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ON THE JACOBI DIFFERENTIAL OPERATORS ASSOCIATED TO MINIMAL ISOMETRIC IMMERSIONS OF SYMMETRIC SPACES INTO SPHERES III

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Contents

Introduction

This is a continuation of our first and second papers [5]. In this paper we shall study on the spectra of the Jacobi differential operator $\tilde{S}$ for minimally immersed spheres into spheres.

Computing the matrix expressions of the linear mappings S_σ, defined in subsection 5.2 of our first paper [5], we show that every eigenvalue of the Jacobi differential operator $\tilde{S}$ is an algebraic number (Theorem 10.4.4, 11.4.4 and 12.3.3), however not a rational number in general. This suggests us that $\tilde{S}$ will not be described only by Casimir operators. We give a lower bound for the nullity of $\tilde{S}$(Theorem 10.6.2 and 11.6.2). In particular, for the minimally immersed 2-dimensional sphere S^2, the nullity is explicitly computed (Theorem 12.4.1) and we show that the nullity is equal to twice the Killing nullity (Theorem 12.4.3).

We shall denote by [I] (resp. by [II]) our first paper [5] (resp. our second paper [5]) for short. We retain the definitions and notation in [I] and [II].

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10. Minimal immersions of $(2h-1)$-dimensional sphere $S^{2h-1}(h\geqq 2)$

In this section we assume that $G=SO(2h)$ and $K=SO(2h-1)$, $h\geqq 2$. The assumptions and the notation are the same as in section 9 of [II]. And in this paper, we will not distinguish G-modules and representations of G.

10.1. In this subsection we consider the full equivariant minimal isometric immersion $F: (S^{2h-1}, c\langle\ ,\ \rangle) \to S$ induced from the second real spherical representation ρ_2 of (G, K). Then by the formula of Freudenthal (cf. Takeuchi [6] p. 205) and Proposition 3.2.1 of [I], we have

$$c = \frac{4h}{2h-1}. \tag{10.1.1}$$

Therefore it follows from Remark 8.3.1 of [II] that the Jacobi differential operator S on $C^\infty(G; (V^N)^C)_K$ is given by

$$S = -\frac{2h-1}{4h}\left(\sum_{i=1}^{n+p} E_i E_i + 8h\, 1_{C^\infty(G;\,(V^N)^C)_K}\right). \tag{10.1.2}$$

Therefore for each $[\sigma] \in D(G; K, \rho^N)$ the operator S acts on $\mathfrak{v}_{[\sigma]}(N(S^{2h-1})^C)$ as a scalar, which will be denoted by $c(\sigma)$. We have by Proposition 9.2.1 of [II]

$$V^C = V_0 + V_1 + V_2,$$

where V_i is the irreducible K-submodule of V^C with the highest weight $i\phi_{h-1}$. Hence

$$(V^0)^C = V_0,\ (V^T)^C = V_1,\ (V^N)^C = V_2. \tag{10.1.3}$$

Theorem 10.1.1. *Let $F: (S^{2h-1}, c\langle\ ,\ \rangle) \to S$, $F(xK) = \rho_2(x)F(o)$, be the full equivariant minimal isometric immersion induced from $\rho = \rho_2$.*

(1) *We have*

$$D(G; K, \rho^N) = \left\{[\sigma] \in D(G);\ \Lambda_\sigma = s\phi_{h-1} + t\phi_h \text{ with } |s| \leqq 2 \text{ and } t \geqq 2\right\},$$

where Λ_σ is the highest weight of the complex irreducible representation σ of G. The multiplicity of each $[\sigma] \in D(G; K, \rho^N)$ is equal to 1.

(2) *We have for $[\sigma] \in D(G; K, \rho^N)$ with $\Lambda_\sigma = s\phi_{h-1} + t\phi_h$*

$$c(\sigma) = \frac{2h-1}{4h}\{s(s+2h-4) + t(t+2h-2) - 8h\}.$$

(3) *The cases where $c(\sigma) \leqq 0$ are the followings:*

	$c(\sigma)$	Λ_σ
$h = 2$	< 0	$2\phi_2,\ \pm\phi_1 + 2\phi_2,\ \pm 2\phi_1 + 2\phi_2,\ 3\phi_2$
	$= 0$	$\pm\phi_1 + 3\phi_2$
$h > 2$	< 0	$2\phi_h,\ \phi_{h-1} + 2\phi_h,\ 2\phi_{h-1} + 2\phi_h,\ 3\phi_h$
	$= 0$	$\phi_{h-1} + 3\phi_h$

Proof. (1) We have the assertion by Proposition 9.2.1 of [II], the Frobenius reciprocity (cf. Takeuchi [6] p. 16) and (10.1.3).

(2) We have the equality by (10.1.2) and the formula of Freudenthal.

(3) We obtain the table from (2) by easy computations. Q.E.D.

REMARK 10.1.1. It follows from the above theorem and Proposition 3.4.2 of [I] that the nullity of F is equal to its Killing nullity.

REMARK 10.1.2. (a) The case $h=2$: Every eigenspace of S is decomposed into at most two G-irreducible components. If $c(\sigma)=c(\sigma')$ with $\sigma\neq\sigma'$ and $\Lambda_\sigma=s\phi_1+t\phi_2$, then $s\neq 0$ and $\Lambda_{\sigma'}=-s\phi_1+t\phi_2$.

(b) The case $h>2$: Every eigenspace of S is G-irreducible.

10.2. Let σ: $G\to GL(W)$ be an irreducible unitary representation with the highest weight $k\phi_h(k>0)$, and c_σ the eigenvalue of the Casimir operator of σ. We have by Proposition 9.2.1 of [II]

$$W=\sum_{i=0}^{k} W_i\,,$$

where W_i is the irreducible K-submodule of W with the highest weight $i\phi_{h-1}$. We shall compute $c(\sigma)^i{}_j$, $i, j=0,1,\cdots,k$, in subsection 6.3 of [II]. It follows from the degree formula of Weyl (cf. Takeuchi [6] p. 157) that

$$\dim W_i=\frac{(i+2h-4)!(2i+2h-3)}{i!(2h-3)!}\,. \tag{10.2.1}$$

If the K-module $\mathfrak{p}^C\otimes W_i$ contains the irreducible K-module W_p, then we have $i=p-1$, p or $p+1$ by (9.4.1) of [II]. Therefore we have by (2) of Lemma 6.2.3 and (a) of Proposition 6.3.7 of [II]

$$c(\sigma)^i{}_j=0 \qquad \text{for } i, j=0, 1, \cdots, k \text{ with } |i-j|>1\,. \tag{10.2.2}$$

We have

Proposition 10.2.1.

$$\begin{cases} c(\sigma)^{k-i}{}_{k-i}=(k-i)\,(k+2h-i-3)\,, \\ c(\sigma)^{k-i}{}_{k-i-1}=\dfrac{(i+1)\,(k-i)\,(2k+2h-i-3)}{2k+2h-2i-3}\,, \\ c(\sigma)^{k-i-1}{}_{k-i}=\dfrac{(i+1)\,(k+2h-i-4)\,(2k+2h-i-3)}{2k+2h-2i-5} \end{cases} \tag{10.2.3}$$

$$\textit{for } i=0, 1, \cdots, k-1\,.$$

Proof. We shall prove the proposition by the induction on i. We have by (3) of Lemma 6.3.4 of [II]

(10.2.4) $$c(\sigma;\mathfrak{k})^i{}_i = i(i+2h-3) \qquad i=0,1,\cdots,k\,.$$

(a) The case $h=2$: Note that the space W_i is the irreducible K-submodule of W with the highest weight $i\phi_1$. Put $H_{\phi_1}=[X_{\phi_1}, X_{-\phi_1}]$ (see subsection 9.1 of [II]). Then $H_{\phi_1}=\sqrt{-1}\,\phi_1$, and $\mathfrak{g}_{X\phi_1}{}^C=\{X_{\phi_1}, Y_{\phi_1}, H_{\phi_1}\}_C$ is a Lie subalgebra of $\mathfrak{g}^C$ by Lemma 7.2.2 of [II]. Considering W as a $\mathfrak{g}_{X\phi_1}{}^C$-module, let $W=\sum_{i=0}^{k} V^{2i}$ be the decomposition of (7.3.4) of [II]. Let $w_i\in W_i$ be an $i\phi_1$-weight vector with $|w_i|=1$ and let $w_i=\sum_{q=i}^{k}\sqrt{a_{i\,;\,q}}\,w_{i\,;\,q}$ with $w_{i\,;\,q}\in V^{2q}$, $|w_{i\,;\,q}|=1$ and $a_{i\,;\,q}\geqq 0$. Then we have $\sum_{q=i}^{k} a_{i\,;\,q}=1, i=0,1,\cdots,k$. Since the vector w_k is contained in V^{2k} and $|\phi_1|=1$, we have by (7.3.6) of [II]

$$|d\sigma(X_{-\phi_1})w_k|^2 = k\,.$$

It follows from Lemma 6.2.2 and (9.4.1) of [II] (applied to K) that the vector $d\sigma(X_{-\phi_1})w_k$ is contained in the subspace W_k+W_{k-1} of W. Let f_k(resp. f_{k-1}) be a K-homomorphism of $\mathfrak{p}^C\otimes W_k$ to W_k(resp. to W_{k-1}) with the property of f_0 in subsection 6.4 of [II]. It follows from (1) of Proposition 6.4.2 of [II] that there exist complex numbers $d^k{}_k$ and $d^{k-1}{}_k$ such that

$$d\sigma(X_{-\phi_1})w_k = d^k{}_k f_k(X_{-\phi_1}\otimes w_k)+d^{k-1}{}_k f_{k-1}(X_{-\phi_1}\otimes w_k)\,.$$

Then we have by Lemma 9.4.5 of [II] (applied to K)

(10.2.5) $$|f_k(X_{-\phi_1}\otimes w_k)|^2=\frac{1}{k+1},\ |f_{k-1}(X_{-\phi}\otimes w_k)|^2=\frac{2k-1}{2k+1}\,.$$

It follows from (6.4.1) and (2) of Lemma 6.3.4 of [II] that

(10.2.6) $$|d^k{}_k|^2 = c(\sigma;\mathfrak{p})^k{}_k,\quad |d^{k-1}{}_k|^2 = c(\sigma)^{k-1}{}_k\,.$$

Therefore we have the following equalities by the above arguments, Lemma 6.3.2, (6.3.10) of [II] and (10.2.2):

(10.2.7) $$\begin{cases} \dfrac{1}{k+1}c(\sigma;\mathfrak{p})^k{}_k+\dfrac{2k-1}{2k+1}c(\sigma)^{k-1}{}_k = k\,, \\ \dim W_{k-1}\,c(\sigma)^{k-1}{}_k = \dim W_k\,c(\sigma)^k{}_{k-1}\,, \\ c(\sigma;\mathfrak{k})^k{}_k+c(\sigma;\mathfrak{p})^k{}_k+c(\sigma)^k{}_{k-1}=-c_\sigma = k(k+2)\,. \end{cases}$$

We have by (10.2.1) and (10.2.4)

$$c(\sigma;\mathfrak{p})^k{}_k = 0\,,\ \ c(\sigma)^k{}_{k-1} = k,\ \ c(\sigma)^{k-1}{}_k = \frac{k(2k+1)}{2k-1}\,.$$

Therefore the formulas (10.2.3) are valid for $i=0$. Suppose that the equalities

(10.2.3) hold for i-1 with $i<k$. The vector $d\sigma(X_{\phi_1})w_{k-i}$ belongs to the K-weight $(k-i+1)\phi_1$, and hence it follows from (9.4.1) of [II] that it is contained in W_{k-i+1}. Then there exists a complex number $d'^{k-i+1}{}_{k-i}$ such that

$$d\sigma(X_{\phi_1})w_{k-i} = d'^{k-i+1}{}_{k-i}w_{k-i+1}\,. \tag{10.2.8}$$

By (2) of Lemma 6.3.4 and (6.4.1) of [II], we have

$$|d'^{k-i+1}{}_{k-i}|^2 = c(\sigma)^{k-i+1}{}_{k-i}\,.$$

Comparing the V^{2q}-components of the both sides of (10.2.8), we have by (7.3.6) of [II]

$$\begin{cases} \frac{1}{2}\,i(2k-i+1)a_{k-i\,;\,k} = c(\sigma)^{k-i+1}{}_{k-i}\,a_{k-i+1\,;\,k}\,, \\ \frac{1}{2}\,(i-1)(2k-i)a_{k-i\,;\,k-1} = c(\sigma)^{k-i+1}{}_{k-i}\,a_{k-i+1\,;\,k-1}\,, \\ \cdots\cdots \\ (k-i+1)a_{k-i\,;\,k-i+1} = c(\sigma)^{k-i+1}{}_{k-i}\,a_{k-i+1\,;\,k-i+1}\,. \end{cases} \tag{10.2.9}$$

We have by (7.3.6) of [II] and (10.2.9)

$$\begin{aligned} |d\sigma(X_{-\phi_1})w_{k-i}|^2 &= \sum_{j=0}^{i}\frac{1}{2}(2k-i-j)\,(i-j+1)a_{k-i\,;\,k-j} \\ &= \sum_{j=0}^{i-1}\frac{1}{2}\,(2k-i-j)\,(i-j+1)a_{k-i\,;\,k-j}+(k-i)\,(1-\sum_{j=0}^{i-1}a_{k-i\,;\,k-j}) \\ &= k-i+\sum_{j=0}^{i-1}\frac{1}{2}\,(i-j)\,(2k-i-j+1)a_{k-i\,;\,k-j} \\ &= k-i+\sum_{j=0}^{i-1}c(\sigma)^{k-i+1}{}_{k-i}a_{k-i+1\,;\,k-j} \\ &= k-i+c(\sigma)^{k-i+1}{}_{k-i}\,. \end{aligned}$$

It follows from (9.4.1) of [II] that the vector $d\sigma(X_{-\phi_1})w_{k-i}$ is contained in $W_{k-i+1}+W_{k-i}+W_{k-i-1}$. Let v_{k-i+1}(resp. v_{k-i} and v_{k-i-1}) be the W_{k-i+1}-component (resp. the W_{k-i}-component and the W_{k-i-1}-component) of $d\sigma(X_{-\phi_1})w_{k-i}$. Then we have the followings by (2) of Lemma 6.3.4, (6.4.1) and Lemma 9.4.6 of [II]:

$$\begin{cases} |v_{k-i+1}|^2 = \dfrac{1}{(k-i+1)\,(2k-2i+1)}\,c(\sigma)^{k-i+1}{}_{k-i}\,, \\ |v_{k-i}|^2 = \dfrac{1}{k-i+1}\,c(\sigma\,;\,\mathfrak{p})^{k-i}{}_{k-i}\,, \\ |v_{k-i-1}|^2 = \dfrac{2k-2i-1}{2k-2i+1}\,c(\sigma)^{k-i-1}{}_{k-i}\,. \end{cases}$$

Therefore we have the following equalities by the above arguments, Lemma

6.3.2, (6.3.10) of [II] and (10.2.2)

$$(10.2.10)\quad \begin{cases} \dfrac{1}{(k-i+1)(2k-2i+1)}c(\sigma)^{k-i+1}{}_{k-i}+\dfrac{1}{k-i+1}c(\sigma;\mathfrak{p})^{k-i}{}_{k-i} \\ \qquad\qquad +\dfrac{2k-2i-1}{2k-2i+1}c(\sigma)^{k-i-1}{}_{k-i}=k-i+c(\sigma)^{k-i+1}{}_{k-i}\,, \\ \dim W_{k-i-1}\,c(\sigma)^{k-i-1}{}_{k-i}=\dim W_{k-i}\,c(\sigma)^{k-i}{}_{k-i-1}\,, \\ c(\sigma)^{k-i}{}_{k-i+1}+c(\sigma;\mathfrak{k})^{k-i}{}_{i-k}+c(\sigma;\mathfrak{p})^{k-i}{}_{k-i}+c(\sigma)^{k-i}{}_{k-i-1} \\ \qquad\qquad =k(k+2)\,. \end{cases}$$

Applying the assumptions of the induction, (10.2.1) and (10.2.4), we obtain the equalities (10.2.3) for i.

