamma\subset\Omega_N\text{ and (iv)}'\text{ holds for all }s\in S\}\,.$$

If N is large enough

$$Q(\cup\alpha_N)>1-\left(H_0\frac{1}{N^2}+\varepsilon\right)$$

$$Q(\cup\alpha'_N)>1-H_0\frac{1}{N^2}$$

for some H_0 and (3) holds.

Given $\varepsilon>0$ put $Q_\varepsilon=g_\varepsilon\,m\times P$ we have:

Lemma 7.6 (basic). *Given $\eta>0$ there exists an integer $M=M(\eta)=M(\varepsilon,\eta)$ such that for each $m\geqq 0$, one can find a collection $\beta_{M+m}\subset\xi_0^{M+m}$ with*

(1) $T^m(A\times\Gamma)\in\xi_0^M$ *for any* $A\times\Gamma\in\beta_{M+m}$.

(2) $Q(\cup\beta_{M+m})>1-(2\varepsilon+\eta)$.

(3) $$\left|\frac{Q_\varepsilon(T^m(D))}{Q_\varepsilon(T^m(A\times\Gamma))}-\frac{Q_\varepsilon(D)}{Q_\varepsilon(A\times\Gamma)}\right|\leqq\eta\,\frac{Q_\varepsilon(D)}{Q_\varepsilon(A\times\Gamma)}$$

for any measurable $D\subset A\times\Gamma\in\beta_{M+m}$ whenever $Q_\varepsilon(A\times\Gamma)>0$ and $Q_\varepsilon(T^m(A\times\Gamma))>0$.

Proof. By Lemma 7.3 one can see that (1) will hold for $A\times\Gamma$ unless at least one of the sets $\bar{A},\overline{f_{i_0}A},\cdots,\overline{f_{i_{m-1}}\cdots f_{i_0}A}$ intersects $\{a_i^{(s)}\}_{i=0}^{k_s}\,{}_{s=1}^{q}$ if $\Gamma\subset[i_0,i_1,\cdots,i_{m-1}]$. For $0\leqq k\leqq m$, the total Q-measure of all atoms $A\times\Gamma\in\xi_0^{M+m-k}$ satisfying $\bar{A}\cap\{a_i^{(s)}\}_{i=0,}^{k_s}\,{}_{s=1}^{q}\neq\phi$ and $\Gamma\subset\Omega_{M+m-k}$ is at most $2(r+1)H_1\,e^{-2\sqrt{M+m-k}}$ since $m(A)\leqq\tau^{M+m-k}$. The total Q-measure of all atoms $A\times\Omega\in\xi_0^{M+m-k}$ with $\Gamma\not\subset\Omega_{M+m-k}$ is at most $Ce^{-\sqrt{M+m-k}}$. Since $f_{i_{k-1}}\cdots f_{i_0}A\times\sigma^k\Gamma$ is a subset of some atom in ξ_0^{M+m-k} if $A\times\Gamma\in\xi_0^{M+m}$ and since Q is T-invariant, the total Q-measure of all atoms $A\times\Gamma\in\xi_0^{M+m}$ with $T^k(A\times\Gamma)\in\xi_0^{M+m-k}$ for all $0\leqq k\leqq m$ is at least

$$1-2(r+1)\,H_1\sum_{n\geqq M}e^{-H_2\sqrt{n}}-C'\sqrt{M}\,e^{-\sqrt{M}}\,.$$

Put

$$\tilde{\beta} = \{A\times\Gamma\in\alpha_{M+m}\colon T^k(A\times\Gamma)\in\xi_0^{M+m-k},\ 0\leqq k\leqq m\}\ ,$$
$$\bar{\alpha}_N = \{A\times\Gamma\in\xi_0^N\colon A\times\Gamma\notin\alpha_N\}\ ,$$
$$\bar{\alpha}'_N = \{A\times\Gamma\in\xi_0^N\colon A\times\Gamma\notin\alpha'_N\}\ ,$$

