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ON COMPLETE KÄHLER MANIFOLDS WITH FAST CURVATURE DECAY

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0. Introduction

We call (M, o) a Riemannian manifold with a pole iff M is a Riemannian manifold and $\exp_o$: $T_oM \to M$ is a global diffeomorphism. We define the radial curvatures at $x \in M$ as the sectional curvatures of all the 2-dimensional planes in T_xM which are tangent to the unique geodesic joining the pole o to x, and write $r(x)$ for the distance function from o. Suppose now our (M, o) satisfies the following conditions:

(0.1) There exist C^∞ functions k, K: $[0, \infty) \to [0, \infty)$ such that

1. $-K(r(x)) \leqq$ all the radial curvatures at $x \leqq k(r(x))$,

2. $\int_0^\infty sK(s)ds < \infty$,

3. $\int_0^\infty sk(s)ds \leqq 1$.

In this paper, we shall prove the following theorem.

Main Theorem. *Let (M, o) be an n-dimensional complete Kähler manifold with a pole o satisfying condition* (0.1) *($n \geqq 2$). Moreover assume that there exists a C^∞ function H: $[0, \infty] \to [0, \infty)$ such that*

(0.2) $\int_0^\infty sH(s)ds < \infty$,

and

(0.3) $-H(r(x)) \leqq$ *tha Ricci curvature at* $x \leqq H(r(x))$.

Then there exists a positive constant γ_0 depending only on $K(s)$ such that if

$$\int_0^\infty sk(s)ds < \gamma_0 ,$$

M is biholomophic to C^n.

It was conjectured by Greene and Wu that if M is an n-dimensional complete, simply connected Kähler manifold satisying

$$-A(1+r(x))^{-2-\varepsilon} \leqq \text{the sectional curvatures at } x \leqq 0\,,$$

then M should be biholomorphic to C^n. This conjecture was verified by Siu and Yau (cf. [10]). In [3], Greene and Wu generalized the above result to the case:

$$-K(r(x)) \leqq \text{the sectional curvatures at } x \leqq 0\,,$$

where K is the function of (0.1) and moreover non-increasing on $[a, \infty)$ for some $a>0$. In view of the above results and several facts of [3], it has been conjectured if a complete Kähler manifold with a pole (M, o) satisfies condition (0.1), then M should be biholomorphic to C^n. In fact, Mok, Siu, and Yau have shown (M, o) is biholomorphic to C^n in the case:

$$-A_\varepsilon(1+r(x))^{-2-\varepsilon} \leqq \text{the sectional curvatures at } x \leqq A_\varepsilon(1+r(x))^{-2-\varepsilon}\,,$$

where A_ε is a sufficiently small constant depending only on ε (cf. [7]). Recently Kasue has proved Main Theorem under the stronger condition than (0.3):

$$-A(1+r(x))^{-2-\delta} \leqq \text{the Ricci curvature at } x \leqq A(1+r(x))^{-2-\delta}\,,$$

where A is a constant independent of δ, then M is biholomophic to C^n.

Combining the argument in [7], [10], and Kasue's unpublished result cited above, the proof of our Main Theorem follows from the following observation:

1. We construct a bounded solotion to the equation:

$$\Delta u(x) = \theta(r(x))$$

where θ is a C^∞ function such that there exists a C^∞ function

$$h\colon [0, \infty) \to [0, \infty)$$

satisfying $|\theta(x)| \leqq h(r(x))$ and $\int_0^\infty sh(s)ds < \infty$.

2. We construct a bounded non-vanishing holomorphic n-form on M more directly than [7].

We would like to express sincere thanks to our adviser Prof. T. Ochiai, to Dr. A. Kasue for his kindness to show us his unpublised result cited above, and to Prof. J. Kazdan who informed us Lemma 2.1 and its corollary which simplify considerably our original proof of Proposition 2.6.

1. Preliminaires

Let (M, o) be an n-dimensional complete Kähler manifold with a pole satisfying condition (0.1) ($n\geqq 2$). We recall several known results which will be used later.