(b) The case $h>2$: It follows from (9.4.1) of [II] that the K-module $\mathfrak{p}^C\otimes W_i$ does not contain the irreducible K-module W_i. Therefore by (3) of Lemma 6.2.3, Proposition 6.3.7 of [II] and (10.2.4), we have

$$(10.2.11)\qquad c(\sigma)^i{}_i=c(\sigma;\mathfrak{k})^i{}_i=i(i+2h-3)\,.$$

We have the following equalities by Lemma 6.3.2, (6.3.10) of [II] and (10.2.2):

$$(10.2.12)\qquad \begin{cases} \dim W_{k-1}\,c(\sigma)^{k-1}{}_k=\dim W_k\,c(\sigma)^k{}_{k-1}\,, \\ c(\sigma)^k{}_k+c(\sigma)^k{}_{k-1}=-c_\sigma=k(k+2h-2)\,. \end{cases}$$

We have by (10.2.1) and (10.2.11)

$$c(\sigma)^k{}_{k-1}=k,\quad c(\sigma)^{k-1}{}_k=\frac{(k+2h-4)(2k+2h-3)}{2k+2h-5}\,.$$

Therefore the formulas (10.2.3) are valid for $i=0$. Suppose that the equalities (10.2.3) hold for $i-1$ with $i<k$. We have the following equalities by Lemma 6.3.2, (6.3.10) of [II] and (10.2.2):

$$(10.2.13)\qquad \begin{cases} \dim W_{k-i-1}\,c(\sigma)^{k-i-1}{}_{k-i}=\dim W_{k-i}\,c(\sigma)^{k-i}{}_{k-i-1}\,, \\ c(\sigma)^{k-i}{}_{k-i+1}+c(\sigma)^{k-i}{}_{k-i}+c(\sigma)^{k-i}{}_{k-i-1}=k(k+2h-2)\,. \end{cases}$$

We have the equalities (10.2.3) by the assumptions of the induction, (10.2.1) and (10.2.11). Q.E.D.

10.3. In this subsection let $\sigma\colon G\to GL(W)$ be an irreducible unitary representation with the highest weight $s\phi_{h-1}+t\phi_h$, $s\neq 0$, and c_σ the eigenvalue of the Casimir operator of σ.

We shall first consider the case $h=2$. Then we have by Proposition 9.2.1 of [II]

$$W=\sum_{|s|\leq i\leq t}W_i\,,$$

where W_i is the irreducible K-submodule of W with the highest weight $i\phi_1$. We shall compute $c(\sigma)^i{}_j$, $i, j=|s|, |s|+1, \cdots, t$. We have in the same way as for (10.2.2) and (10.2.4)

$$(10.3.1)\quad \begin{cases} c(\sigma)^i{}_j = 0 & \text{for } i, j = |s|, |s|+1, \cdots, t \text{ with } |i-j|>1, \\ c(\sigma; \mathfrak{k})^i{}_i = i(i+1) & \text{for } i = |s|, |s|+1, \cdots, t. \end{cases}$$

We have

Proposition 10.3.1. (a) *If $|s|=t$, we have*

$$c(\sigma)^t{}_t = 2t(t+1).$$

(b) *If $|s|<t$, we have for $i=0, 1, \cdots, t-|s|-1$*

$$(10.3.2)\quad \begin{cases} c(\sigma; \mathfrak{p})^{t-i}{}_{t-i} = \dfrac{s^2(t+1)^2}{(t-i)(t-i+1)}, \\ c(\sigma)^{t-i}{}_{t-i-1} = \dfrac{(i+1)(2t-i+1)(t-s-i)(s+t-i)}{(t-i)(2t-2i+1)}, \\ c(\sigma)^{t-i-1}{}_{t-i} = \dfrac{(i+1)(2t-i+1)(t-s-i)(s+t-i)}{(t-i)(2t-2i-1)}. \end{cases}$$

Proof. (a) Since $W=W_t$, we have by (6.3.10) of [II]

$$c(\sigma)^t{}_t = -c_\sigma = 2t(t+1).$$

(b) We shall prove the above equalities (10.3.2) in the similar way to the proof (a) of Proposition 10.2.1. Let $w_i \in W_i$ be an $i\phi_1$-weight vector with $|w_i|=1$, $i=|s|, |s|+1, \cdots, t$. Considering W as a $\mathfrak{g}_{X_{\phi_1}}{}^C$-module, we obtain the following equalities in the similar way to (10.2.7):

$$\begin{cases} |d\sigma(X_{-\phi_1})w_t|^2 = t = \dfrac{1}{t+1}\, c(\sigma; \mathfrak{p})^t{}_t + \dfrac{2t-1}{2t+1}\, c(\sigma)^{t-1}{}_t, \\ \dim W_{t-1}\, c(\sigma)^{t-1}{}_t = \dim W_t\, c(\sigma)^t{}_{t-1}, \\ c(\sigma; \mathfrak{k})^t{}_t + c(\sigma; \mathfrak{p})^t{}_t + c(\sigma)^t{}_{t-1} = s^2 + t(t+2). \end{cases}$$

We have by (10.2.1) and (10.3.1)

$$\begin{cases} c(\sigma; \mathfrak{p})^t{}_t = \dfrac{s^2(t+1)}{t}, \\ c(\sigma)^t{}_{t-1} = \dfrac{(t-s)(s+t)}{t}, \\ c(\sigma)^{t-1}{}_t = \dfrac{(2t+1)(t-s)(s+t)}{t(2t-1)}. \end{cases}$$

Therefore the equalities (10.3.2) are valid for $i=0$. Suppose that the equalities

(10.3.2) hold for $i-1$ with $i<t-|s|-1$. We obtain the following equalities in the similar way to (10.2.10):

$$\begin{cases} |d\sigma(X_{-\phi_1})w_{t-i}|^2 = t-i+c(\sigma)^{t-i+1}{}_{t-i} \\ \quad = \dfrac{1}{(t-i+1)(2t-2i+1)}c(\sigma)^{t-i+1}{}_{t-i}+\dfrac{1}{t-i+1}c(\sigma;\mathfrak{p})^{t-i}{}_{t-i} \\ \qquad\qquad + \dfrac{2t-2i-1}{2t-2i+1}c(\sigma)^{t-i-1}{}_{t-i}, \\ \dim W_{t-i-1}\, c(\sigma)^{t-i-1}{}_{t-i} = \dim W_{t-i}\, c(\sigma)^{t-i}{}_{t-i-1}, \\ c(\sigma)^{t-i}{}_{t-i+1}+c(\sigma;\mathfrak{k})^{t-i}{}_{t-i}+c(\sigma;\mathfrak{p})^{t-i}{}_{t-i}+c(\sigma)^{t-i}{}_{t-i-1} \\ \qquad = s^2+t(t+2). \end{cases}$$

Applying the assumptions of the induction, (10.2.1) and (10.3.1), we have the equalities (10.3.2). Q.E.D.

Next we shall consider the case $h>2$. We have by Proposition 9.2.1 of [II]

$$W=\sum_{0\leqq p\leqq s\leqq q\leqq t} W_{p,q},$$

where $W_{p,q}$ is the irreducible K-submodule of W with the highest weight $p\phi_{h-2}+q\phi_{h-1}$. We shall compute $c(\sigma)^{0,i}{}_{0,j}, i,j=s,s+1,\cdots,t$. If the K-module $\mathfrak{p}^C\otimes W_{p,q}$ contains the irreducible K-module $W_{0,i}$, then we have by Lemma 9.2.4 of [II]

$$p=0 \text{ or } 1,$$

and

$$\begin{cases} q=i-1 \text{ or } i+1 & \text{if } p=0, \\ q=i & \text{if } p=1. \end{cases}$$

Therefore by (2) of Lemma 6.2.3 and (a) of Proposition 6.3.7 of [II], we have

$$(10.3.3)\qquad \begin{cases} c(\sigma)^{0,i}{}_{0,j}=0 & \text{for } i,j=s,s+1,\cdots,t \text{ with } |i-j|>1, \\ c(\sigma)^{0,i}{}_{1,j}=0 & \text{for } i,j=s,s+1,\cdots,t \text{ with } i\neq j, \\ c(\sigma)^{0,i}{}_{p\,j}=0 & \text{for } i,j=s,s+1,\cdots,t \text{ and } p>1. \end{cases}$$

We have

Proposition 10.3.2. (a) *If $s=t$, we have*

$$\begin{cases} c(\sigma)^{0,t}{}_{0,t}=t(t+2h-3), \\ c(\sigma)^{0,t}{}_{1,t}=t(t+2h-3), \\ c(\sigma)^{1,t}{}_{0,t}=\dfrac{(t+1)(t+2h-4)}{2h-3}. \end{cases}$$

(b) *If $s<t$, we have for $i=0,1,\cdots,t-s-1$*

$$(10.3.4)\quad \begin{cases} c(\sigma)^{0,t-i}{}_{0,t-i} = (t-i)(t+2h-i-3)\,, \\ c(\sigma)^{0,t-i}{}_{0,t-i-1} = \dfrac{(i+1)(2t+2h-i-3)(t-s-i)(s+t+2h-i-4)}{(t+2h-i-4)(2t+2h-2i-3)}\,, \\ c(\sigma)^{0,t-i-1}{}_{0,t-i} = \dfrac{(i+1)(2t+2h-i-3)(t-s-i)(s+t+2h-i-4)}{(t-i)(2t+2h-2i-5)}\,, \\ c(\sigma)^{0,t-i}{}_{1,t-i} = \dfrac{s(s+2h-4)(t+1)(t+2h-3)}{(t-i+1)(t+2h-i-4)}\,, \\ c(\sigma)^{1,t-i}{}_{0,t-i} = \dfrac{s(s+2h-4)(t+1)(t+2h-3)}{(2h-3)(t-i)(t+2h-i-3)}\,. \end{cases}$$

Proof. We have by the degree formula of Weyl

$$(10.3.5)\qquad \dim W_{1,i} = \frac{(i+2h-5)!\,i(i+2h-3)(2i+2h-3)}{(i+1)!(2h-4)!}\,.$$

We have the following in the similar way to (10.2.11):

$$(10.3.6)\qquad c(\sigma)^{0,i}{}_{0,i} = i(i+2h-3) \qquad i = s, s+1, \cdots, t\,.$$

(a) We have the following equalities by (6.3.10), Lemma 6.3.2 of [II] and (10.3.3):

$$\begin{cases} c(\sigma)^{0,t}{}_{1,t} + c(\sigma)^{0,t}{}_{0,t} = -c_\sigma = 2t(t+2h-3)\,, \\ \dim W_{0,t}\, c(\sigma)^{0,t}{}_{1,t} = \dim W_{1,t}\, c(\sigma)^{1,t}{}_{0,t}\,. \end{cases}$$

Therefore we have by (10.3.6), (10.2.1) and (10.3.5)

$$\begin{cases} c(\sigma)^{0,t}{}_{1,t} = t(t+2h-3)\,, \\ c(\sigma)^{1,t}{}_{0,t} = \dfrac{(t+1)(t+2h-4)}{2h-3}\,. \end{cases}$$

(b) We shall prove the equalities (10.3.4) in the similar way to the proof (a) of Proposition 10.2.1. Put $H_{\phi_{h-1}} = [X_{\phi_{h-1}}, X_{-\phi_{h-1}}]$ and $\mathfrak{g}_{X_{\phi_{h-1}}}{}^C = \{X_{\phi_{h-1}}, X_{-\phi_{h-1}}, H_{\phi_{h-1}}\}_C$. Let $w_i \in W_{0,i}$ be an $i\phi_{h-1}$-weight vector with $|w_i| = 1$, $i = s, s+1, \cdots, t$. Considering W as a $\mathfrak{g}_{X_{\phi_{h-1}}}{}^C$-module, we obtain the following equalities in the similar way to (10.2.7):

$$\begin{cases} |d\sigma(X_{-\phi_{h-1}})w_t|^2 = t \\ \quad = \dfrac{t(2h-3)}{(t+1)(t+2h-4)}\, c(\sigma)^{1,t}{}_{0,t} + \dfrac{t(2t+2h-5)}{(t+2h-4)(2t+2h-3)}\, c(\sigma)^{0,t-1}{}_{0,t}\,, \\ \dim W_{1,t}\, c(\sigma)^{1,t}{}_{0,t} = \dim W_{0,t}\, c(\sigma)^{0,t}{}_{1,t}\,, \\ \dim W_{0,t-1}\, c(\sigma)^{0,t-1}{}_{0,t} = \dim W_{0,t}\, c(\sigma)^{0,t}{}_{0,t-1}\,, \\ c(\sigma)^{0,t}{}_{1,t} + c(\sigma)^{0,t}{}_{0,t} + c(\sigma)^{0,t}{}_{0,t-1} = -c_\sigma \\ \qquad\qquad = s(s+2h-4) + t(t+2h-2)\,. \end{cases}$$

We have by (10.2.1), (10.3.5) and (10.3.6)

$$\begin{cases} c(\sigma)^{0,t}{}_{0,t-1} = \dfrac{(t-s)\,(s+t+2h-4)}{t+2h-4}\,, \\ c(\sigma)^{0,t-1}{}_{0,t} = \dfrac{(2t+2h-3)\,(t-s)\,(s+t+2h-4)}{t(2t+2h-5)}\,, \\ c(\sigma)^{0,t}{}_{1,t} = \dfrac{s(s+2h-4)\,(t+2h-3)}{t+2h-4}\,, \\ c(\sigma)^{1,t}{}_{0,t} = \dfrac{s(s+2h-4)(t+1)}{t(2h-3)}\,. \end{cases}$$