and put

$$\begin{aligned}\tilde{\beta}' &= T^{-m}\,\bar{\alpha}_M \cup \bar{\alpha}_{M=m}\\ &= \{B\colon B=T^{-m}\,C \text{ with } C\in\bar{\alpha}_M \text{ or } B\in\bar{\alpha}_{M+m}\}\ ,\\ \tilde{\beta}'_k &= T^{-m+k}\,\bar{\alpha}'_{M+m}\\ &= \{T^{-m+k}\,B\colon B\in\bar{\alpha}'_{M+m}\}\ ,\\ \beta_{M+m} &= \{B\in\tilde{\beta}\colon B\notin\tilde{\beta}'_k \text{ for all } 0\leqq k\leqq m \text{ and } B\notin\tilde{\beta}'\}\ .\end{aligned}$$

Then we have

$$\begin{aligned}Q(\cup\beta_{M+m})>1-2(r+1)\,H_1\sum_{n\geqq M}e^{-H_2\sqrt{n}}-C'\,\sqrt{M}\,e^{-\sqrt{M}}\\ -2\varepsilon-H_0\frac{1}{n^2}-H_0\frac{1}{(M+m)^2}\\ -H_0\sum_{n\geqq M}\frac{1}{n^2}\,.\end{aligned}$$

Now we have only to show (3). Assume that $A\times\Gamma\in\beta_{M+m}$, $\Gamma=[i_0, i_1, \cdots, i_{M+m}]$, $Q_\varepsilon(A\times\Gamma)>0$ and $Q_\varepsilon(T^m(A\times\Gamma))>0$. In the first place, consider the case $D=\tilde{A}\times\tilde{\Gamma}$ where $\tilde{A}\subset A$ and $\tilde{\Gamma}\subset\Gamma$ is a cylinder set.

Writing g for g_ε and set $\nu=gm$, we have

$$\begin{aligned}&\nu(f_{i_{m-1}}\cdots f_{i_0}\,\tilde{A})\\ &\quad=\int_{f_{i_{m-1}}\cdots f_{i_0}\,\tilde{A}}g(x)\,dm\\ &\quad=\int_{\tilde{A}}p(x)\,g(x)\,dm\,,\end{aligned}$$

where $p(x)=\dfrac{g(f_{i_{m-1}}\cdots f_{i_0}\,x)\,|(f_{i_{m-1}}\cdots f_{i_0})'(x)|}{g(x)}$

So

$$\frac{p(x)}{p(y)}=\frac{g(f_{i_{m-1}}\cdots f_{i_0}\,x)\,g(y)}{g(f_{i_{m-1}}\cdots f_{i_0}\,y)\,g(x)}\prod_{j=0}^{m-1}\frac{\phi_{i_j}(f_{i_j}\cdots f_{i_0}\,y)}{\phi_{i_j}(f_{i_j}\cdots f_{i_0}\,x)}$$

Since $\beta_{M+m}\subset\alpha_{M+m}\cap T^{-m}\alpha_M\cap T^{-k}\alpha'_{M+m}$ we have

$$\left|\frac{g(f_{i_{m-1}}\cdots f_{i_0}\,x)}{g(f_{i_{m-1}}\cdots f_{i_0}\,y)}-1\right|<\frac{2}{M^2}\,,$$
$$\left|\frac{g(x)}{g(y)}-1\right|<\frac{2}{(M+m)^2}\,,$$

and

$$\left|\frac{\phi_{i_j}(f_{i_j}\cdots f_{i_0}y)}{\phi_{i_j}(f_{i_j}\cdots f_{i_0}x)}-1\right|<\frac{2}{(M+m-j)^2}\,.$$

Because $\lim_{M\to\infty}\prod_M^{\infty}\left(1\pm\frac{2}{n^2}\right)=1$,

$$\left|\frac{p(x)}{p(y)}-1\right|<\eta_0 \text{ if } M \text{ is sufficiently large.}$$