Fact 1.1 (cf. [4], p. 678, Fact 2.1). *Define C^∞ functions $f(t)$, $F(t)$ by*

(1.2) $f''+kf=0,\quad f(0)=0,\quad f'(0)=1\,,$

(1.3) $F''-KF=0,\ F(0)=0,\ F'(0)=1\,.$

Then there exist constants μ and λ satisfying the following inequalities:

(1.4) $\mu\leqq f'(t)\leqq 1$ *and* $\mu t\leqq f(t)\leqq t\,,$

(1.5) $1\leqq F'(t)\leqq\lambda$ *and* $t\leqq F(t)\leqq\lambda t\,,$

(1.6) $1-\int_0^\infty sk(s)ds\leqq\mu\leqq 1\,,$

(1.7) $1\leqq\lambda\leqq exp\{\int_0^\infty sk(s)ds\}\,.$

Using the results of ([4], p. 679, Lemma 2.1) and ([3], Th.C), one can obtain the following inequalities by simple computation.

Fact 1.8. *Let $f(t)$ and $F(t)$ be as in Fact 1.1. Set*

$$s(t)=exp\left\{\int_1^t\frac{dr}{f(r)}\right\}.$$

Then

(1.9) *$(log\ s)\ (r(x))$ is plurisubharmonic on M,*

(1.10) *$s^2(r(x))$ is a C^∞ strictly plurisubharmonic function on M*

and

$$2\left(F'\frac{s^2}{f^2}\right)(r(x))\Omega\geqq L(s^2(r(x))\geqq 2\left(f'\frac{s^2}{f^2}\right)(r(x))\Omega\,,$$

(1.11) *for arbitrary $p>0$,*

$$L(log(1+s^p)\ (r(x)))\geqq min.\left\{\frac{p^2s^p}{2(1+s^p)^2f^2},\ \frac{ps^pf'}{(1+s^p)f^2}\right\}(r(x))\Omega\,,$$

(1.12) $t\leqq s(t)\leqq t^{1/\mu}\ (t\geqq 1)$

and

$$t^{1/\mu}\leqq s(t)\leqq t\ (0\leqq t\leqq 1)$$

where Ω is the Kähler form of the given Kähler metic G and L is the Levi-form. Note that M is, in particular, a Stein manifold.

We call a differential operator on an open set U of R^{2n}

$$A = \sum_{ij} \frac{\partial}{\partial x_i}\left(a^{ij}\frac{\partial}{\partial x_j}\right), \ a^{ij} \in C^\infty(U)$$

uniformlly elliptic iff there is a positive number η (which is called as *uniform ellipticity* of A) such that for any $x \in U$ and for any tangent vector $X \in T_x(R^{2n})$,

$$\eta^{-1}\sum_i X_i^2 \leqq \sum_{ij} a^{ij}(x) X_i X_j \leqq \eta \sum_j X_j^2 .$$

Fact 1.13 (cf. [3], p. 56, Th.C and p. 80). *The exponential map*

$$exp_o \colon T_o(M) \to M$$

is a quasi isometry and the real operator

$$\sqrt{|g|}\Delta = \sum_{ij} \frac{\partial}{\partial x_i}\left(\sqrt{|g|}g^{ij}\frac{\partial}{\partial x_j}\right)$$

is uniformlly elliptic with respect to the coordinates

$exp_o \colon T_o(M) \to M$. Here

$Re\ G = \sum_{ij} g_{ij} dx^i dx^j$,

(g^{ij}) *is the inverse of* (g_{ij}) ,

$|g| = det(g_{ij})$.

From now on, we call the global coordinates $\exp_o \colon T_o(M) \to M$ as the *natural coordinates*.

Fact 1.14 (cf. [4], p. 678, Th.1). *Let $E \to M$ be a holomorphic line bundle with a hermitian fibre metric h. Suppose the Chern form $\omega = -(\sqrt{-1}/2\pi)\partial\bar\partial \log h$ of the hermitian line bundle $\{E, h\}$ satisfies the condition*

(1.15) $\quad ||\omega(x)|| \leqq v(r(x)) \qquad (x \in M)$

where $v(t)$ is a non-negative C^∞ function on $[0, \infty)$ satisfying

(1.16) $\quad \int_0^\infty sv(s)ds < \infty$.

Then there exists a positive number ν_0 such that if σ is a non-zero holomorphic section of E over M satisfying

(1.17) $\quad ||\sigma(x)|| \leqq C(1+r(x))^\nu$

on M for some constant C and some $0 < \nu < \nu_0$, then σ is nowhere zero on M.