Therefore the equalities (10.3.4) are valid for $i=0$. Suppose that the equalities (10.3.4) hold for $i-1$ with $i<t-s-1$. We obtain the following equalities in the similar way to (10.2.10):

$$\begin{cases} |d\sigma(X_{-\phi_{h-1}})w_{t-i}|^2 = t-i+c(\sigma)^{0,t-i+1}{}_{0,t-i} \\ \quad = \dfrac{2h-3}{(t-i+1)\,(2t+2h-2i-3)}\,c(\sigma)^{0,t-i+1}{}_{0,t-i} \\ \quad + \dfrac{(t-i)\,(2h-3)}{(t-i+1)\,(t+2h-i-4)}\,c(\sigma)^{1,t-i}{}_{0,t-i} \\ \quad + \dfrac{(t-i)\,(2t+2h-2i-5)}{(t+2h-i-4)\,(2t+2h-2i-3)}\,c(\sigma)^{0,t-i-1}{}_{0,t-i}\,, \\ \dim W_{1,t-i}\,c(\sigma)^{1,t-i}{}_{0,t-i} = \dim W_{0,t-i}\,c(\sigma)^{0,t-i}{}_{1,t-i}\,, \\ \dim W_{0,t-i-1}\,c(\sigma)^{0,t-i-1}{}_{0,t-i} = \dim W_{0,t-i}\,c(\sigma)^{0,t-i}{}_{0,t-i-1}\,, \\ c(\sigma)^{0,t-i}{}_{0,t-i+1}+c(\sigma)^{0,t-i}{}_{1,t-i}+c(\sigma)^{0,t-i}{}_{0,t-i}+c(\sigma)^{0,t-i}{}_{0,t-i-1} \\ \qquad = s(s+2h-4)+t(t+2h-2)\,. \end{cases}$$

Applying the assumptions of the induction, (10.2.1), (10.3.5) and (10.3.6), we obtain the equalities (10.3.4). Q.E.D.

10.4. In the rest of this section we consider the full equivariant minimal isometric immersion $F\colon (S^{2h-1}, c\langle\ ,\ \rangle)\to S$ induced from the *k-th* real spherical representation $\rho=\rho_k$ of (G,K), $k=2,3,\cdots$. Then by the formula of Freudenthal and Proposition 3.2.1 of [I], we have

(10.4.1) $$c = \frac{k(k+2h-2)}{2h-1}\,.$$

We have by Proposition 9.2.1 of [II]

$$V^C = V_0+V_1+\cdots+V_k\,,$$

where V_i is the irreducible K-submodule of V^C with the highest weight $i\phi_{h-1}$.

Hence

(10.4.2) $$(V^0)^C = V_0,\ (V^T)^C = V_1,\ (V^N)^C = \sum_{i=2}^{k} V_i\,.$$

It follows from Corollary for Proposition 9.2.1 and the argument in subsection 6.5 of [II] that there exist complex numbers c_i, $i=0,1,\cdots,k$, such that

(10.4.3) $$\sum_{i=1}^{n+p} \{d\rho(E_i)\,(d\rho(E_i)*)^N\}^N|_{V_i} = c_i 1_{V_i}\,.$$

Then we have

Lemma 10.4.1.

$$c_i = \begin{cases} 0 & \textit{if } i=0,1\,, \\ -\{k(k+2h-2)-\dfrac{2(k-1)(k+2h-1)}{2h+1}\} & \textit{if } i=2\,, \\ -k(k+2h-2) & \textit{if } i>2\,. \end{cases}$$

Proof. We obtain the above equalities by Proposition 6.5.1, Proposition 6.3.8 of [II], Proposition 10.2.1 and (10.2.2). Q.E.D.

It follows from Proposition 9.2.1 of [II] and the Frobenius reciprocity that

$$D(G;K,\rho^N) = \{[\sigma]\in D(G);\ \Lambda_\sigma = s\phi_{h-1}+t\phi_h,\ |s|\leqq k,\ 2\leqq t\}$$

and that the multiplicity of the above $[\sigma]\in D(G;K,\rho^N)$ is equal to $\mathrm{Min}\{k-1, k-|s|+1, t-1, t-|s|+1\}$. We have

Lemma 10.4.2. *Let $\sigma\colon G\to GL(W)$ be a complex irreducible representation with $[\sigma]\in D(G;K,\rho^N)$ and $\Lambda_\sigma=t\phi_h$. Then there exists a basis $\{\omega'_0,\omega'_1,\cdots,\omega'_d\}$ of $(W^*\otimes V^C)_0$ such that $\{\omega'_2,\omega'_3,\cdots,\omega'_d\}$ is a basis of $(W^*\otimes(V^N)^C)_0$ and that*

$$L(\sigma^*,\rho)\omega'_i = \frac{(2h+i-4)(k+2h+i-3)(t+2h+i-3)}{2h+2i-5}\,\omega'_{i-1}$$
$$+i(2h+i-3)\omega'_i+\frac{(i+1)(k-i)(t-i)}{2h+2i-1}\,\omega'_{i+1}$$
$$\textit{for } i=0,1,\cdots,d\,,$$

where $d=Min\{k,t\}$ and $\omega'_{-1}=\omega'_{d+1}=0$.

Proof. We may choose orthonormal bases $\{v_{i;1}, v_{i;2},\cdots, v_{i;n(i)}\}$ of V_i, $i=0,1,\cdots,k$, with the following properties: Each of the vectors $v_{i;\alpha}$, $\alpha=1,2,\cdots,n(i)$, is a weight vector of the K-module V_i, the vector $v_{i;1}$ is an $i\phi_{h-1}$-weight vector, and

$$d\rho(X_{\phi_{h-1}})v_{i;1} = \sqrt{c(\rho)^{i+1}{}_i}\, v_{i+1;1} \quad i=0,1,\cdots,k-1\,.$$

In fact take an arbitrary unit vector in V_0 as $v_{0;1}$. Then by (9.4.1), (2) of Lemma 6.3.4 and (6.4.1) of [II], $d\rho(X_{\phi_{h-1}})v_{0;1}$ is a ϕ_{h-1}-weight vector in V_1 and $|d\rho(X_{\phi_{h-1}})v_{0;1}|^2=c(\rho)^1{}_0$. Then $c(\rho)^1{}_0\neq 0$ by Proposition 10.2.1. Put

$$v_{1;1}=\sqrt{\frac{1}{c(\rho)^1{}_0}}\,d\rho(X_{\phi_{h-1}})v_{0;1}\,,$$

and choose an orthonormal basis $\{v_{1;1},v_{1;2},\cdots,v_{1;n(1)}\}$ of V_1 in such a way that each $v_{1;\alpha}$ is a weight vector of V_1. Now we may choose inductively orthonormal bases $\{v_{i;1},v_{i;2},\cdots,v_{i;n(i)}\}$ of V_i with the above property. We have by Proposition 9.2.1 of [II]

$$W=\sum_{j=0}^{t}W_j\,,$$

where W_j is the irreducible K-submodule of W with the highest weight $j\phi_{h-1}$. Then we may choose orthonormal bases $\{w_{j;1},w_{j;2},\cdots,w_{j;n(j)}\}$ of W_j, $j=0,1,\cdots,t$, and unitary K-isomorphisms a_i: $V_i\to W_i$, $i=0,1,\cdots,d$, such that

$$\begin{cases} d\sigma(X_{\phi_{h-1}})w_{j;1}=\sqrt{c(\sigma)^{j+1}{}_j}\,w_{j+1;1} & \text{for } j=0,1,\cdots,t-1\,,\\ a_i(v_{i;\alpha})=w_{i;\alpha} \quad \text{for } i=0,1,\cdots,d \text{ and } \alpha=1,2,\cdots,n(i)\,. \end{cases}$$

Put

$$\omega_i=\sum_{\alpha=1}^{n(i)}w_{i;\alpha}{}^*\otimes v_{i;\alpha}\qquad i=0,1,\cdots,d\,,$$

$$E_0=\begin{pmatrix}0 & -1 & 0\\ 1 & & \\ 0 & & 0\end{pmatrix}\in\mathfrak{p}\,.$$

Take an orthonormal basis $\{E_1,E_2,\cdots,E_p\}$ of $\mathfrak{k}$. Then the basis $\{E_0,E_1,\cdots,E_p,X_{\phi_j},X_{-\phi_j};j=1,2,\cdots,h-1\}$ of $\mathfrak{g}^C$ satisfies the assumption of Proposition 6.3.9 of [II]. Considering the weights to which the vectors $d\rho(E_0)v_{i;\alpha}$, $d\rho(X_\lambda)v_{i;\alpha}$ belong, $i=0,1,\cdots,d-1$, $\alpha=1,2,\cdots,n(i)$, $\lambda=\pm\phi_1,\pm\phi_2,\cdots,\pm\phi_{h-1}$, we have

$$\begin{cases} C_{j,i,\alpha}^{i+1,1}=0 & \text{for } j=0,1,\cdots,p\,,\\ C_{\lambda,i,\alpha}^{i+1,1}=0 & \text{unless } \lambda=\phi_{h-1} \text{ and } \alpha=1\,,\\ C_{\phi_{h-1},i,1}^{i+1,1}=\sqrt{c(\rho)^{i+1}{}_i},\ D_{\phi_{h-1},i,1}^{i+1,1}=\sqrt{c(\sigma)^{i+1}{}_i}\,, \end{cases}$$

where $C_{i,h\alpha}^{j\beta}$, $C_{\lambda,h\alpha}^{j\beta}$, $D_{i,h\alpha}^{j\beta}$, $D_{\lambda,h\alpha}^{j\beta}$ are those in subsection 6.3 of [II], but for the representations ρ and σ. Therefore by (a) of Proposition 6.3.9 of [II] and Proposition 10.2.1, we have

$$\begin{aligned} c(\sigma^*,\rho)^{i+1}{}_i&=\sqrt{c(\rho)^{i+1}{}_i c(\sigma)^{i+1}{}_i}\\ &=\frac{i+1}{2h+2i-1}\sqrt{(k-i)(k+2h+i-2)(t-i)(t+2h+i-2)}\,. \end{aligned}$$

We have by Lemma 6.3.2 of [II] and (10.2.1)

$$c(\sigma^*, \rho)^i{}_{i+1} = \frac{2h+i-3}{2h+2i-3}\sqrt{(k-i)(k+2h+i-2)(t-i)(t+2h+i-2)}\,.$$

We have by (9.4.1) of [II] and the proof of Proposition 10.2.1

$$\begin{cases} C_{0,i\alpha}{}^{i\beta} = 0 & \alpha, \beta = 1, 2, \cdots, n(i)\,, \\ C_{\lambda,i\alpha}{}^{i\beta} = 0 & \lambda = \pm\phi_1, \pm\phi_2, \cdots, \pm\phi_{h-1}, \alpha, \beta = 1, 2, \cdots, n(i)\,. \end{cases}$$

Therefore we have by (b) of Proposition 6.3.9 of [II]

$$c(\sigma^*, \rho)^i{}_i = c(\sigma^*, \rho; \mathfrak{k})^i{}_i\,.$$

Since a_i: $V_i \to W_i$ is a unitary K-isomorphism, we have

$$C_{j,i\alpha}{}^{i\beta} = D_{j,i\alpha}{}^{i\beta} \qquad j = 1, 2, \cdots, p, \;\; \alpha, \beta = 1, 2, \cdots, n(i)\,.$$

Therefore by (b) of Proposition 6.3.9 and (3) of Lemma 6.3.4 of [II], we have

$$c(\sigma^*, \rho)^i{}_i = c(\rho; \mathfrak{k})^i{}_i = i(2h+i-3)\,.$$

It follows from (9.4.1), (2) of Lemma 6.2.3 and (a) of Proposition 6.3.7 of [II] that

$$c(\sigma^*, \rho)^i{}_j = 0 \qquad \text{for } i, j = 0, 1, \cdots, d, \text{ with } |i-j| > 1\,.$$

Put

$$\omega'_i = \sqrt{(k-i)!(k+2h+i-3)!(t-i)!(t+2h+i-3)!}\,\omega_i\,.$$

Then the basis $\{\omega'_0, \omega'_1, \cdots, \omega'_d\}$ of $(W^* \otimes V^C)_0$ has the required property.
Q.E.D.

Lemma 10.4.3. *Let σ: $G \to GL(W)$ be a complex irreducible representation with $[\sigma] \in D(G; K, \rho^N)$ and $\Lambda_\sigma = s\phi_{h-1} + t\phi_h$, $s \neq 0$. Then there exists a basis $\{\omega'_{|s|}, \omega'_{|s|+1}, \cdots, \omega'_d\}$ of $(W^* \otimes V^C)_0$ such that $\{\omega'_m, \omega'_{m+1}, \cdots, \omega'_d\}$ is a basis of $(W^* \otimes (V^N)^C)_0$ and that*

$$\begin{aligned} L(\sigma^*, \rho)\omega'_i = {} & \frac{(k+2h+i-3)(t+2h+i-3)(i-s)(s+2h+i-4)}{i(2h+2i-t)}\,\omega'_{i-1} \\ & + i(2h+i-3)\omega'_i + \frac{(i+1)(k-i)(t-i)}{2h+2i-1}\,\omega'_{i+1} \\ & \textit{for } i = |s|, |s|+1, \cdots, d\,, \end{aligned}$$

where $d = \mathrm{Min}\{k, t\}$, $m = \mathrm{Max}\{2, |s|\}$ and $\omega'_{|s|-1} = \omega'_{d+1} = 0$.