Thus we have

$$\begin{aligned}
&\frac{Q_\varepsilon(T^m(\tilde{A}\times\tilde{\Gamma}))}{Q_\varepsilon(T^m(A\times\Gamma))}\\
&=\frac{\nu(f_{i_{m-1}}\cdots f_{i_0}\tilde{A})\,P(\sigma^m\tilde{\Gamma})}{\nu(f_{i_{m-1}}\cdots f_{i_0}A)\,P(\sigma^m\Gamma)}\\
&=\frac{\nu(f_{i_{m-1}}\cdots f_{i_0}\tilde{A})\,P(\tilde{\Gamma})}{\nu(f_{i_{m-1}}\cdots f_{i_0}A)\,P(\Gamma)}\\
&=\frac{\int_{\tilde{A}}p(x)\,g(x)\,dm\,P(\tilde{\Gamma})}{\int_A p(x)\,g(x)\,dm\,P(\Gamma)}\\
&\leqq\frac{p(y)\int_{\tilde{A}}g(x)\,dm+\eta_0\,p(y)\int_{\tilde{A}}g(x)\,dm}{p(y)\int_A g(x)\,dm-\eta_0\,p(y)\int_A g(x)\,dm}\,\frac{P(\tilde{\Gamma})}{P(\Gamma)}\\
&=\frac{1+\eta_0}{1-\eta_0}\,\frac{\nu(\tilde{A})}{\nu(A)}\,\frac{P(\tilde{\Gamma})}{P(\Gamma)}\\
&=\frac{1+\eta_0}{1-\eta_0}\,\frac{Q_\varepsilon(\tilde{A}\times\tilde{\Gamma})}{Q_\varepsilon(A\times\Gamma)}\,.
\end{aligned}$$

Next if $D=\bigcup_i\tilde{A}_i\times\tilde{\Gamma}_i$ is a finite disjoint union of $\tilde{A}_i\times\tilde{\Gamma}_i$ with $\tilde{A}_i\subset A$ and a cylinder $\tilde{\Gamma}_i\subset\Gamma$, then we have

$$\frac{Q_\varepsilon(T^mD)}{Q_\varepsilon(T^m(A\times\Gamma))}\leqq\frac{1+\eta_0}{1-\eta_0}\,\frac{Q_\varepsilon(D)}{Q_\varepsilon(A\times\Gamma)}\,. \tag{7.1}$$

To prove this it suffices to show the following:

If $D_1\subset A\times\Gamma$, $D_2\subset A\times\Gamma$ and $D_1\cap D_2=\phi$, then $T^mD_1\cap T^mD_2=\phi$.

If $(x,\omega)\in T^mD_1\cap T^mD_2$, then there is a unique $z\in A$ with $f_{i_{m-1}}\cdots f_{i_0}z=x$ and a unique $\omega'\in\Gamma\cap\sigma^{-m}\omega$ with $\sigma^n\omega'=\omega$, that is, $\omega'\in[i_0, i_1, \cdots, i_{m-1}]$. Hence $(z,\omega')\in D_1\cap D_2$.

$$\text{Put}\quad \mathcal{E}=\{D\in\mathcal{B}(A\times\Gamma)\colon (7.1)\text{ holds}\}\,.$$

It is easy to see that $\mathcal{E}$ is a monotone class. Hence we have now proved that (7.1) holds for every measurable $D\subset A\times\Gamma$.

We can show that

$$\frac{Q_\varepsilon(T^m D)}{Q_\varepsilon(T^m(A\times\Gamma))}\geqq\frac{1-\eta_0}{1+\eta_0}\frac{Q_\varepsilon(D)}{Q_\varepsilon(A\times\Gamma)}$$

for every measurable $D\subset A\times\Gamma$ in the same manner. This implies (3).

Proof of Theorem 7.1. Given $\varepsilon>0$ choose g_ε as Lemma 7.4 and $M=M(\varepsilon)$ as Lemma 7.6.

Put

$$\begin{aligned}\bar\beta_{M+m}=\{B\in\beta_{M+m}\colon &Q_\varepsilon(B)>0,\ Q_\varepsilon(T^m B)>0\\ &Q(B)-Q_\varepsilon(B)<\sqrt{\varepsilon}\ Q(B)\ \text{ and}\\ &Q(T^m B)-Q_\varepsilon(T^m B)<\sqrt{\varepsilon}\ Q(T^m B)\}\end{aligned}$$

Then we have $Q(\cup\bar\beta_{M+m})>1-5\varepsilon-2\sqrt{\varepsilon}$.