Fact 1.18 (Hörmander's ($\bar\partial$-L^2) method) (cf. [2], AI-53). *Let M be a Stein manifold and $E \to M$ a holomorphic line bundle with a hermitian fibre metric h. Let φ be a plurisubharmonic function on M. Assume that there exists a positive continuous function $c(x)$ on M satisfying*

(1.19) $Ric(M)+c_1(E, h)\geqq c\Omega$

where $Ric(M)$ is the Ricci form of (M, G) and $c_1(E, h)$ is the Chern form of $\{E, h\}$. If an E-valued $C^\infty(0, p)$-form σ on M satisfies $(p\geqq 1)$

(1.20) $$\bar{\partial}\sigma = 0 \text{ and } \int_M \frac{\langle\sigma, \sigma\rangle_h}{c} e^{-\varphi} < \infty ,$$

then there exists uniquely an E-valued $C^\infty(0, p\text{-}1)$ form ψ on M such that

(1.21) $$\bar{\partial}\psi = \sigma \text{ aud } \int_M \langle\psi, \psi\rangle_h e^{-\varphi} \leqq \int_M \frac{\langle\sigma, \sigma\rangle_h}{c} e^{-\varphi} .$$

The definition of $\langle\ ,\ \rangle_h$ is as follows:
if we write locally σ as $\sigma_j e_j$ on U_j where σ_j is a $(0, p)$-form and e_j is a local holomorphic section of E, then $\langle\sigma, \sigma\rangle_h(x)$ is defined as

$$\langle\sigma, \sigma\rangle_h(x) = \langle\sigma_j, \sigma_j\rangle (x)h(e_j, \bar{e}_j) (x)$$

where $\langle\ ,\ \rangle$ is the inner-product of $(0, p)$-form induced by the metric of M.

Fact 1.22 (cf. [6], p. 67 (7.9)). *Let A be a uniformlly elliptic operator on $R^{2n}(n\geqq 2)$. Then there exists the Green's function G_A of A on R^{2n} satisfying the following inequality*:

(1.23) $$-\frac{c(n, \eta)^{-1}}{|x-y|^{2n-2}} \leqq G_A(x, y) \leqq -\frac{c(n, \eta)}{|x-y|^{2n-2}}$$

where $c(n, \eta)$ is a positive constant depending only on n and η (uniform ellipticity of A).

Fact 1.24 (Moser's submean value inequality: cf. [8], p. 462, Th.1). *Let A be a uniformlly elliptic operator on $B(2R)=\{x\in R^{2n}\colon |x|<2R\}$. Assume $v\in W^{1,2}(B(2R))$ satisfies*

(1.25) $$\int_{B(2R)} \sum_{ij} a^{ij} \frac{\partial v}{\partial x_i} \frac{\partial \phi}{\partial x_j} dx \leqq 0$$

for any $\phi\geqq 0$ in $\mathcal{D}(B(2R))$. Then

(1.26) $$||v^2||_{\infty, B(R)} \leqq \frac{c(n, \eta)}{R^{2n}} \int_{B(2R)} v^2$$

where $c(n, \eta)$ is a constant as in (1.22).

Fact 1.27 (Moser's Harnack inequality: cf. [9], p. 578, Th.1). *Let V be as in* (1.24) *and u a positive C^∞ function defined on $B(2R)$ such that $Au=0$ on $B(2R)$. Then*

$$(1.28) \qquad \sup_{B(R)} u \leqq c(n, \eta) \inf_{B(R)} u$$

where $c(n, \eta)$ *is a constant as in* (1.22). *In particular, from* (1.28), *one can easily obtain the following "Liouville theorem": Assume* A *is a uniformlly elliptic operator on* R^{2n}. *If a positive* C^∞*-function* u *on* R^{2n} *satisfies* $Au=0$, *then* u *is a constant.*