Proof. Choose the orthonormal bases $\{v_{i;1}, v_{i;2}, \cdots, v_{i;n(i)}\}$ of V_i in the proof of Lemma 10.4.2. We have by Proposition 9.2.1 of [II]

$$W=\begin{cases}\sum_{0\leq p\leq s\leq q\leq t} W_{p,q} & \text{if } h>2\,,\\ \sum_{|s|\leq p\leq t} W_p & \text{if } h=2\,,\end{cases}$$

where $W_{p,q}$(resp. W_p) is the irreducible K-submodule of W with the highest weight $p\phi_{h-2}+q\phi_{h-1}$(resp. with the highest weight $p\phi_1$). We may choose orthonormal bases $\{w_{j\,;\,1}, w_{j\,;\,2}, \cdots, w_{j\,;\,n(j)}\}$ of $W_{(0),j}$, $j=|s|, |s|+1 \cdots, d$ and unitary K-isomorphisms a_i: $V_i \to W_{(0),i}$ $i=|s|$ $|s|+1, \cdots, d$, such that

$$\begin{cases} d\sigma(X_{\phi_{h-1}})w_{j\,;\,1}=\sqrt{c(\sigma)^{(0),j+1}{}_{(0),j}}\, w_{j+1\,;\,1} & j=|s|, |s|+1, \cdots, d\,,\\ a_i(v_{i\,;\,\alpha})=w_{i\,;\,\alpha} \qquad i=|s|, |s|+1, \cdots, d, & \alpha=1, 2, \cdots, n(i)\,. \end{cases}$$

Here $W_{(0),j}$(resp. $c(\sigma)^{(0),j+1}{}_{(0),j}$) means $W_{0,j}$(resp. $c(\sigma)^{0,j+1}{}_{0,j}$) if $h>2$, and W_j (resp. $c(\sigma)^{j+1}{}_j$) if $h=2$. Put $\omega_i=\sum_{\alpha=1}^{n(i)} w_{i\,;\,\alpha}{}^*\otimes v_{i\,;\,\alpha}$, $i=|s|, |s|+1, \cdots, d$. Applying Proposition 10.2.1, 10.3.1 and 10.3.2, we have the following equalities in the similar way to the proof of Lemma 10.4.2:

$$\begin{cases} c(\sigma^*, \rho)^{i+1}{}_i=\sqrt{c(\rho)^{i+1}{}_i c(\sigma)^{(0),i+1}{}_{(0),i}}\\ \quad =\sqrt{\dfrac{(i+1)(k-i)(k+2h+i-2)(t-i)(t+2h+i-2)(i-s+1)(s+2h+i-3)}{(2h+i-3)(2h+2i-1)^2}}\,,\\ c(\sigma^*, \rho)^i{}_{i+1}\\ \quad =\sqrt{\dfrac{(2h+i-3)(k-i)(k+2h+i-2)(t-i)(t+2h+i-2)(i-s+1)(s+2h+i-3)}{(i+1)(2h+2i-3)^2}}\,,\\ c(\sigma^*, \rho)^i{}_i=i(2h+i-3)\,,\\ c(\sigma^*, \rho)^i{}_j=0 \qquad\qquad \text{if } |i-j|>1\,. \end{cases}$$

Put

$$\omega'_i=\sqrt{\frac{(k-i)!(k+2h+i-3)!(t-i)!(t+2h+i-3)!(i-s)!}{i!(2h+i-4)!}}\prod_{j=|s|}^{i}(s+2h+j-4)\omega_i\,.$$

Then the basis $\{\omega'_{|s|}, \omega'_{|s|+1}, \cdots, \omega'_d\}$ of $(W^*\otimes V^C)_0$ has the required property.
Q.E.D.

Theorem 10.4.4. *Let F: $(S^{2h-1}, c\langle\ ,\ \rangle)\to S$, $F(xK)=\rho_k(x)F(o)$, be the full equivariant minimal isometric immersion induced from $\rho=\rho_k$, $k, h\geqq 2$. Then we have*

(1) *Every eigenvalue of the Jacobi differential operator $\tilde{S}$ is an algebraic number.*

(2) *For any $[\sigma]\in D(G; K, \rho^N)$ the multiplicity of every eigenspace of S in $\mathfrak{v}_{[\sigma]}(N(S^{2h-1})^C)$ is equal to 1. (Recall that the operator S leaves $\mathfrak{v}_{[\sigma]}(N(S^{2h-1})^C)$ invariant.)*

Proof. By virtue of Theorem 3 of [I], it is sufficient to show that for any

$[\sigma]\in D(G;K,\rho^N)$ every eigenvalue of the operator S_σ in subsection 5.2 of [I] is an algebraic number and that every eigenspace of S_σ is of dimension 1. Let W be the representation space of σ. Put

$$a=-\frac{4(k-1)(k+2h-1)}{2h+1}.$$

(a) The case $\Lambda_\sigma=t\phi_h$; Let $\{\omega'_0,\omega'_1,\cdots,\omega'_d\}$ be the basis of $(W^*\otimes V^C)_0$ in Lemma 10.4.2. Put for $i=0,1,\cdots,d$

$$\begin{cases} a^{i-1}{}_i=-\dfrac{2(2h+i-4)(k+2h+i-3)(t+2h+i-3)}{2h+2i-5}, \\ a^i{}_i=t(i+2h-2)-2i(2h+i-3), \\ a^{i+1}{}_i=-\dfrac{2(i+1)(k-i)(t-i)}{2h+2i-1}. \end{cases}$$

Let A be the matrix expression of the linear mapping S_σ of $(W^*\otimes(V^N)^C)_0$ with respect to the basis $\{\omega'_2,\omega'_3,\cdots,\omega'_d\}$. Then by (10.4.1), Lemma 10.4.1, Lemma 10.4.2 and (5.2.3) of [I], we have

$$A=\frac{2h-1}{k(k+2h-2)}\begin{pmatrix} a^2{}_2+a & a^2{}_3 & & & 0 \\ a^3{}_2 & a^3{}_3 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & a^{d-1}{}_d \\ 0 & & & a^d{}_{d-1} & a^d{}_d \end{pmatrix}.$$

Therefore all eigenvalues of S_σ are algebraic numbers. Since $a^{i+1}{}_i\neq 0$, $i=2,3,\cdots,d-1$, each eigenspace of S_σ is of dimension 1.

(b) The case $\Lambda_\sigma=s\phi_{h-1}+t\phi_h$ with $s\neq 0$: Let $\{\omega'_{|s|},\omega'_{|s|+1},\cdots,\omega'_d\}$ be the basis of $(W^*\otimes V^C)_0$ in Lemma 10.4.3. Put for $i=|s|,|s|+1,\cdots,d$

$$\begin{cases} b^{i-1}{}_i=-\dfrac{2(k+2h+i-3)(t+2h+i-3)(i-s)(s+2h+i-4)}{i(2h+2i-5)}, \\ b^i{}_i=s(s+2h-4)+t(t+2h-2)-2i(2h+i-3), \\ b^{i+1}{}_i=-\dfrac{2(i+1)(k-i)(t-i)}{2h+2i-1}. \end{cases}$$

Let B be the matrix expression of S_σ with respect to the basis $\{\omega'_m,\omega'_{m+1},\cdots,\omega'_d\}$. Then we have the followings by (10.4.1), Lemma 10.4.1, Lemma 10.4.3 and (5.2.3) of [I]:

[1] The case $|s|=1, 2$:

$$B=\frac{2h-1}{k(k+2h-2)}\begin{pmatrix} b^2{}_2+a & b^2{}_3 & & & 0 \\ b^3{}_2 & b^3{}_3 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & b^{d-1}{}_d \\ 0 & & & b^d{}_{d-1} & b^d{}_d \end{pmatrix}.$$

[2] The case $|s|>2$:

$$B=\frac{2h-1}{k(k+2h-2)}\begin{pmatrix} b^{|s|}{}_{|s|} & b^{|s|}{}_{|s|+1} & & & 0 \\ b^{|s|+1}{}_{|s|} & b^{|s|+1}{}_{|s|+1} & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & b^{d-1}{}_d \\ 0 & & & b^d{}_{d-1} & b^d{}_d \end{pmatrix}.$$

Therefore we obtain the required assertion in the same way as (a). Q.E.D.

REMARK 10.4.1. The eigenvalues of $\tilde{S}$ are not necessarily rational. For example, if $k=h=3$ and $\Lambda_\sigma=4\phi_3$, the eigenvalues of S_σ are not rational.

10.5. In the rest of this section, we shall compute eigenvalues of the operator S_σ for $[\sigma]\in D(G;K,\rho^N)$ with $\Lambda_\sigma=s\phi_{h-1}+t\phi_h$, $|s|\geqq 3$, and give some estimates for the nullity of F. For this purpose we prepare a proposition on the decomposition of tensor products.

In this subsection we denote by W_h the Weyl group of $G=SO(2h)$ with respect to the Cartan subalgebra $\mathfrak{t}$, and by $\delta_h=\phi_2+2\phi_3+\cdots+(h-1)\phi_h$ the half sum of all positive roots of $\mathfrak{g}^C=\mathfrak{so}(2h, \boldsymbol{C})$, $h\geqq 2$. Let $\mathfrak{S}_h$ be the symmetric group of degree k, P_h the family of all subsets of the set $N=\{1, 2, \cdots, h\}$, and P'_h the family of all subsets consisting of even elements of N. We consider $\mathfrak{S}_h$ and P_h as subgroups of $GL(\mathfrak{t})$ in the following manner: If $\tau\in\mathfrak{S}_h$,

$$\tau(\phi_i)=\phi_{\tau(i)} \qquad i=1,2,\cdots,h\,.$$

If $\tau\in P_h$,

$$\tau(\phi_i)=\begin{cases}\phi_i & \text{if } i\notin\tau\,,\\ -\phi_i & \text{if } i\in\tau\,.\end{cases}$$

Then we have

$$W_h=\mathfrak{S}_h\times P'_h \text{ (semi-direct product)}.$$

For an element $\tau\in W_h$ we define a non-negative integer $a_h(\tau)$ as follows:

$$a_h(\tau)=\sum_{i=1}^h |b_i(\tau)|\,,$$

where $\delta_h-\tau(\delta_h)=\sum_{i=1}^h b_i(\tau)\phi_i$. We claim that $a_h(\tau)$ is even. We shall first show this for $\tau \in \mathfrak{S}_h$. Put

$$\begin{cases} N_0 = \{i \in N;\ \tau(i) = i\}\ , \\ N_1 = \{i \in N;\ \tau(i),\ \tau^{-1}(i) < i\}\ , \\ N_2 = \{i \in N;\ \tau(i),\ \tau^{-1}(i) > i\}\ , \\ N_3 = \{i \in N;\ \tau^{-1}(i) < i < \tau(i)\}\ , \\ N_4 = \{i \in N;\ \tau(i) < i < \tau^{-1}(i)\}\ . \end{cases}$$

Then N is a disjoint union of $N_0, N_1, \cdots, N_4$. Since $a_h(\tau) = \sum_{i=1}^{h} |\tau(i) - i|$, we have

$$a_h(\tau) = \sum_{i \in N_1 \cup N_4} i + \sum_{i \in N_2 \cup N_3} \tau(i) - \sum_{i \in N_1 \cup N_4} \tau(i) - \sum_{i \in N_2 \cup N_3} i\ .$$

If $i \in N_1 \cup N_4$(resp. $i \in N_2 \cup N_3$), then $\tau(i) \in N_2 \cup N_4$(resp. $\tau(i) \in N_1 \cup N_3$). Therefore we have

$$a_h(\tau) = 2\Big(\sum_{i \in N_1} i - \sum_{i \in N_2} i\Big)\ ,$$

which is an even integer. Next let $\tau = \tau_1 \tau_2 \in W_h$ with $\tau_1 \in \mathfrak{S}_h$ and $\tau_2 \in P'_h$. Then we have

$$a_h(\tau) = \sum_{i \notin \tau_2} |\tau_1(i) - i| + \sum_{i \in \tau_2} (\tau_1(i) + i - 2)\ . \tag{10.5.1}$$

If we put $m_i = \mathrm{Min}\ \{i, \tau_1(i)\}$, then

$$\begin{aligned} a_h(\tau) &= \sum_{i \notin \tau_2} |\tau_1(i) - i| + \sum_{i \in \tau_2} \{|\tau_1(i) - i| + 2(m_i - 1)\} \\ &= a_h(\tau_1) + 2 \sum_{i \in \tau_2} (m_i - 1)\ . \end{aligned}$$

Therefore $a_h(\tau)$ is an even integer.

Put

$$\begin{cases} W'_h = \{\tau \in W_h;\ \tau(\phi_h) = \phi_h\}\ , \\ W''_h = \{\tau \in W'_h;\ \tau(\phi_{h-1}) = \phi_{h-1}\}\ . \end{cases}$$

We may identify this subgroup W'_h(resp. W''_h) of W_h with the group W_{h-1} (resp. with the subgroup W'_{h-1} of W_{h-1}). Under this identification we have for $\tau \in W'_h$.

$$a_{h-1}(\tau) = a_h(\tau)\ . \tag{10.5.2}$$

Lemma 10.5.1. *Suppose that $h \geqq 3$. We have for a non-negative integer i*

$$\sum_{\substack{\tau \in W'_h \\ 2i - a_h(\tau) \geqq 0}} \det(\tau) \frac{\left(h + \frac{2i - a_h(\tau)}{2} - 2\right)\left(h + \frac{2i - a_h(\tau)}{2} - 3\right) \cdots \left(\frac{2i - a_h(\tau)}{2} + 1\right)}{(h-2)!}$$
$$= 1\ .$$

Proof. We shall prove the lemma by the induction on h. If $h=3$, straightforward calculations show that the equality is valid. Suppose that the equality holds for $h-1$. Put for $\tau\in W_h$

$$K_h(i,\tau)$$
$$=\det(\tau)\frac{\left(h+\frac{2i-a_h(\tau)}{2}-2\right)\left(h+\frac{2i-a_h(\tau)}{2}-3\right)\cdots\left(\frac{2i-a_h(\tau)}{2}+1\right)}{(h-2)!}$$

The subgroup W'_h is decomposed to left cosets modulus its subgroup W''_h in the following way:

$$\begin{aligned}(10.5.3)\quad W'_h=&W''_h\cup(h-2,h-1)W''_h\cup\{h-2,h-1\}W''_h\\ &\cup(h-2,h-1)\{h-2,h-1\}W''_h\cup\bigcup_{j=1}^{h-3}{}''(j,h-1)W''_h\\ &\cup\bigcup_{j=1}^{h-3}(j,h-1)\{h-2,h-1\}W''_h,\end{aligned}$$

where (i,j) (resp. $\{i,j\}$) denotes the transposition of i and j(resp. the subset of N consisting of i and j). Applying (10.5.1), we see easily that

$$(10.5.4)\qquad a_h((h-2,h-1)\tau)=a_h(\tau)+2\qquad\text{for }\tau\in W''_h.$$

Therefore if $2i-a_h(\tau)\geqq 0$ for $\tau\in(h-2,h-1)W''_h$, then $(h-2,h-1)\tau\in W''_h$ and $2i-a_h((h-2,h-1)\tau)>0$. Suppose that $\tau\in W''_h$, $2i-a_h(\tau)\geqq 0$ and $2i-a_h((h-2,h-1)\tau)<0$. Then since $a_h(\tau)$ is even, it follows from (10.5.4) that $2i-a_h(\tau)=0$. Hence

$$K_h(i,(h-2,h-1)\tau)=0.$$

Therefore we have

$$\begin{aligned}&\sum_{\substack{\tau\in W''_h\\ 2i-a_h(\tau)\geqq 0}}K_h(i,\tau)+\sum_{\substack{\tau\in(h-2,h-1)W''_h\\ 2i-a_h(\tau)\geqq 0}}K_h(i,\tau)\\ =&\sum_{\substack{\tau\in W''_h\\ 2i-a_h(\tau)\geqq 0}}\{K_h(i,\tau)+K_h(i,(h-2,h-1)\tau)\}.\end{aligned}$$