In fact, $Q(\bigcup_{Q_\varepsilon(B)=0} B)<Q(I\times\Omega)-Q_\varepsilon(I\times\Omega)<\varepsilon$ and the total Q-measure of all atoms $A\in\xi_0^M$ with $Q_\varepsilon(A)=0$ is at most ε. Since Q is T-invariant and $T^m B$ is contained in some atom $A\in\xi_0^M$ if $B\in\xi_0^{M+m}$, the total Q-measure of all atoms $B\in\xi_0^{M+m}$ with $Q_\varepsilon(T^m B)=0$ is also at most ε. Next the total Q-measure of all atoms $B\in{}_0^{M+m}$ with $Q(B)-Q_\varepsilon(B)\geqq\sqrt{\varepsilon}\,Q(B)$ is at most $\sqrt{\varepsilon}$ since $\varepsilon>Q(\cup B)-Q_\varepsilon(\cup B)\geqq\sqrt{\varepsilon}\ Q(\cup B)$, where $\cup B$ means the union of all atoms $B\in\xi_0^{M+m}$ with $Q(B)-Q_\varepsilon(B)\geqq\sqrt{\varepsilon}\,Q(B)$. And the total Q-measure of all atoms $A\in\xi_0^M$ with $Q(A)-Q_\varepsilon(A)\geqq\sqrt{\varepsilon}\,Q(A)$ is at most $\sqrt{\varepsilon}$. Since Q is T-invariant the total Q-measure of all atoms $B\in\xi_0^M$ with $Q(T^m B)-Q_\varepsilon(T^m B)\geqq\sqrt{\varepsilon}\,Q(T^m B)$ is at most $\sqrt{\varepsilon}$.

For families of finitely many disjoint measurable sets ξ_1 and ξ_2, put

$$D(\xi_1,\xi_2)=\sum_{A\in\xi_1}\sum_{B\in\xi_2}|Q(A\cap B)-Q(A)\,Q(B)|\,.$$

Notice that $\bar\xi_i\subset\xi_i$ $i=1, 2$, then

$$D(\xi_1,\xi_2)\leqq 2(2-Q(\cup\bar\xi_1)-Q(\cup\bar\xi_2))+D(\bar\xi_1,\bar\xi_2)\,.$$

In oder to prove that ξ is a weak Bernoulli generator for T, we have to estimate $D(\xi_0^{M+m},\xi_{M+m+N}^{2M+2m+N})$. From the above

$$\begin{aligned}&D(\xi_0^{M+m},\xi_{M+m+N}^{2M+2m+N})\leqq 10\varepsilon+4\sqrt{\varepsilon}+D(\bar\beta_{M+m},\xi_{M+m+N}^{2M+2m+N})\,.\\ &D(\bar\beta_0^{M+m},\xi_{M+m+N}^{2M+2m+N})\\ &=\sum_{B\in\bar\beta_{M+m}}\ \sum_{C\in\xi_{M+m+N}^{2M+2m+N}}|Q(B\cap C)-Q(B)\,Q(C)|\\ &\leqq\sum_B Q(B)\sum_C\left|\frac{Q(B\cap C)}{Q(B)}-\frac{Q(T^m(B\cap C))}{Q(T^m B)}\right|\end{aligned}$$

$$+\sum_B Q(B)\sum_C\left|\frac{Q(T^m(B\cap C))}{Q(T^mB)}-Q(T^mC)\right|$$
$$=D_1(N)+D_2(N)\,.$$