2. The solution of Poisson's equation

Lemma 2.1. *Let* $h(|x|)\in C^\infty(R^{2n})$ *depend only on* $r=|x|$ *for* $x\in R^{2n}(n\geqq 2)$, *and* $\Delta_0=\sum_i\left(\frac{\partial}{\partial x_i}\right)^2$ *the usual Laplacian on* R^{2n}. *If* $h(s)$ *satisfies*

$$\int_0^\infty sh(s)ds<\infty\,,$$

then there exists a solution to the equation $\Delta_0 v=h$ *such that*

$$(2.2) \qquad v(x) = -\frac{1}{2(n-1)\omega_{2n-1}}\int_{R^{2n}}\frac{h(|y|)}{|x-y|^{2n-2}}dy\,,$$

$$(2.3) \qquad v(x)-v(o) = \frac{1}{2(n-1)}\int_0^{|x|} sh(s)\left[\left(1-\frac{s}{|x|}\right)^{2n-2}\right]ds\,,$$

where ω_{2n-1} *is the volume of the unit sphere of* R^{2n}.

Proof. It is easy to see that the integral in the right hand side of (2.2) is finite. Then it is well known that $\Delta_0 v=h$. And moreover the solution to $\Delta_0 v=h$ is unique up to harmonic fuctions. But then both v and h depend only on r, so the equation $\Delta_0=h$ becomes an O.D.E.

$$h = \Delta_0 v = v_{rr}+\frac{2n-1}{r}v_r = \frac{1}{r^{2n-1}}(r^{2n-1}v_r)_r\,.$$

This can be integrated explicitly, just by integrating

$$r^{2n-1}v_r(r) = \int_0^r s^{2n-1}h(s)ds\,,$$

and we obtain

$$v_r(r) = \frac{1}{r^{2n-1}}\int_0^r s^{2n-1}h(s)ds\,.$$

Therefore

$$\begin{aligned} v(r) &= v(0)+\int_0^r\frac{1}{t^{2n-1}}\left[\int_0^t s^{2n-1}h(s)ds\right]dt \\ &= v(0)+\frac{1}{2(n-1)}\int_0^r h(s)s^{2n-1}\left(\frac{1}{s^{2n-2}}-\frac{1}{r^{2n-2}}\right)ds\,. \end{aligned}$$

Q.E.D.

From (2.3), the following is obvious.

Corollary 2.4. *If $h \geqq 0$, the function $v(x)$ of (2.2) satisfies*

$$(2.5) \qquad 0 \leqq v(x) - v(o) \leqq \frac{1}{2n-2} \int_0^{|x|} sh(s)ds .$$

Proposition 2.6. *Let θ be a C^∞ function on M and assume that there exists a C^∞ function h: $[0, \infty) \to [0, \infty)$ such that*

$$(2.7) \qquad |\theta(x)| \leqq h(r(x)) \quad and \quad \int_0^\infty sh(s)ds < \infty .$$

Then there exists a bounded C^∞ function u on M satisfying

$$(2.8) \qquad \Delta u = \theta .$$

Proof. Using the natural coordinates, set

$$A = \sum_{ij} \frac{\partial}{\partial x_i}\left(\sqrt{|g|}g^{ij}\frac{\partial}{\partial x_j}\right) = \sqrt{|g|}\Delta .$$

By (1.13), A is uniformlly elliptic on R^{2n} $(=T_oM)$. Let G_A be the Green's function of A on R^{2n} as in (1.22). Then

$$u(x) := \int_{R^{2n}} G_A(x, y) (\sqrt{|g|}\theta)(y)dy$$

is a bounded solution of (2.8). In fact, by (1.13) and the assumption on θ, we obtain

$$|\sqrt{|g|}\theta(y)| \leqq C_1 h(|y|) ,$$

where C_1 is a constant. By (1.23),

$$\left|\int_{R^{2n}} G_A(x, y) (\sqrt{|g|}\theta)(y)dy\right| \leqq C_2 \int_{R^{2n}} \frac{h(|y|)}{|x-y|^{2n-2}} dy ,$$

but from (2.1) and (2.4), the right hand side is bounded by a constant independent of x. Q.E.D.