And we have by (10.5.4) and (10.5.2)

$$K_h(i,\tau)+K_h(i,(h-2,h-1)\tau)=K_{h-1}(i,\tau)\qquad\text{for }\tau\in W''_h.$$

Therefore we have by the assumption of the induction

$$\begin{aligned}(10.5.5)\qquad&\sum_{\substack{\tau\in W''_h\\ 2i-a_h(\tau)\geqq 0}}K_h(i,\tau)+\sum_{\substack{\tau\in(h-2,h-1)W''_h\\ 2i-a_h(\tau)\geqq 0}}K_h(i,\tau)\\ =&\sum_{\substack{\tau\in W'_{h-1}\\ 2i-a_{h-1}(\tau)\geqq 0}}K_{h-1}(i,\tau)=1.\end{aligned}$$

Applying (10.5.1), we have

$$a_h((h-2, h-1)\tau) = a_h(\tau) \qquad \text{for } \tau \in \{h-2, h-1\} W''_h .$$

Therefore

$$\begin{aligned}(10.5.6)\qquad & \sum_{\substack{\tau\in\{h-2,h-1\}W''_h \\ 2i-a_h(\tau)\geqq 0}} K_h(i,\tau) + \sum_{\substack{\tau\in(h-2,h-1)\{h-2,h-1\}W''_h \\ 2i-a_h(\tau)\geqq 0}} K_h(i,\tau) \\ &= \sum_{\substack{\tau\in\{h-2,h-1\}W''_h \\ 2i-a_h(\tau)\geqq 0}} \{K_h(i,\tau)+K_h(i,(h-2,h-1)\tau)\} \\ &= 0 .\end{aligned}$$

Suppose that $j=1, 2, \cdots, h-3$. Then since $(h-2, h-1)(j, h-1)=(j, h-1)(j, h-2)$, it follows that if τ is contained in $(j, h-1)W''_h$ (resp. in $(j, h-1)\{h-2, h-1\} W''_h$), $(h-2, h-1)\tau$ is also contained in $(j, h-1)W''_h$ (resp. in $(j, h-1)\{h-2, h-1\} W''_h$). Applying (10.5.1), we have

$$\begin{aligned}& a_h((h-2, h-1)\tau) = a_h(\tau) \\ & \quad \text{for } \tau\in(j, h-1)W''_h \cup (j, h-1)\{h-2, h-1\} W''_h .\end{aligned}$$

Therefore we have

$$\begin{aligned}(10.5.7)\qquad \sum_{\substack{\tau\in(j,h-1)W''_h \\ 2i-a_h(\tau)\geqq 0}} K_h(i,\tau) &= \sum_{\substack{\tau\in(j,h-1)\{h-2,h-1\}W'' \\ 2i-a_h(\tau)\geqq 0}} K_h(i,\tau) \\ &= 0 .\end{aligned}$$

We obtain the lemma by (10.5.3), (10.5.5), (10.5.6) and (10.5.7). Q.E.D.

Proposition 10.5.2. *Let $\rho_j\colon G\to GL(W_j)$ and $\sigma\colon G\to GL(W)$ be complex irreducible representations with the highest weights $j\phi_h$ and $s\phi_{h-1}+t\phi_h$ respectively. Then the tensor product $\sigma^*\otimes\rho_j$ contains a spherical representation of (G,K), if and only if $j\geqq|s|$. The highest weights of the spherical representations contained in $\sigma^*\otimes\rho_j$ are the followings:*

$$(j+t-|s|-2i)\phi_h \qquad i=0, 1, \cdots, \mathrm{Min}\{j-|s|, t-|s|\} .$$

Proof. We have by Proposition 9.2.1 of [II]

$$\dim(W^*\otimes W_j)_0 = \begin{cases} 0 & \text{if } j<|s| , \\ \mathrm{Min}\{j-|s|+1,\ t-|s|+1\} & \text{if } j\geqq|s| .\end{cases}$$

Therefore the tensor product $\sigma^*\otimes\rho_j$ contains a spherical representation, if and only if $j\geqq|s|$. In the representation space of a spherical representation of (G,K), the subspace of K-fixed vectors is of dimension 1(cf. Takeuchi [6] p. 104). Therefore the sum of the multiplicities of the spherical representations contained in $\sigma^*\otimes\rho_j$ is equal to $\mathrm{Min}\{j-|s|+1, t-|s|+1\}$. Let $\psi_\Lambda\colon G\to GL(V_\Lambda)$ be a complex irreducible representation with the highest weight Λ, and m_Λ the multiplicity of ψ_Λ in $\tau^*\otimes\rho_j$. Then we have (cf. Chevalley [2] p. 188)

$$m_\Lambda = \int_G \chi_{\sigma^*\otimes\rho_j}\bar{\chi}_{\psi_\Lambda}\, dx\,,$$

where dx is the normalized Haar measure of G and $\chi_{\sigma^*\otimes\rho_j}$ (resp. χ_{ψ_Λ}) is the character of $\sigma^*\otimes\rho_j$(resp. of ψ_Λ). Suppose that ψ_Λ is a spherical representation. Since the characters χ_{ρ_j} and χ_{ψ_Λ} are real valued by Remark 3.2.2 of [I], we have

$$\begin{aligned} m_\Lambda &= \overline{\int_G \chi_{\sigma^*\otimes\rho_j}\bar{\chi}_{\psi_\Lambda} dx} = \int_G \chi_\sigma \bar{\chi}_{\rho_j}\chi_{\psi_\Lambda} dx \\ &= \int_G \chi_\sigma \chi_{\rho_j}\bar{\chi}_{\psi_\Lambda} dx = \int_G \chi_{\sigma\otimes\rho_j}\bar{\chi}_{\psi_\Lambda} dx\,. \end{aligned}$$

Therefore m_Λ is equal to the multiplicity of ψ_Λ in $\sigma\otimes\rho_j$. On the other hand we have Lemma 9.1.1 of [II]

$$m_\Lambda = \sum_{\tau\in W_h} \det(\tau)\, m(\Lambda+\delta_h-\tau(s\phi_{h-1}+t\phi_h+\delta_h); j\phi_h)\,.$$

We consider the case of $\Lambda=(j+t-|s|-i)\phi_h$ with $0\leqq i\leqq \mathrm{Min}\{2(j-|s|), (2(t-|s|)\}$. Then

$$j+t-|s|-i\geqq |s|\,.$$

For $\tau\in W_h$ we define a non-negative integer $c(\tau)$ by

$$c(\tau) = \sum_{k-1}^{h} |c_k(\tau)|\,,$$

where $(j+t-|s|-i)\phi_h+\delta_h-\tau(s\phi_{h-1}+t\phi_h+\delta_h)=\sum_{k=1}^{h} c_k(\tau)\phi_i$. Let $\tau=\tau_1\tau_2\in W_h$ with $\tau_1\in \mathfrak{S}_h$ and $\tau_2\in P'_h$. If $\tau_1(\phi_h)\neq\phi_h$, we have

$$\begin{aligned} c(\tau) &\geqq |c_{\tau_1(h)}(\tau)|+|c_h(\tau)| \\ &\geqq |t+h-1-(h-2)|+|j+t-|s|-i+h-1-(|s|+h-2)| \\ &= 2(t-|s|)-i+j+2\geqq j+2\,. \end{aligned}$$

If $\tau(\phi_h)=-\phi_h$, we have

$$c(\tau)\geqq |c_h(\tau)| = 2t-|s|-i+j+2h-2\geqq j+2\,.$$

Therefore unless $\tau(\phi_h)=\phi_h$, we have by Proposition 9.3.2 of [II]

$$m((j+t-|s|-i)\phi_h+\delta_h-\tau(s\phi_{h-1}+t\phi_h+\delta_h); j\phi_h) = 0\,.$$

Hence we have

$$\begin{aligned} (10.5.8)\quad & m_{(j+t-|s|-i)\phi_h} \\ &= \sum_{\tau\in W'_h} \det(\tau)\, m((j+t-|s|-i)\phi_h+\delta_h-\tau(s\phi_{h-1}+t\phi_h+\delta_h); j\phi_h)\,. \end{aligned}$$

(a) The case $h=2$: It follows from (10.5.8) that

$$m_{(j+t-|s|-i)\phi_2}=m(-s\phi_1+(j-|s|-i)\phi_2; j\phi_2)\,.$$

Applying Proposition 9.3.2 of [II], we have

$$m_{(j+t-|s|-i)\phi_2}=\begin{cases}0 & \text{if } i \text{ is odd,}\\ 1 & \text{if } i \text{ is even.}\end{cases}$$

Therefore $\sigma^*\otimes\rho_j$ contains spherical representations ψ_Λ, $\Lambda=(j+t-|s|-2i)\phi_2$, $i=0,1,\cdots,\mathrm{Min}\{j-|s|, t-|s|\}$.

(b) The case $h>2$: Let $\tau\in W'_h$. If $j-s-i\geqq 0$, we have

$$j-c(\tau)=j-(s+a_h(\tau)+j-s-i)=i-a_h(\tau)\,.$$

If $j-s-i<0$, we have

$$j-c(\tau)=j-(s+a_h(\tau)+s+i-j)=2j-2s-i-a_h(\tau)\,.$$

Recall that $a_h(\tau)$ is even. If i is odd, then we have by Proposition 9.3.2 of [II] and (10.5.8)

$$m_{(j+t-s-i)\phi_h}=0\,.$$

If $j-s-i\geqq 0$ and i is even, we have by (10.5.8), Proposition 9.3.2 of [II] and Lemma 10.5.1

$$\begin{aligned}m_{(j+t-s-i)\phi_h}&=\sum_{\substack{\tau\in W'_h\\ i-a_h(\tau)\geqq 0}}\det(\tau)\,({}_hH_{(i-a_h(\tau))/2}-{}_hH_{(i-a_h(\tau)-2)/2})\\&=1\,.\end{aligned}$$

If $j-s-i<0$ and i is even, we have in the same way as above

$$m_{(j+t-s-i)\phi_h}=1\,.$$

Therefore $\sigma^*\otimes\rho_j$ contains spherical representations ψ_Λ, $\Lambda=(j+t-s-2i)\phi_h$, $i=0,1,\cdots,\mathrm{Min}\{j-s, t-s\}$.

Since the sum of the multiplicities of the spherical representations contained in $\sigma^*\otimes\rho_j$ is equal to $\mathrm{Min}\{j-|s|+1, t-|s|+1\}$, we obtain the proposition.

Q.E.D.

10.6. We consider again the full equivariant minimal isometric immersion $F\colon(S^{2h-1}, c\langle\ ,\ \rangle)\to S$, $F(xK)=\rho_k(x)F(o)$, induced from $\rho=\rho_k$, $k=2,3,\cdots$.

Let $\sigma\colon G\to GL(W)$ be a complex irreducible representation with $[\sigma]\in D(G; K, \rho^N)$, $\Lambda_\sigma=s\phi_{h-1}+t\phi_h$. We define a linear mapping T_σ of $(W^*\otimes V^C)_0$ by

$$T_\sigma=-(c_\sigma 1_{W^*\otimes V^C}+2L(\sigma^*, \rho))\,.$$

Since $c_{\sigma^*}=c_\sigma$, we have by (5.2.1) of [I]

$$T_\sigma = -(C_{\sigma^*\otimes\rho}-c_\rho 1_{W^*\otimes V^C}) , \tag{10.6.1}$$

where $C_{\sigma^*\otimes\rho}$ is the Casimir operator of the tensor product $\sigma^*\otimes\rho$. It follows from Proposition 10.5.2 that there exists a basis $\{\psi_0,\psi_1,\cdots,\psi_m\}$ of $(W^*\otimes V^C)_0$, $m=\mathrm{Min}\{k-|s|,t-|s|\}$, such that every ψ_i is a K-fixed vector in the irreducible G-submodule of $W^*\otimes V^C$ with the highest weight $(k+t-|s|-2i)\phi_h$. Therefore it follows from (10.6.1) and the formula of Freudenthal that the eigenvalues of T_σ are given by

$$(t-|s|-2i)(2k+t+2h-|s|-2i-2) \qquad i=0,1,\cdots,m . \tag{10.6.2}$$

Suppose that $[\sigma]\in D(G;K,\rho^N)$ and $|s|\geqq 3$. Then we have by Proposition 9.2.1 of [II] and (10.4.2)

$$(W^*\otimes V^C)_0=(W^*\otimes(V^N)^C)_0 .$$

And we have by Lemma 10.4.1

$$1_{W^*}\otimes\sum_{i=1}^{n+p}\{d\rho(E_i)(d\rho(E_i)^*)^N\}^N{}_{|(W^*\otimes(V^N)^C)_0}=c_\rho 1_{(W^*\otimes(V^N)^C)_0} .$$

Therefore it follows from Lemma 5.2.2 of [I] that the operator S_σ of $(W^*\otimes(V^N)^C)_0$ coincides with $\frac{1}{c}T_\sigma$. Hence we have the following theorem by (10.6.2).

Theorem 10.6.1. *Let $F:(S^{2h-1},c\langle\ ,\ \rangle)\to S$, $F(xK)=\rho_k(x)F(o)$, be the full equivariant minimal isometric immersion induced from $\rho=\rho_k$, $k=3,4,\cdots$. Suppose that $[\sigma]\in D(G;K,\rho^N)$ and $\Lambda_\sigma=s\phi_{h-1}+t\phi_h$ with $|s|\geqq 3$. Then the eigenvalues of S_σ are given by*

$$\frac{2h-1}{k(k+2h-2)}(t-|s|-2i)(2k+t+2h-|s|-2i-2)$$
$$i=0,1,\cdots,\mathrm{Min}\{k-|s|,t-|s|\} .$$

Let U_0 be the 0-eigenspace of the operator S in $C^\infty(G;(V^N)^C)_K$. Put

$$\Pi_0=\{[\sigma]\in D(G;K,\rho^N);\ S_\sigma \text{ has an eigenvalue } 0\} .$$

Then it follows from (2) of Theorem 10.4.4 that U_0 is decomposed into a direct sum of the irreducible G-submodules of U_0 as follows:

$$U_0=\sum_{[\sigma]\in\Pi_0}U_{[\sigma]} ,$$

where $U_{[\sigma]}$ is the irreducible G-submodule of U_0 with the highest weight Λ_σ. The following theorem gives a lower bound for the nullity of the minimal immersion F.