$$D_1(N)\leqq\sum_B Q(B)\sum_C\left\{\frac{Q(B\cap C)}{Q(B)}-\frac{Q_\varepsilon(B\cap C)}{Q(B)}\right\}$$
$$+\sum_B Q(B)\sum_C\left\{\frac{Q_\varepsilon(B\cap C)}{Q_\varepsilon(B)}-\frac{Q_\varepsilon(B\cap C)}{Q(B)}\right\}$$
$$+\sum_B Q(B)\sum_C\left|\frac{Q_\varepsilon(B\cap C)}{Q_\varepsilon(B)}-\frac{Q_\varepsilon(T^m(B\cap C))}{Q_\varepsilon(T^mB)}\right|$$
$$+\sum_B Q(B)\sum_C\left\{\frac{Q_\varepsilon(T^m(B\cap C))}{Q_\varepsilon(T^mB)}-\frac{Q_\varepsilon(T^m(B\cap C))}{Q(T^mB)}\right\}$$
$$+\sum_B Q(B)\sum_C\left\{\frac{Q(T^m(B\cap C))}{Q(T^mB)}-\frac{Q_\varepsilon(T^m(B\cap C))}{Q(T^mB)}\right\}$$
$$=D_{1,1}+D_{1,2}+D_{1,3}+D_{1,4}+D_{1,5}\,.$$

Clearly $D_{1,1}<\varepsilon$.

$$D_{1,2}\leqq\sum_B Q(B)\,Q_\varepsilon(B)\,\frac{Q(B)-Q_\varepsilon(B)}{Q(B)\,Q_\varepsilon(B)}$$
$$=\sum_B\,(Q(B)-Q_\varepsilon(B))$$
$$<\varepsilon\,.$$

By Lemma 7.6,

$$D_{1,3}\leqq\sum_B\,(Q(B)\sum_C\varepsilon\,\frac{Q_\varepsilon(B\cap C)}{Q_\varepsilon(B)}$$
$$<\varepsilon$$
$$D_{1,4}=\sum_B Q(B)\,Q_\varepsilon(T^mB)\,\frac{Q(T^mB)-Q_\varepsilon(T^mB)}{Q_\varepsilon(T^mB)\,Q(T^mB)}$$
$$<\sum_B Q(B)\,\sqrt{\varepsilon}$$
$$\leqq\sqrt{\varepsilon}$$
$$D_{1,5}<\sqrt{\varepsilon}\quad\text{in the same way as }D_{14}\,.$$

Thus $D_1(N)\leqq 3\varepsilon+2\sqrt{\varepsilon}$

Now we prove that $D_2(N)\to 0$ $(N\to\infty)$.

$$D_2(N)=\sum_{B\in\bar\beta_{M+m}}Q(B)\sum_{C\in\xi^{2M+2m+N}_{M+m+N}}\left|\frac{Q(T^m(B\cap C))}{Q(T^mB)}-Q(T^mC)\right|$$
$$=\sum_B Q(B)\sum_C\left|\frac{Q(T^mB\cap T^mC)}{Q(T^mB)}-Q(T^mC)\right|$$

$$\leqq \sum_{B\in\xi_0^M} \sum_{C\in\xi_{M+m+N}^{2M+2m+N}=T^{-N}\xi_{M+m}^{2M+2m}} |Q(B\cap C)-Q(B)\,Q(C)|$$

$$=\sum_B \sum_{C=T^{-N}\bar{C}} \left|\int_{T^{-N}\bar{C}} (I_B-Q(B))\,dQ\right|$$

$$\leqq \sum_B \int |\mathcal{L}_Q^N I_B-Q(B)|\,dQ$$

The last term goes to 0 as $N\to\infty$ from Proposition 3.2.

8. Examples and remarks

EXAMPLE 8.1 (Ito and Tanaka [4]). Consider the case $S=\{1, 2, \cdots, l\}$, $\pi\{i\}=p_i>0$, $\sum_{i=1}^{l} p_i=1$ and f_i is given by

$$f_i(x)=\begin{cases} a_i x & \left(0\leqq x<\dfrac{1}{2}\right) \\ -a_i(x-1) & \left(\dfrac{1}{2}\leqq x\leqq 1\right)\end{cases}$$

where $0<a_i<2$ $i=1, 2, \cdots, l$.