Using a local holomorphic coordinates $(z_1, \cdots, z_n)$, the Ricci tensor R is locally expressed as

$$R = -\sum_{\alpha\beta} \frac{\partial^2 \log|G|}{\partial z_\alpha \partial \bar{z}_\beta} dz^\alpha d\bar{z}^\beta$$

where $G = \sum_{\alpha\beta} G_{\alpha\bar{\beta}} dz^\alpha d\bar{z}^\beta$ and $|G| = det(G_{\alpha\bar{\beta}})$. Set

$$(2.9) \qquad \phi = -\sum_{\alpha\beta} G^{\alpha\bar{\beta}} \frac{\partial^2 \log|G|}{\partial z_\alpha \partial \bar{z}_\beta}$$

where $(G^{\alpha\bar{\beta}})$ is the inverse of $(G_{\alpha\bar{\beta}})$. Note that 2ϕ is the scalar curvature of Re G. The following is obvious from (0.3) and (2.6).

Corollary 2.10. *There exists a bounded C^∞ function u on M satisfying*

$$\Delta u = \phi \,. \tag{2.11}$$

So u is unique up to constants.

3. Construction of non-vanishing bounded holomorphic n-form

The following is obvious.

Lemma 3.1. *Let $E \to M$ be a hermitian vector bundle with a fibre metric h. Assume that σ is a non-zero holomorphic section of E over M such that*

$$\Delta |\sigma|_h \geqq 0 \qquad on \;\; B_G(R) - V \tag{3.2}$$

where $B_G(R) = \{x \in M: r(x) < R\}$, and *$V = \{x \in M: |\sigma|_h(x) = 0\}$. Then*

$$|\sigma|_h \in W^{1,2}(B_G(R)) \,, \tag{3.3}$$

and

$$\Sigma_{ij} \int_{B_G(R)} \sqrt{|g|}\, g^{ij} \frac{\partial |\sigma|_h}{\partial x_i} \frac{\partial \phi}{\partial x_j} \leqq 0 \tag{3.4}$$

for any $\phi \geqq 0$, $\phi \in \mathcal{D}(B_G(R))$ with respect to the natural coordinates.

Let v be a real valued C^∞ function on M. Then we define a fibre metric $|\;\;|_v$ on the canonical line bundle K_M of M by

$$|\;\;|_v^2 = |\;\;|_{K_M}^2 e^{-v} \tag{3.5}$$

where $|\;\;|_{K_M}$ is the fibre metric on K_M induced by G.

The next lemma immediately follows from Poincaré-Lelong's formula.

Lemma 3.6. *Let u be the function of* (2.11). *Then for any holomorphic n-form ξ,*

$$\Delta \log |\xi|_{2u} = 0 \qquad \text{on } \; M - V \tag{3.7}$$

where $V = \{x \in M: \xi(x) = 0\}$. In particular

$$\Delta |\xi|_{2u} \geqq 0 \qquad \text{on } \; M - V \,. \tag{3.8}$$

Proposition 3.9. *For any positive number ν, there exists a non-zero holomorphic n-form ξ on M such that*

$$|\xi|_{K_M}(x) \leqq C(1 + r(x))^{n(1/\mu - 1) + \nu/\mu} \tag{3.10}$$

where C is a constant and μ is as in (1.1).

Proof. Let $\{U, (z_1, \cdots, z_n)\}$ be a local holomorphic coordinates around o and assume $B_G(\varepsilon)\Subset U$. We choose $\rho\in\mathcal{D}(R)$ satisfying

$$\rho(r)=1 \quad \text{if } |r|\leqq 2^{-1}\varepsilon\,,$$
$$\rho(r)=0 \quad \text{if } |r|\geqq \varepsilon\,,$$

and

$$0\leqq\rho\leqq 1\,.$$

Set

$$\alpha=\nu\log(1+s^2)\,,\quad \varphi=2n\log s$$

where s is the function in (1.8), and we choose a fibre metric $|\ \ |_\alpha$ on K_M. By direct computation using (1.1), we obtain

$$Ric(M)+c_1(K_M, |\ \ |_\alpha)\geqq c\Omega\,,$$

where c is a positive continuous function on M.
Set

$$g(x)=\rho(r(x))dz_1\wedge\cdots\wedge dz_n\,,$$

then, by (1.18), we obtain a $C^\infty(n, 0)$-form on M with

$$\bar\partial\eta=\bar\partial g \tag{3.11}$$

and

$$\int_M|\eta|_\alpha^2 e^{-\varphi}\leqq\int_M|\bar\partial g|_\alpha^2\frac{e^{-\varphi}}{c}<\infty\,. \tag{3.12}$$