Theorem 10.6.2. *Put*

$$\Pi'_0 = \left\{ [\sigma] \in D(G; K, \rho^N);\ \Lambda_\sigma = s\phi_{h-1} + t\phi_h \ \textit{satisfies} \begin{array}{l} \text{(i)}\quad |s| = 1 \ \textit{or} \ |s| \geqq 3 \\ \textit{and} \\ \text{(ii)}\quad |s| + t \leqq 2k \ \textit{and} \ t - |s| \ \textit{is even} \end{array} \right\}.$$

Then we have

$$\Pi'_0 \subset \Pi_0 .$$

If $[\sigma]$ *is contained in* Π_0 *and satisfies the above condition* (i), *then* $[\sigma]$ *is contained in* Π'_0.

Proof. We have for $[\sigma] \in D(G; K, \rho^N)$ with $\Lambda_\sigma = s\phi_{h-1} + t\phi_h$ and for $i = 0, 1, \cdots, \mathrm{Min}\,\{k - |s|, t - |s|\}$

$$(10.6.3) \qquad 2k + t + 2h - |s| - 2i - 2 \geqq t + |s| + 2h - 2 > 0 .$$

If $|s| + t > 2k$, we have for $i = 0, 1, \cdots, \mathrm{Min}\,\{k - |s|, t - |s|\}$

$$(10.6.4) \qquad t - |s| - 2i \geqq t - |s| - 2(k - |s|) = |s| + t - 2k > 0 .$$

Suppose that $|s| + t \leqq 2k$. Then we have

$$|s| + t \leqq 2k,\ 2t ,$$

and hence

$$(10.6.5) \qquad t - |s| \leqq 2(k - |s|),\ 2(t - |s|) .$$

(a) The case where $[\sigma] \in D(G; K, \rho^N)$ and $|s| \geqq 3$: Suppose that $[\sigma]$ satisfies the condition (ii). It follows from Theorem 10.6.1 and (10.6.5) that $[\sigma]$ is contained in Π_0. Conversely if $[\sigma]$ is contained in Π_0, it follows from Theorem 10.6.1, (10.6.3) and (10.6.4) that $|s| + t \leqq 2k$ and that $t - |s|$ is even.

(b) The case where $[\sigma] \in D(G; K, \rho^N)$ and $|s| = 1$: Take the basis $\{\omega'_1, \omega'_2, \cdots, \omega'_d\}$ of $(W^* \otimes V^C)_0$ in Lemma 10.4.3. Let B' be the matrix expression of T_σ with respect to this basis, and let $b^{i-1}{}_i$, $b^i{}_i$, $b^{i+1}{}_i$ and B denote the same ones as in the proof of Theorem 10.4.4. Then we have

$$B' = \begin{pmatrix} b^1{}_1 & b^1{}_2 & & & & 0 \\ b^2{}_1 & b^2{}_2 & \ddots & & & \\ & b^3{}_2 & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & \ddots & b^{d-1}{}_d \\ 0 & & & & b^d{}_{d-1} & b^d{}_d \end{pmatrix}.$$

Since

$$b^1{}_1 = (t-1)(t+2h-1),\ b^1{}_2 = -(k+2h-1)(t+2h-1),$$
$$b^2{}_1 = -\frac{4(k-1)(t-1)}{2h-1},$$

we have

$$\det B' = (t-1)(t+2h-1)\times$$
$$\det\begin{pmatrix} 1 & -(k+2h-1) & & & & 0 \\ -\dfrac{4(k-1)}{2h+1} & b^2{}_2 & b^2{}_3 & & & \\ & b^3{}_2 & b^3{}_3 & \ddots & & \\ & & b^4{}_3 & \ddots & \ddots & \\ & & & \ddots & \ddots & b^{d-1}{}_d \\ 0 & & & & b^d{}_{d-1} & b^d{}_d \end{pmatrix}$$
$$= (t-1)(t+2h-1)\det(cB).$$

On the other hand we have by (10.6.2)

$$\det B' = \prod_{i=0}^{d-1}(t-2i-1)(2k+t+2h-2i-3).$$

Since $t \geqq 2$ and $t+2h-1>0$, we have

$$(10.6.6)\quad \det(cB) = \frac{2k+t+2h-3}{t+2h-1}\prod_{i=1}^{d-1}(t-2i-1)(2k+t+2h-2i-3).$$

Applying (10.6.3), (10.6.4), (10.6.5) and (10.6.6), we obtain the assertion in the same way as in (a). Q.E.D.

REMARK 10.6.1. Suppose that $k=3$. Computing the matrices A and B for $|s|=2$ in the proof of Theorem 10.4.4, we have by the above theorem

$$\Pi_0 = \Pi'_0.$$

REMARK 10.6.2. (1) Suppose that $k=3$. Applying Proposition 3.4.2 of [I], we see by the above remark that the nullity of F coincides with its Killing nullity.

(2) Suppose that $k=4$. Then the sum of dim $U_{[\sigma]}$, $[\sigma]\in\Pi'_0$, is greater than the Killing nullity of F. Therefore the nullity is greater than the Killing nullity.

11. Minimal immersions of $2h$-dimensional sphere $S^{2h}(h\geqq 2)$

In this section we assume that $G=SO(2h+1)$ and $K=SO(2h)$, $h\geqq 2$. The assumptions and the notation are the same as in section 9 of [II].

11.1. In this subsection we consider the full equivariant minimal isometric immersion $F: (S^{2h}, c\langle\ ,\ \rangle) \to S$ induced from the second real spherical representation ρ_2 of (G, K). Then we have by the formula of Freudenthal and Proposition 3.2.1 of [I]

$$c = \frac{2h+1}{h}. \tag{11.1.1}$$

Therefore it follows from Remark 8.3.1 of [II] that the operator S on $C^\infty(G; (V^N)^C)_K$ is given by

$$S = -\frac{h}{2h+1}\left(\sum_{i=1}^{n+p} E_i E_i + 4(2h+1) 1_{C^\infty(G;(V^N)^C)_K}\right). \tag{11.1.2}$$

Hence for every $[\sigma] \in D(G; K, \rho^N)$ the operator S acts on $\mathfrak{v}_{[\sigma]}(N(S^{2h})^C)$ as a scalar, which will be denoted by $c(\sigma)$. We have by Proposition 9.2.1 of [II]

$$(V^0)^C = V_0,\ (V^T)^C = V_1,\ (V^N)^C = V_2, \tag{11.1.3}$$

where V_i is the irreducible K-submodule of V^C with the highest weight $i\phi_h$. We have

Theorem 11.1.1. *Let $F: (S^{2h}, c\langle\ ,\ \rangle) \to S$, $F(xK) = \rho_2(x)F(o)$, be the full equivariant minimal isometric immersion induced from $\rho = \rho_2$.*

(1) *We have*

$$D(G; K, \rho^N) = \{[\sigma] \in D(G);\ \Lambda_\sigma = s\phi_{h-1} + t\phi_h \text{ with } 0 \leqq s \leqq 2 \text{ and } t \geqq 2\},$$

where Λ_σ is the highest weight of the complex irreducible representation σ of G. The multiplicity of each $[\sigma] \in D(G; K, \rho^N)$ is equal to 1.

(2) *We have for $[\sigma] \in D(G; K, \rho^N)$ with $\Lambda_\sigma = s\phi_{h-1} + t\phi_h$*

$$c(\sigma) = \frac{h}{2h+1}\{s(s+2h-3) + t(t+2h-1) - 4(2h+1)\}.$$

(3) *The cases where $c(\sigma) \leqq 0$ are the followings:*

$c(\sigma)$	Λ_σ
< 0	$2\phi_h,\ \phi_{h-1}+2\phi_h,\ 2\phi_{h-1}+2\phi_h,\ 3\phi_h$
$= 0$	$\phi_{h-1}+3\phi_h$

Proof. Applying Proposition 9.2.1 of [II], the Frobenius reciprocity and the formula of Freudenthal, we obtain the theorem in the similar way to Theorem 10.1.1. Q.E.D.

REMARK 11.1.1. It follows from the above theorem and Proposition 3.4.2 of [I] that the nullity of F is euqal to its Killing nullity.

REMARK 11.1.2. (a) The case $h=2$: Every eigenspace of S is G-irreducible.

(b) The case $h>2$: The eigenspace corresponding to the eigenvalue $\frac{h(3h^2-9h-4)}{2h+1}$ is decomposed into two G-irreducible components, which have the highest weights $h\phi_h$ and $2\phi_{h-1}+(h-1)\phi_h$. The other eigenspaces are G-irreducible.

11.2. Let $\sigma: G\to GL(W)$ be an irreducible unitary representation with the highest weight $k\phi_h$, $k>0$. We have by Proposition 9.2.1 of [II]

$$W=\sum_{i=0}^{k} W_i\,,$$

where W_i is the irreducible K-submodule of W with the highest weight $i\phi_h$. We shall compute $c(\sigma)^i{}_j$, $i, j=0, 1, \cdots, k$. It follows from (9.4.1), (2) of Lemma 6.2.3 and (a) of Proposition 6.3.7 of [II] that

$$c(\sigma)^i{}_j=0 \quad \text{for } i,j=0,1,\cdots,k \text{ with } |i-j|>1\,. \tag{11.2.1}$$

We have

Proposition 11.2.1

$$\begin{cases} c(\sigma)^{k-i}{}_{k-i}=(k-i)\,(k+2h-i-2)\,,\\ c(\sigma)^{k-i}{}_{k-i-1}=\dfrac{(i+1)(k-i)(2k+2h-i-2)}{2(k+h-i-1)}\,,\\ c(\sigma)^{k-i-1}{}_{k-i}=\dfrac{(i+1)\,(k+2h-i-3)\,(2k+2h-i-2)}{2(k+h-i-2)}\,. \end{cases}$$

Proof. It follows from (9.4.1), (3) of Lemma 6.2.3, Proposition 6.3.7 and (3) of Lemma 6.3.4 of [II] that

$$c(\sigma)^i{}_i=c(\sigma\,;\mathfrak{k})^i{}_i=i(i+2h-2)\,. \tag{11.2.2}$$

Applying Lemma 6.3.2, (6.3.10) of [II], (11.2.1) and (11.2.2), we obtain the proposition in the similar way to the proof of (b) of Proposition 10.2.1. Q.E.D.

11.3. Let $\sigma: G\to GL(W)$ be an irreducible unitary representation with the highest weight $s\phi_{h-1}+t\phi_h$, $s>0$. We have by Proposition 9.2.1 of [II]

$$W=\begin{cases} \displaystyle\sum_{|p|\leq s\leq q\leq t} W_{p,q} & \text{if } h=2\,,\\ \displaystyle\sum_{0\leq p\leq s\leq q\leq t} W_{p,q} & \text{if } h>2\,, \end{cases}$$

where $W_{p,q}$ is the irreducible K-submodule of W with the highest weight $p\phi_{h-1}+q\phi_h$. We shall compute $c(\sigma)^{0,i}{}_{0,j}$, $i, j=s, s+1, \cdots, t$. It follows from Lemma 9.2.4, (2) of Lemma 6.2.3 and (a) of Proposition 6.3.7 of [II] that

Proof. We have in the same way as in (11.2.2)

$$c(\sigma)^{0,i}{}_{0,i} = i(i+2h-2)\,. \tag{11.3.2}$$

(a) Applying Lemma 6.3.2, (6.3.10) of [II], (11.3.1) and (11.3.2), we obtain the equalities in the similar way to the proof of (a) of Proposition 10.3.2.

(b) Put $H_{\phi_h}=[X_{\phi_h}, X_{-\phi_h}]$ and $\mathfrak{g}_{X\phi_h}{}^C=\{X_{\phi_h}, X_{-\phi_h}, H_{\phi_h}\}_C$. Considering W as a $\mathfrak{g}_{X\phi_h}{}^C$-module, we obtain the equalities in the similar way to the proof of (b) of Proposition 10.3.2. Q.E.D.

1.4. In the rest of this section we consider the full equivariant minimal isometric immersion $F: (S^{2h}, c\langle\ ,\ \rangle)\rightarrow S$ induced from the *k-th* real spherical representation $\rho=\rho_k$ of (G, K), $k=2, 3, \cdots$. Then we have by the formula of Freudenthal and Proposition 3.2.1 of [I]

$$c = \frac{k(k+2h-1)}{2h}\,. \tag{11.4.1}$$

We have by Proposition 9.2.1 of [II]

$$(V^0)^C = V_0,\ (V^T)^C = V_1,\ (V^N)^C = \sum_{i=2}^{k} V_i\,, \tag{11.4.2}$$

where V_i is the irreducible K-submodule of V^C with the highest weight $i\phi_h$. It follows from Corollary for Proposition 9.2.1 and the argument in subsection 6.5 of [II] that there exist complex numbers c_i, $i=0, 1, \cdots, k$, such that

$$\sum_{i=1}^{n+p} \{d\rho(E_i)\,(d\rho(E_i)*)^N\}^N{}_{|V_i} = c_i 1_{V_i}\,.$$

Then we have the following lemma by Proposition 6.5.1, Proposition 6.3.8 of [II], Proposition 11.2.1 and (11.2.1).

Lemma 11.4.1.

$$c_i = \begin{cases} 0 & \text{if } i = 0,\ 1\,, \\ -\left\{k(k+2h-1)-\dfrac{(k-1)\,(k+2h)}{h+1}\right\} & \text{if } i = 2\,, \\ -k(k+2h-1) & \text{if } i > 2\,. \end{cases}$$

It follows from Proposition 9.2.1 of [II] and the Frobenius reciprocity that

$$D(G; K, \rho^N) = \{[\sigma]\in D(G); \Lambda_\sigma = s\phi_{h-1}+t\phi_h,\ 0\leqq s\leqq k,\ 2\leqq t\}\,,$$

and that the multiplicity of the above $[\sigma]\in D(G; K, \rho^N)$ is equal to Min $\{k-1, k-s+1, t-1, t-s+1\}$. We have

Lemma 11.4.2. *Let $\sigma: G\rightarrow GL(W)$ be a complex irreducible representation with $[\sigma]\in D(G; K, \rho^N)$ and $\Lambda_\sigma=t\phi_h$. Then there exists a basis $\{\omega'_0, \omega'_1, \cdots, \omega'_d\}$ of*

$(W^*\otimes V^C)_0$ *such that* $\{\omega'_2, \omega'_3, \cdots, \omega'_d\}$ *is a basis of* $(W^*\otimes(V^N)^C)_0$ *and that*

$$L(\sigma^*, \rho)\omega'_i = \frac{(2h+i-3)(k+2h+i-2)(t+2h+i-2)}{2(h+i-2)}\omega'_{i-1} + i(2h+i-2)\omega'_i + \frac{(i+1)(k-i)(t-i)}{2(h+i)}\omega'_{i+1} \quad \text{for } i=0,1,\cdots,d,$$

where $d=\mathrm{Min}\{k,t\}$ and $\omega'_{-1}=\omega'_{d+1}=0$.