(1) T has an $(m\times P)$-absolutely continuous invariant probability measure if and only if

$$\prod_{i=1}^{l} a_i^{\,p_i}>1 \tag{8.1}$$

(2) Assume (8.1). Then the $(m\times P)$-absolutely continuous T-invariant probability measure is ergodic.

(3) Assume (8.1) and let Q be the above. If T satisfies that $Q(T^n(A\times\Omega))\to 1$ for any $A\in\mathcal{B}(I)$ with $Q(A\times\Omega)>0$, then (T, Q) is weakly Bernoulli.

Proof of (1). In this case $\alpha(s)=a_s^{-1}$, so (A.1) is equivalent to (8.1). Only if part can be proved in the same way as Proposition 1.3 and Proposition 1.4 in [4].

Proof of (2). From Lemma 5.4 the number of $(m\times P)$-absolutely continuous ergodic probability measures is less than the number of the m-absolutely continuous ergodic invariant probability measures for f_i with $d_{f_i}>1$. And each f_i has at most one m-absolutely continuous invariant probability measure since f_i's are unimodal.

Proof of (3). Let Ψ be an eigenfunction of U_T with $U_T\Psi=\lambda\Psi$. From Lemma 5.4, we can find $A\in\mathcal{B}(I)$ such that Ψ takes constant value on $A\times\Omega$. We may assume that $\Psi=1$ on $A\times\Omega$ Then $Q(T^n(A\times\Omega))\to 1$ implies $\Psi=1$ a.s. (Q) and $\lambda=1$. Therefore (T, Q) is weakly Bernoulli by Theorem 7.1.

EXAMPLE 8.2. Let f_0 be the identity function on I. Consider the case $S=$

$\{0, 1, \cdots, q\}$, $\pi\{i\}>0$ for all $i\in S$. If (A.1) is satisfied, then caeh ergodic component of T is weakly Bernoulli one. In fact, if $U_T\Psi=\lambda\Psi$, then $U_{f_0}\Psi=\lambda\Psi$ by Lemma 5.4. Thus $\lambda=1$.

REMARK 8.1. In general, we can say that we can expect nice ergodic properties of T if the family $\{f_s\}_{s\in S}$ consists of transformations having distinct spectral types one another.

EXAMPLE 8.3. Consider the case: $S=\mathbf{R}$, $\pi=N(a, 1)$ and

$$f_s(x)=\begin{cases} e^s x & (s\leqq 0) \\ (e^s+\delta)\,x & (\log n\leqq s\leqq \log (n+1-\delta) \\ (n+1)\,x & (\log (n+1-\delta)\leqq s<\log (n+1)\end{cases} \qquad \text{mod } 1$$

for $n\in\mathbf{N}$, where δ is a positive constant with $0<\delta<1$. Then T has a unique $(m\times P)$-absolutely continuous invariant probability measure Q and the dynamical system (T, Q) is exact if $a>\log 2$. In fact, $\alpha(s)\leqq e^{-s}$ and

$$\beta_1(s)=\begin{cases} 2\,e^{-s} & (s\leqq 0) \\ \dfrac{2}{e^s+\delta-n} & (\log n\leqq s<\log (n+1-\delta)) \\ 2 & (\log (n+1-\delta)\leqq s<\log (n+1)\end{cases}$$

So $\alpha\in L^1(\pi)$ and

$$\begin{aligned}\int \log \alpha\, d\pi &< -\int s\, d\pi \\ &= -a \\ &< -\log 2\end{aligned}$$

$$\begin{aligned}\int \beta_1(s)\,\pi(ds) &= 2\int_{-\infty}^{0} e^{-s}\,\pi(ds) \\ &\quad +2\sum_{n=1}^{\infty}\int_{\log n}^{\log (n+1-\delta)} \frac{1}{e^s+\delta-n}\,\pi(ds) \\ &\quad +2\sum_{n=1}^{\infty}\int_{\log (n+1-\delta)}^{\log (n+1)} \pi(ds) \\ &<\infty\end{aligned}$$

Thus assumption (A.1) and (A.2) are valid with $N=1$.