Because of the singularity of φ at o, we must have $\eta(o)=0$. So

$$\xi=g-\eta \tag{3.13}$$

is a non-zero holomorphic n-form and satisfies

$$\xi(o)=dz_1\wedge\cdots\wedge dz_n \ \text{ and } \ \bar\partial\xi=0\,.$$

Since g has a compact support,

$$\int_M\frac{|\xi|_{K_M}^2}{(1+s^2)^\nu(1+s)^{2n}}<\infty\,.$$

In the following, C_i denotes a constant independent of ξ. Let u be the function (2.11). Then

$$\int_M\frac{|\xi|_{2u}^2}{(1+s^2)^\nu(1+s)^{2n}}\leqq C_1\int_M\frac{|\xi|_{K_M}^2}{(1+s^2)^\nu(1+s)^{2n}}\,.$$

From (3.1), (3.8), and (1.24),

$$\sup_{B_G(R)} |\xi|^2_{2u} \leqq C_2 R^{-2n} \int_{B_G(2R)} |\xi|^2_{2u}$$
$$\leqq C_3 R^{-2n}(1+s(2R))^{2\nu}(1+s(2R))^{2n} \int_M \frac{|\xi|^2_{2u}}{(1+s^2)^\nu (1+s)^{2n}}.$$

Here by (1.12), we obtain

$$\sup_{B_G(R)} |\xi|^2_{2u} \leqq C_4(1+R)^{2n(1/\mu-1)+2\nu/\mu}.$$

Because u is bounded

$$|\xi|_{K_M}(x) \leqq C_5(1+r(x))^{n(1/\mu-1)+\nu/\mu}. \qquad \text{Q.E.D.}$$

The following lemma is proved in Moser's paper (cf. [9]).

Lemma 3.14. *There exists a positive number* δ_0 *such that if* u *is a harmonic function on* M *satisfying*

(3.15) $\quad u(x) \leqq C_1(r(x)+C_2)^\delta \quad$ *for some* $0<\delta<\delta_0$

where C_i *are constants, then*

(3.16) $\quad u = u(o)$ *identically.*

Proposition 3.17. *There exists a positive number* γ_1 *such that if*

$$\int_0^\infty sk(s)ds < \gamma_1,$$

then there exists a holomorphic n-form ξ *on* M *satisfying*

$$0 < C^{-1} \leqq |\xi|_{K_M} \leqq C$$

where C *is a constant.*

Proof. Choose any $\gamma_1 < 1$ so that

$$n\left(\frac{1}{1-\gamma_1}-1\right) < \nu_0$$

where ν_0 is the number of (1.14), and take $\nu>0$ so that

$$n\left(\frac{1}{1-\gamma_1}-1\right)+\frac{\nu}{1-\gamma_1} < \gamma_0.$$

Let ξ be the holomorphic n-form of (3.10). If $\int_0^\infty sk(s)ds<\gamma_1$, then

$$n\left(\frac{1}{\mu}-1\right)+\frac{\nu}{\mu} < \nu_0$$

therefore ξ is nowhere zero on M by (1.14). Hence $\log|\xi|_{K_M}$ is well-defined everywhere M and

$$\Delta(\log|\xi|^2_{K_M}-2u)=0 \qquad \text{on } M.$$

Using the natural coordinates, set $A=\sqrt{|g|}\Delta$ and

$$v=\log|\xi|^2_{K_M}-2u.$$

Then A is uniformlly elliptic and $Av=0$. By (3.10)

$$v(x)\leqq C\{1+\log(1+r(x))\}$$

where C is a constant. From (3.14), v is a constant. Since u is unifoımlly bounded on M, we get the conclusion. Q.E.D.

4. Construction of a biholomorphic map

Let $T^*(M)$ be the holomorphic cotangent bundle of M and w a real C^∞ function on M. We define the fibre metric $\langle\!\langle\ ,\ \rangle\!\rangle_w$ on $T^*(M)$ by

(4.1) $$\langle\!\langle\ ,\ \rangle\!\rangle_w=\langle\ ,\ \rangle e^{-w}$$

where $\langle\ ,\ \rangle$ is the fibre metric on $T^*(M)$ induced by G.