Proof. We may choose orthonormal bases $\{v_{i;1}, v_{i;2}, \cdots, v_{i;n(i)}\}$ of V_i and $\{w_{j;1}, w_{j;2}, \cdots, w_{j;n(j)}\}$ of W_j, and unitary K-isomorphisms $a_i\colon V_i\to W_i$, $i, j=1, \cdots, d$, such that

$$\begin{cases} d\rho(X_{\phi_h})v_{i;1} = \sqrt{c(\rho)^{i+1}{}_i}\, v_{i+1;1}, \\ d\sigma(X_{\phi_h})w_{j;1} = \sqrt{c(\sigma)^{j+1}{}_j}\, w_{j+1;1}, \\ a_i(v_{i;\alpha}) = w_{i;\alpha} \qquad \alpha = 1, 2, \cdots, n(i). \end{cases}$$

Put $\omega_i=\sum_{\alpha=1}^{n(i)} w_{i;\alpha}{}^*\otimes v_{i;\alpha}$ $i=0, 1, \cdots, d$. Then applying Proposition 6.3.9, Lemma 6.3.2, (3) of Lemma 6.3.4 of [II] and Proposition 11.2.1, we have the following equations in the similar way to the proof of Lemma 10.4.2:

$$\begin{cases} c(\sigma^*, \rho)^{i+1}{}_i = \dfrac{(i+1)}{2(h+i)}\sqrt{(k-i)(k+2h+i-1)(t-i)(t+2h+i-1)}, \\ c(\sigma^*, \rho)^i{}_{i+1} = \dfrac{2h+i-2}{2(h+i-1)}\sqrt{(k-i)(k+2h+i-1)(t-i)(t+2h+i-1)}, \\ c(\sigma^*, \rho)^i{}_i = i(2h+i-2), \\ c(\sigma^*, \rho)^i{}_j = 0 \qquad i, j=0, 1, \cdots, d \text{ with } |i-j|>1. \end{cases}$$

Put

$$\omega'_i = \sqrt{(k-i)!(k+2h+i-2)!(t-i)!(t+2h+i-2)!}\,\omega_i.$$

Then the basis $\{\omega'_0, \omega'_1, \cdots, \omega'_d\}$ of $(W^*\otimes V^C)_0$ has the required property.
Q.E.D.

Lemma 11.4.3. *Let* $\sigma\colon G\to GL(W)$ *be a complex irreducible representation with* $[\sigma]\in D(G; K, \rho^N)$ *and* $\Lambda_\sigma=s\phi_{h-1}+t\phi_h$, $s>0$. *Then there exists a basis* $\{\omega'_s, \omega'_{s+1}, \cdots, \omega'_d\}$ *of* $(W^*\otimes V^C)_0$ *succh that* $\{\omega'_m, \omega'_{m+1}, \cdots, \omega'_d\}$ *is a basis of* $(W^*\otimes(V^N)^C)_0$ *and that*

$$L(\sigma^*, \rho)\omega'_i = \frac{(k+2h+i-2)(t+2h+i-2)(i-s)(s+2h+i-3)}{2i(h+i-2)}\omega'_{i-1} + i(2h+i-2)\omega'_i + \frac{(i+1)(k-i)(t-i)}{2(h+i)}\omega'_{i+1}$$

$$for\ \ i = s, s+1, \cdots, d\,,$$

where $d = Min\{k, t\}$, $m = Max\{2, s\}$ *and* $\omega'_{s-1} = \omega'_{d+1} = 0$.

Proof. We may choose orthonormal bases $\{v_{i\,;\,1}, v_{i\,;\,2}, \cdots, v_{i\,;\,n(i)}\}$ of V_i and $\{w_{j\,;\,1}, w_{j\,;\,2}, \cdots, w_{j\,;\,n(j)}\}$ of $W_{0,j}$, and unitary K-isomorphisms $a_i\colon V_i \to W_{0,i}$, $i, j = s, s+1, \cdots, d$, such that

$$\begin{cases} d\rho(X_{\phi_h})v_{i\,;\,1} = \sqrt{c(\rho)^{i+1}{}_i}\, v_{i+1\,;\,1}\,, \\ d\sigma(X_{\phi_h})w_{j\,;\,1} = \sqrt{c(\sigma)^{j+1}{}_j}\, w_{j+1\,;\,1}\,, \\ a_i(v_{i\,;\,\alpha}) = w_{i\,:\,\alpha} \qquad \alpha = 1, 2, \cdots, n(i)\,. \end{cases}$$

Put $\omega_i = \sum_{\alpha=1}^{n(i)} w_{i\,;\,\alpha}{}^* \otimes v_{i\,;\,\alpha}$, $i = s, s+1, \cdots, d$. Applying Proposition 11.2.1 and Proposition 11.3.1, we have the following equalities in the similar way to the proof of Lemma 10.4.2:

$$\begin{cases} c(\sigma^*, \rho)^{i+1}{}_i \\ \quad = \sqrt{\dfrac{(i+1)(k-i)(k+2h+i-1)(t-i)(t+2h+i-1)(i-s+1)(s+2h+i-2)}{4(h+i)^2(2h+i-2)}}\,, \\ c(\sigma^*, \rho)^{i}{}_{i+1} \\ \quad = \sqrt{\dfrac{(2h+i-2)(k-i)(k+2h+i-1)(t-i)(t+2h+i-1)(i-s+1)(s+2h+i-2)}{4(i+1)(h+i-1)^2}}\,, \\ c(\sigma^*, \rho)^{i}{}_{i} = i(2h+i-2)\,, \\ c(\sigma^*, \rho)^{i}{}_{j} = 0 \qquad \text{if } |i-j| > 1\,. \end{cases}$$

Put

$$\omega'_i = \sqrt{\frac{(k-i)!\,(k+2h+i-2)!\,(t-i)!\,(t+2h+i-2)!\,(i-s)!}{i!\,(2h+i-3)!}}\ \prod_{j=s}^{i}(s+2h+j-3)\omega_i\,.$$

Then the basis $\{\omega'_s, \omega'_{s+1}, \cdots, \omega'_d\}$ of $(W^* \otimes V^C)_0$ has the required property.
Q.E.D.

Theorem 11.4.4. *Let* $F\colon (S^{2h}, c\langle\ ,\ \rangle) \to S$, $F(xK) = \rho_k(x)F(o)$, *be the full equivariant minimal isometric immersion induced from* $\rho = \rho_k$, $h \geqq 2$, $k = 2, 3, \cdots$. *Then we have*

(1) *Every eigenvalue of the Jacobi differential operator* $\tilde{S}$ *is an algebraic number.*

(2) *For any* $[\sigma] \in D(G; K, \rho^N)$, *the multiplicity of every eigenspace of* $\tilde{S}$ *in* $\mathfrak{v}_{[\sigma]}(N(S^{2h})^C)$ *is equal to* 1.

Proof. By virtue of Theorem 3 of [I], it is sufficient to show that for any $[\sigma] \in D(G; K, \rho^N)$ every eigenvalue of the operator S_σ is an algebraic number and that every eigenspace of S_σ is of dimension 1. Let W be the representation space of σ. Put

$$a=-\frac{2(k-1)(k+2h)}{h+1}.$$

(a) The case $\Lambda_\sigma=t\phi_h$: Let $\{\omega'_0,\omega'_1,\cdots,\omega'_d\}$ be the basis of $(W^*\otimes V^C)_0$ in Lemma 11.4.2. Put for $i=0,1,\cdots,d$

$$\begin{cases} a^{i-1}{}_i=-\dfrac{(2h+i-3)(k+2h+i-2)(t+2h-i-2)}{h+i-2}, \\ a^i{}_i=t(t+2h-1)-2i(2h+i-2), \\ a^{i+1}{}_i=-\dfrac{(i+1)(k-i)(t-i)}{h+i}. \end{cases}$$

Let A be the matrix expression of the linear mapping S_σ with respect to the basis $\{\omega'_2,\omega'_3,\cdots,\omega'_d\}$. Then we have by (11.4.1), Lemma 11.4.1, Lemma 11.4.2 and (5.2.3) of [I]

$$A=\frac{2h}{k(k+2h-1)}\begin{pmatrix} a^2{}_2+a & a^2{}_3 & & & 0 \\ a^3{}_2 & a^3{}_3 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & a^{d-1}{}_d \\ 0 & & & a^d{}_{d-1} & a^d{}_d \end{pmatrix}.$$

Therefore all eigenvalues of S_σ are algebraic numbers. Since $a^{i+1}{}_i\neq 0$, $i=2,3,\cdots,d-1$, each eigenspace of S_σ is of dimension 1.

(b) The case $\Lambda_\sigma=s\phi_{h-1}+t\phi_h$, $s>0$: Let $\{\omega'_s,\omega'_{s+1},\cdots,\omega'_d\}$ be the basis of $(W^*\otimes V^C)_0$ in Lemma 11.4.3. Put for $i=s,s+1,\cdots,d$

$$\begin{cases} b^{i-1}{}_i=-\dfrac{(k+2h+i-2)(t+2h+i-2)(i-s)(s+2h+i-3)}{i(h+i-2)}, \\ b^i{}_i=s(s+2h-3)+t(t+2h-1)-2i(2h+i-2), \\ b^{i+1}{}_i=-\dfrac{(i+1)(k-i)(t-i)}{h+i}. \end{cases}$$

Let B be the matrix expression of S_σ with respect to the basis $\{\omega'_m,\omega'_{m+1},\cdots,\omega'_d\}$. Then we have the followings by (11.4.1), Lemma 11.4.1, Lemma 11.4.3 and (5.2.3) of [I]:

[1] The case $s=1,2$:

$$B=\frac{2h}{k(k+2h-1)}\begin{pmatrix} b^2{}_2+a & b^2{}_3 & & & 0 \\ b^3{}_2 & b^3{}_3 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & b^{d-1}{}_d \\ 0 & & & b^d{}_{d-1} & b^d{}_d \end{pmatrix}.$$

[2] The case $s>2$:

$$B=\frac{2h}{k(k+2h-1)}\begin{pmatrix} b^s{}_s & b^s{}_{s+1} & \cdot & & 0 \\ b^{s+1}{}_s & b^{s+1}{}_{s+1} & \cdot & \cdot & \\ & \cdot & \cdot & \cdot & \cdot \\ & & \cdot & \cdot & b^{d-1}{}_d \\ 0 & & & b^d{}_{d-1} & b^d{}_d \end{pmatrix}.$$

Therefore we obtain our assertion. Q.E.D.

11.5. In this subsection the notation W_h, $\mathfrak{S}_h$, P_h and $a_h(\tau)$ are the same as in subsection 10.5. We have

$$W_h=\mathfrak{S}_h\times P_h \text{ (semi-direct product).}$$

Let $\tau=\tau_1\tau_2\in W_h$ with $\tau_1\in\mathfrak{S}_h$ and $\tau_2\in P_h$. Then we have

$$a_h(\tau)=\sum_{i\notin\tau_2}|\tau_1(i)-i|+\sum_{i\in\tau_2}(\tau_1(i)+i-1)\,. \tag{11.5.1}$$

Identifying the subgroup W'_h of W_h with the group W_{h-1}, we have for $\tau\in W'_h$

$$a_{h-1}(\tau)=a_h(\tau)\,. \tag{11.5.2}$$

Lemma 11.5.1. *Suppose that $h\geqq 2$. We have for a non-negative integer i*

$$\sum_{\substack{\tau\in W'_h\\ i-a_h(\tau)\geqq 0}}\det(\tau)\frac{\left(h+\left[\frac{i-a_h(\tau)}{2}\right]-1\right)\left(h+\left[\frac{i-a_h(\tau)}{2}\right]-2\right)\cdots\left(\left[\frac{i-a_h(\tau)}{2}\right]+1\right)}{(h-1)!}$$
$$=\begin{cases}1 & \text{if } i \text{ is even,}\\ 0 & \text{if } i \text{ is odd.}\end{cases}$$

Proof. We obtain the lemma by the induction on h in the similar way to the proof of Lemma 10.5.1. Q.E.D.

Proposition 11.5.2. *Let $\rho_j\colon G\to GL(W_j)$ and $\sigma\colon G\to GL(W)$ be complex irreducible representations with the highest weight $j\phi_h$ and $s\phi_{h-1}+t\phi_h$ respectively. Then the tensor product $\sigma^*\otimes\rho_j$ contains a spherical representation of (G,K), if and only if $s\leqq j$. The highest weights of the spherical representations contained in $\sigma^*\otimes\rho_j$ are the followings:*

$$(j+t-s-2i)\phi_h \qquad i=0,1,\cdots,Min\{j-s,\,t-s\}\,.$$

Proof. We have the followings in the similar way to the proof of Proposition 10.5.2.

(a) The tensor product $\sigma^*\otimes\rho_j$ contains a spherical representation of (G,K), if and only if $s\leqq j$.

(b) The sum of the multiplicities of the spherical representations contained in $\sigma^*\otimes\rho_j$ is equal to $\mathrm{Min}\{j-s+1, t-s+1\}$.

(c) Let ψ_Λ be a spherical representation of (G,K) and m_Λ the multiplicity of ψ_Λ in $\sigma^*\otimes\rho_j$. Then m_Λ is equal to the multiplicity of ψ_Λ in $\sigma\otimes\rho_j$.

(d) If $\Lambda=(j+t-s-i)\phi_h$ and $0\leqq i\leqq\mathrm{Min}\{2(j-s), 2(t-s)\}$, we have

$$m_{(j+t-s-i)\phi_h}=\sum_{\tau\in W'_h}\det(\tau)m((j+t-s-i)\phi_h+\delta_h-\tau(s\phi_{h-1}+t\phi_h+\delta_h); j\phi_h).$$

Therefore we have by Proposition 9.3.2 of [II]

$$m_{(j+t-s-i)\phi_h}=\begin{cases}\displaystyle\sum_{\substack{\tau\in W'_h\\ i-a_h(\tau)\geqq 0}}\det(\tau)_h H_{[\{i-a_h(\tau)\}/2]} & \text{if } j-s-i\geqq 0,\\ \displaystyle\sum_{\substack{\tau\in W'_h\\ 2j-2s-i-a_h(\tau)\geqq 0}}\det(\tau)_h H_{[\{2j-2s-i-a_h(\tau)\}/2]} & \text{if } j-s-i<0.\end{cases}$$

Applying Lemma 11.5.1, we obtain the proposition. Q.E.D.