REMARK 8.2. One can easily see that our results are based on the inequality (3.1). So if the family $\{f_s\}_{s\in S}$ satisfies the smae type inequality as (3.1), we can get the same results as above. For example, if there is a constant $k\geqq 1$ and there are measurable functions $\bar{\beta}_n(s_1, \cdots, s_n)$ with

(8.2) $$\vee \mathcal{L}_{s_n} \cdots \mathcal{L}_{s_1} \phi \leqq k\, \alpha(s_1) \cdots \alpha(s_n) \vee \phi + \beta_n(s_1, \cdots, s_n) \|\phi\|_{1\, m}$$

for $\phi \in BV$ and for $n=1, 2, \cdots$, then under the assumption (A.1) and the following new assumption (A.2′), our arguments still works.

(A.2′) For some $N > (-\log k)\left(\int \log \alpha\, d\pi\right)^{-1}$, $\beta_N(s_1, \cdots, s_N) \in L^1(\pi^N)$ and $\beta_1(s) \in L^1(\pi)$.

EXAMPLE 8.4. Consider the case: $S=\mathbf{R}$, $\pi=N(a, 1)$, and

$$f_s(x) = e^s x \qquad \text{mod } 1 .$$

Then T has a unique $(m \times P)$-absolutely continuous invariant proabbility measure Q and the dynamical system (T, Q) is exact whenever $a>0$.

Proof. Put $f(x)=cx$ mod 1 with $c>0$, and $\phi \in BV$. If $c \leqq 1$, we have $\vee \mathcal{L}_f \phi \leqq 2c^{-1} \vee \phi + 2c^{-1} \|\phi\|_{1,m}$ by (3.1). If $n<c \leqq n+1$, $\mathcal{L}_f \phi = c^{-1} \sum_{i=1}^{n} \phi \circ f_i^{-1} + c^{-1} 1_{[0,c-n]}\, \phi \circ f_{n+1}^{-1}$, where $f_i = f\,|_{((i-1)c^{-1}, ic^{-1})}^{-1}$ for $i=1, 2, \cdots, n$ and $f_{n+1} = f\,|_{(nc^{-1},1)}^{-1}$. Therefore

$$\begin{aligned}
\vee \mathcal{L}_f \phi &\leqq c^{-1} \sum_{n=1}^{n} \bigvee_{(i-1)c^{-1}}^{ic^{-1}} \phi + c^{-1} \vee 1_{[0,c-n]}\, \phi \circ f_{n+1}^{-1} \\
&\leqq c^{-1} \bigvee_{0}^{ic^{-1}} \phi + c^{-1} \bigvee_{nc^{-1}}^{1} \phi + c^{-1} |\phi(1)| \\
&\leqq c^{-1} \vee \phi + c^{-1} \|\phi\|_{\infty,m} \\
&\leqq c^{-1} \vee \phi + c^{-1} \vee \phi + c^{-1} \|\phi\|_{1,m} \\
&= 2c^{-1} \vee \phi + c^{-1} \|\phi\|_{1,m} .
\end{aligned}$$

Consequently

(8.3) $$\vee \mathcal{L}_f \phi \leqq 2c^{-1} \vee \phi + 2c^{-1} \|\phi\|_{1,m} .$$

Note that $f_{s_n} f_{s_{n-1}} \cdots f_{s_1} x = e^{s_1+s_2+\cdots+s^n} x$ mod 1. Put $\beta_n(s_1, s_2, \cdots, s_n) = 2\, e^{-(s_1+s_2+\cdots+s_n)}$. From (8.3), (8.2) is valid with $\beta_n(s_1, s_2, \cdots, s_n)$ and

(8.4) $$\int \beta_n(s_1, s_2, \cdots, s_n)\, \pi^n(ds_1\, ds_2 \cdots ds_n) < \infty .$$

We have

$$\int \log \alpha(s)\, \pi(ds) = \int s\, \pi(ds) = -a < 0 .$$

Thus (A.1) is satisfied. From (8.4), (A.2') is valid.

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