Lemma 4.2. *There exists a bounded real C^∞ function ρ on M such that for any holomorphic 1-form σ on M,*

(4.3) *$\log\|\sigma\|_\rho$ is subharmonic on $M-V$*

where $\|\sigma\|^2_\rho=\langle\!\langle\sigma,\sigma\rangle\!\rangle_\rho$ and $V=\{x\in M\colon \sigma(x)=0\}$.

Proof. From (2.6), there exists a bounded real C^∞ function ρ on M such that

$$\text{the Ricci curvature at } x\geqq\Delta\rho(x).$$

Using Bochner's identity

$$\frac{1}{2}\Delta\|\sigma\|^2_\rho=\|\nabla^\rho\sigma\|^2_\rho+Ric(\sigma^\sharp,\sigma^\sharp)e^{-\rho}+\frac{1}{2}\Delta\rho\|\sigma\|^2_\rho$$

where $\sigma^\sharp$ is the dual of σ and ∇^ρ is the covariant derivative with respect to the metric $\langle\!\langle\ ,\ \rangle\!\rangle_\rho$, we get the conclusion. Q.E.D.

We refer the proof of the following lemma to ([3], p. 43, Th.B).

Lemma 4.4. *Assume that v is a non-negative C^∞ function on M satisfying $\Delta v\geqq 0$. Then*

$$(4.5)\qquad \int_{B_G(\alpha R)} \Delta v \leqq C(n, \lambda, \mu, \alpha) R^{-2} \int_{B_G(R)} v$$

where $0<\alpha<1$ *and* $C(n, \lambda, \mu, \alpha)$ *is a constant depending only on* n, λ, μ, α.

Proposition 4.6. *For any positive number* ν, *there exist holomorphic functions* $f_1, \cdots, f_n$ *on* M *such that*

$$(4.7)\qquad f_i(o)=0 \text{ and } df_i(o)=dz_i,$$

$$(4.8)\qquad |f_i(x)| \leqq C_1(1+r(x))^{\nu+n+1/\mu-n}$$

$$(4.9)\qquad \|df_i(x)\| \leqq C_2(1+r(x))^{\nu/\mu+(n+1)(1/\mu-1)}$$

where C_i *are constants and* $(z_1, \cdots, z_n)$ *are the holomorphic coordinates at* o *of* (3.9).

Proof. Take positive numbers a, b so that $B_G(a) \Subset B_G(b) \Subset U$. Let χ be a C^∞ function on $[0, \infty)$ such that

$$(4.10)\qquad \chi(t) = \begin{cases} 1 & 0 \leqq t \leqq a \\ 0 & t \geqq b \end{cases}$$

and $0 \leqq \chi \leqq 1$.

Set $v_i(x) = \chi(r(x)) z_i$ on U, and $v_i(x)=0$ on $M-U$.

Let ξ be the holomorphic n-form in (3.17), and set

$$(4.11)\qquad \beta = \log|\xi|^2_{K_M} \text{ and } \psi = (2n+2)\log s.$$

We define a fibre metric on the trivial line bundle $E = M \times \boldsymbol{C}$ by

$$(4.12)\qquad h_\nu = e^{\beta - \nu \log(1+s^2)}$$

From (1.11),

$$Ric(M) + c_1(E, h_\nu) \geqq c\Omega,$$

where c is a positive continuous function on M. From (1.18), there exists a C^∞ function u_i on M such that

$$(4.13)\qquad \bar\partial u_i = \bar\partial v_i$$

and

$$(4.14)\qquad \int_M \frac{|u_i|^2 |\xi|^2_{K_M}}{(1+s^2)^\nu s^{2n+2}} \leqq \int_M \frac{\langle \bar\partial v_i, \bar\partial v_i \rangle_{h_\nu}}{c} e^{-\psi} < \infty.$$

Because of the singularity of $e^{-\psi}$ at o,

$$u_i(o) = 0, \quad du_i(o) = 0.$$

Therefore if we set

$$f_i = v_i - u_i ,$$

then

(4.15) $\quad \bar{\partial} f_i = 0,\ f_i(o) = 0, \text{ and } df_i(o) = dz_i .$

Since v_i has a compact support, (4.14) implies

(4.16) $$\int_M \frac{|f_i|^2}{(1+s^2)^\nu (1+s)^{2n+2}} < \infty .$$

Using the natural coordinates, we set $A = \sqrt{|g|}\Delta$. Then from (1.13), A is uniformlly elliptic.
By (3.1) and (1.24),

(4.17) $$\sup_{B_G(2^{-1}r(x))} |f_i|^2 \leqq C_1 r(x)^{-2n} \int_{B_G(r(x))} |f_i|^2$$

$$\leqq C_1 r(x)^{-2n} (1+s(r(x))^2)^\nu (1+s(r(x)))^{2n+2} \int_M \frac{|f_i|^2}{(1+s^2)^\nu (1+s)^{2n+2}} .$$

(by (4.16) and by (1.12))

$$\leqq C_2 (1+r(x))^{2/(\nu+\eta+1/\mu-n)}$$

where C_i are constants. Let ρ be the function of (4.2). Then by (3.1) and (1.24), we obtain

(4.18) $$\sup_{B_G(2^{-1}r(x))} \|df_i\|_\rho^2 \leqq C_3 r(x)^{-2n} \int_{B_G(r(x))} \|df_i\|_\rho^2$$

(since ρ is bounded)

$$\leqq C_4 r(x)^{-2n} \int_{B_G(r(x))} \|df_i\|^2 = C_4 r(x)^{-2n} \int_{B_G(r(x))} \square |f_i|^2$$

(observe $\Delta = 2\square$)

$$= 2^{-1} C_4 r(x)^{-2n} \int_{B_G(r(x))} \Delta |f_i|^2$$

$$\leqq C_5 r(x)^{-(2n+2)} \int_{B_G(2(r(x))} |f_i|^2 \qquad \text{(from (4.4))}$$

$$\leqq C_6 r(x)^{-(2n+2)} \int_0^{2r(x)} (1+t)^{2(\nu+n+1)/\mu-1} dt \qquad \text{(from (4.17))}$$

$$\leqq C_7 (1+r(x))^{2(\nu+n+1/\mu-n-1)}$$

where C_i are constants. Recall that ρ is bounded, then we obtain

$$\|df_i\|(x) \leqq C_8 (1+r(x))^{\nu/\mu+(n+1)(1/\mu-1)} . \qquad \text{Q.E.D.}$$

The proofs of the following lemma and proposition are the same as those of

([7], p. 214, Lemma 3 and p. 215, Proposition).

Lemma 4.19. *There exists a positive number* $\gamma_2 \leqq \gamma_1$ *where* γ_1 *is the constant of* (3.17) *such that if*

$$\int_0^\infty sk(s)ds < \gamma_2,$$

there exists holomorphic functions $f_1, \cdots, f_n$ *on* M *satisfying the following conditions*:

$$df_1 \wedge \cdots \wedge df_n = \xi \tag{4.20}$$

where ξ *is the holomorphic n-form of* (3.9), *and if we define holomorphic vector fields* $\{X_i\}_{1 \leqq i \leqq n}$ *on* M *as*

$$df_i(X_j) = \delta_{ij} \tag{4.21}$$

then

$$|X_i|(x) \leqq C_2 r(x)^{(n-1)\{\nu/\mu+(n+1)(1/\mu-1)\}} \qquad 1 \leqq i \leqq n, \tag{4.22}$$

where C_i *are constants.*

We define a holomorphic map F: $M \to C^n$ by

$$F = (f_1, \cdots, f_n): M \to C^n.$$

Proposition 4.23. *There exists a positive number* $\gamma_0 \leqq \gamma_2$ *where* γ_2 *is the number in* (4.19) *such that if*

$$\int_0^\infty sk(s)ds < \gamma_0,$$

then the holomorphic map F *defined above is a proper map.*

Now we give the proof of our Main Theorem.

Proof of the Main Theorem. If $\int_0^\infty sk(s)ds < \gamma_0$, then from (4.20) and (4.23), F is a covering map. Since C^n is simply connected, F is biholomorphic. Q.E.D.

REMARK. Moreover if the sectional curvature of M does not change the sign, we can conclude that M is flat by using Mok-Siu-Yau's argument in ([10], p. 211).

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