11.6. We consider again the full equivariant minimal isometric immersion $F: (S^{2h}, c\langle\ ,\ \rangle)\to S$, $F(xK)=\rho_k(x)F(o)$, induced from $\rho=\rho_k$, $k=2, 3, \cdots$. Let T_σ, Π_0 and Π'_0 denote the same ones as in subsection 10.6. Let $[\sigma]\in D(G; K, \rho^N)$ and $\Lambda_\sigma=s\phi_{h-1}+t\phi_h$. Then we have in the similar way to (10.6.2) that the eigenvalues of T_σ are

$$(11.6.1)\quad (t-s-2i)(2k+t+2h-s-2i-1)\qquad i=0, 1, \cdots, \mathrm{Min}\{k-s, t-s\}.$$

Suppose that $[\sigma]\in D(G; K, \rho^N)$ and $s\geqq 3$. Then we have that $(W^*\otimes V^C)_0=(W^*\otimes(V^N)^C)_0$ and $S_\sigma=\frac{1}{c}T_\sigma$. Therefore we have

Theorem 11.6.1. *Let $F: (S^{2h}, c\langle\ ,\ \rangle)\to S$, $F(xK)=\rho_k(x)F(o)$, be the full equivariant minimal isometric immersion induced from $\rho=\rho_k$, $k=3, 4, \cdots$. Suppose that $[\sigma]\in D(G; K, \rho^N)$ and $\Lambda_\sigma=s\phi_{h-1}+t\phi_h$ with $s\geqq 3$. Then eigenvalues of S_σ are given by*

$$\frac{2h}{k(k+2h-1)}(t-s-2i)(2k+t+2h-s-2i-1)\qquad i=0, 1, \cdots, \mathrm{Min}\{k-s, t-s\}.$$

Theorem 11.6.2. *We have*

$$\Pi'_0\subset\Pi_0.$$

Suppose that $[\sigma]\in\Pi_0$, $\Lambda_\sigma=s\phi_{h-1}+t\phi_h$. Then if $s=1$ or $s\geqq 3$, $[\sigma]$ is contained in Π'_0.

Proof. Let $[\sigma]\in D(G; K, \rho^N)$ and $\Lambda_\sigma=s\phi_{h-1}+t\phi_h$. Then we have

$$(11.6.2)\quad\begin{cases}2k+t+2h-s-2i-1>0 & \text{for } i=0,1,\cdots,\mathrm{Min}\{k-s, t-s\},\\ t-s-2i>0 & \text{for } i=0,1,\cdots,\mathrm{Min}\{k-s, t-s\} \text{ if } s+t>2k,\\ t-s<2(k-s),\ 2(t-s) & \text{if } s+t\leqq 2k.\end{cases}$$

(a) The case where $[\sigma]\in D(G;K,\rho^N)$ and $s\geqq 3$: Applying Theorem 11.6.1 and (11.6.2), we have that $[\sigma]\in\Pi_0$, if and only if $[\sigma]\in\Pi'_0$.

(b) The case where $[\sigma]\in D(G;K,\rho^N)$ and $s=1$: Take the basis $\{\omega'_1,\omega'_2,\cdots,\omega'_d\}$ of $(W^*\otimes V^C)_0$ in Lemma 11.4.3. Let B' be the matrix expression of T_σ with respect to this basis, and let $b^{i-1}{}_i$, $b^i{}_i$, $b^{i+1}{}_i$ and B denote the same ones as in the proof of Theorem 11.4.4. Then we have in the similar way to the proof (b) of Theorem 10.6.2

$$\det B' = (t-1)\,(t+2h)\det(cB)\,.$$

Therefore we have by (11.6.1)

$$(11.6.3)\qquad \det(cB) = \frac{2k+t+2h-2}{t+2h}\prod_{i=1}^{d-1}(t-2i-1)\,(2k+t+2h-2i-2)\,.$$

Applying (11.6.2) and (11.6.3), we obtain the assertion. Q.E.D.

REMARK 11.6.1. If $k=3$, we have $\Pi_0=\Pi'_0$.

REMARK 11.6.2. (1) If $k=3$, the nullity of F coincides with its Killing nullity.

(2) If $k=4$, the sum of dim $U_{[\sigma]}$, $[\sigma]\in\Pi'_0$, is greater than the Killing nullity of F. Therefore the nullity is greater than the Killing nullity.

12. Minimal immersions of 2-dimensional sphere S^2

In this section we assume that $G=SO(3)$ and $K=SO(2)$. The assumptions and the notation are the same as in section 9 of [II].

12.1. In this subsection we consider the full equivariant minimal isometric immersion $F:(S^2,c\langle\ ,\ \rangle)\to S$ induced from the second real spherical representation ρ_2 of (G,K). Then we have by the formula of Freudenthal and Proposition 3.2.1 of [I]

$$(12.1.1)\qquad c=3\,.$$

It follows from Remark 8.3.1 of [II] that the operator S on $C^\infty(G;(V^N)^C)_K$ is given by

$$(12.1.2)\qquad S=-\frac{1}{3}\Big(\sum_{i=1}^{3}E_iE_i+12\;1_{C^\infty(G\,;\,(V^N)^C)_K}\Big)\,.$$

Therefore for each $[\sigma]\in D(G;K,\rho^N)$ the operator S acts on $\mathfrak{v}_{[\sigma]}(N(S^2)^C)$ as a scalar, which is denoted by $c(\sigma)$. We have by Proposition 9.2.1 of [II]

$$(12.1.3)\qquad (V^0)^C=V_0,\ (V^T)^C=V_{-1}+V_1,\ (V^N)^C=V_{-2}+V_2\,,$$

where V_i is the $i\phi_1$-weight space of V^C relative to $\mathfrak{t}=\mathfrak{k}$.

Theorem 12.1.1. *Let* $F\colon (S^2, c\langle\ ,\ \rangle)\to S,\ F(xK)=\rho_2(x)F(o)$, *be the full equivariant minimal isometric immersion induced from* $\rho=\rho_2$.

(1) *We have*

$$D(G;K,\rho^N) = \{[\sigma]\in D(G);\ \Lambda_\sigma = t\phi_1\ \ t\geqq 2\}\ ,$$

where Λ_σ *is the highest weight of the complex irreducible representation* σ. *The multiplicity of each* $[\sigma]\in D(G;K,\rho^N)$ *is equal to* 2.

(2) *We have for* $[\sigma]\in D(G;K,\rho^N)$ *with* $\Lambda_\sigma=t\phi_1$

$$c(\sigma) = \frac{1}{3}(t^2+t-12)\ .$$

(3) *The cases where* $c(\sigma)\leqq 0$ *are the followings*:

$c(\sigma)$	Λ_σ
<0	$2\phi_1$
$=0$	$3\phi_1$

Proof. Applying Proposition 9.2.1 of [II], the Frobenius reciprocity and the formula of Freudenthal, we obtain the theorem. Q.E.D.

REMARK 12.1.1. It follows from the above theorem and Proposition 3.4.2 of [I] that the nullity of F is equal to twice its Killing nullity.

12.2. Let $\sigma\colon G\to GL(W)$ be an irreducible unitary representation with the highest weight $k\phi_1 (k>0)$, and c_σ the eigenvalue of the Casimir operator of σ. We have by Proposition 9.2.1 of [II]

$$W = W_0+\sum_{i=1}^{k}(W_{-i}+W_i)\ ,$$

where W_i is the $i\phi_1$-weight space of W relative to $\mathfrak{t}=\mathfrak{k}$. We shall compute $c(\sigma)^i{}_j,\ i,j=0,\pm 1,\cdots,\pm k$. It follows from (9.4.1), (2) of Lemma 6.2.3 and (a) of Proposition 6.3.7 of [II] that

$$c(\sigma)^i{}_j = 0 \qquad \text{for } i,j = 0,\pm 1,\cdots,\pm k \text{ with } |i-j|>1\ . \tag{12.2.1}$$

We have

Proposition 12.2.1.

$$\begin{cases} c(\sigma)^{k-i}{}_{k-i} = (k-i)^2\ , \\ c(\sigma)^{k-i}{}_{k-i-1} = c(\sigma)^{k-i-1}{}_{k-i} = \dfrac{(i+1)(2k-i)}{2} \quad \textit{for } i = 0,1,\cdots,2k-1\ . \end{cases} \tag{12.2.2}$$

$$(12.2.3) \qquad c(\sigma)^{k-i}{}_{k-i-1} = c(\sigma)^{-(k-i)}{}_{-(k-i-1)} \qquad \textit{for } i = 0, 1, \cdots, k-1 .$$

Proof. It follows from (9.4.1), (3) of Lemma 6.2.3, Proposition 6.3.7 and (3) of Lemma 6.3.4 of [II] that

$$(12.2.4) \qquad c(\sigma)^i{}_i = c(\sigma; \mathfrak{k})^i{}_i = i^2 .$$

Applying Lemma 6.3.2, (6.3.10) of [II], (12.2.1) and (12.2.4), we obtain the equalities (12.2.2) by the induction on i in the similar way to the proof (b) of Proposition 10.2.1. We have the equality (12.2.3) by (12.2.2). Q.E.D.

12.3. In the rest of this section we consider the full equivariant minimal isometric immersion $F: (S^2, c\langle\ ,\ \rangle) \to S$ induced from the *k-th* real spherical representation $\rho = \rho_k$ of (G, K), $k = 2, 3, \cdots$. Then we have by the formula of Freudenthal and Proposition 3.2.1 of [I]

$$(12.3.1) \qquad c = \frac{k(k+1)}{2} .$$

We have by Proposition 9.2.1 of [II]

$$(12.3.2) \qquad (V^0)^C = V_0,\ (V^T)^C = V_{-1} + V_1,\ (V^N)^C = \sum_{i=2}^{k} (V_{-i} + V_i) ,$$

where V_i is the $i\phi_1$-weight space of V^C. Then $\dim V_i = 1$. It follows from Corollary for Proposition 9.2.1 and the argument in subsection 6.5 of [II] that there exist complex numbers c_i, $i = 0, \pm 1, \cdots, \pm k$, such that

$$\sum_{i=1}^{3} \{d\rho(E_i)\,(d\rho(E_i)*)^N\}^N{}_{|V_i} = c_i\, 1_{V_i} .$$

Then we have the following lemma by Proposition 6.5.1, Proposition 6.3.8 of [II], Proposition 12.2.1 and (12.2.1).

Lemma 12.3.1.

$$c_i = \begin{cases} 0 & \textit{if } i = 0, \pm 1 , \\ -\left\{k(k+1) - \dfrac{(k-1)(k+2)}{2}\right\} & \textit{if } i = \pm 2 , \\ -k(k+1) & \textit{if } |i| > 2 . \end{cases}$$

It follows from Proposition 9.2.1 of [II] and the Frobenius reciprocity that

$$D(G; K, \rho^N) = \{[\sigma] \in D(G);\ \mathrm{V}_\sigma = t\phi_1 \quad t \geqq 2\}$$

and that the multiplicity of the above $[\sigma] \in D(G; K, \rho^N)$ is equal to $2\,\mathrm{Min}\{k-1, t-1\}$. We have

Lemma 12.3.2. *Let $\sigma: G \to GL(W)$ be a complex irreducible representation* *with* $[\sigma]\in D(G;K,\rho^N)$ *and* $\Lambda_\sigma=t\phi_1$. *Then there exists a basis* $\{\omega'_{-d},\cdots,\omega'_{-1},\omega'_0,\omega'_1,\cdots,\omega'_d\}$ *of* $(W^*\otimes V^C)_0$ *such that* $\{\omega'_{-d},\cdots,\omega'_{-2},\omega'_2,\cdots,\omega'_d\}$ *is a basis of* $(W^*\otimes(V^N)^C)_0$ *and that*

$$L(\sigma^*,\rho)\omega_i'=\frac{1}{2}(k+i)(t+i)\omega'_{i-1}+i^2\omega'_i+\frac{1}{2}(k-i)(t-i)\omega'_{i+1}\qquad \text{for } i=0,\pm1,\cdots,\pm d,$$

where $d=Min\{k,t\}$ *and* $\omega'_{-d-1}=\omega'_{d+1}=0$.

Proof. We may choose $i\phi_1$-weight vector v_i of V^C and $j\phi_1$-weight vector w_j of W with unit lengths, $i=0,\pm1,\cdots,\pm k$, $j=0,\pm1,\cdots,\pm t$, such that

$$\begin{cases} d\rho(X_{\phi_1})v_i=\sqrt{c(\rho)^{i+1}{}_i}\,v_{i+1}, \\ d\sigma(X_{\phi_1})w_j=\sqrt{c(\sigma)^{j+1}{}_j}\,w_{j+1}. \end{cases}$$

Put $\omega_i=w_i^*\otimes v_i$, $i=0,\pm1,\cdots,\pm d$. Then $\{\omega_{-d},\cdots,\omega_{-1},\omega_0,\omega_1,\cdots,\omega_d\}$ is a basis of $(W^*\otimes V^C)_0$ and $\{\omega_{-d},\cdots,\omega_{-2},\omega_2,\cdots,\omega_d\}$ is a basis of $(W^*\otimes(V^N)^C)_0$. Then applying Proposition 6.3.9, Lemma 6.3.2, (3) of Lemma 6.3.4 of [II] and Proposition 12.2.1, we have the following equalities in the similar way to the proof of Lemma 10.4.2:

$$\begin{cases} c(\sigma^*,\rho)^{i+1}{}_i=c(\sigma^*,\rho)^i{}_{i+1}=\frac{1}{2}\sqrt{(k-i)(k+i+1)(t-i)(t+i+1)}, \\ c(\sigma^*,\rho)^i{}_i=i^2, \\ c(\sigma^*,\rho)^i{}_j=0 \qquad i,j=0,\pm1,\cdots,\pm d \text{ with } |i-j|>1. \end{cases}$$

Put

$$\omega'_i=\sqrt{(k-i)!(k+i)!(t-i)!(t+i)!}\,\omega_i.$$

Then the basis $\{\omega'_{-d},\cdots,\omega'_{-1},\omega'_0,\omega'_1,\cdots,\omega'_d\}$ of $(W^*\otimes V^C)_0$ has the required property. Q.E.D.

Theorem 12.3.3. *Let* $F\colon(S^2,c\langle\ ,\ \rangle)\to S$, $F(xK)=\rho_k(x)F(o)$, *be the full equivariant minimal isometric immersion induced from* $\rho=\rho_k$, $k=2,3,\cdots$. *Then we have*

(1) *Every eigenvalue of the Jacobi differential operator* $\tilde{S}$ *is an algebraic number.*

(2) *For any* $[\sigma]\in D(G;K,\rho^N)$, *the multiplicity of every eigenspace of* $\tilde{S}$ *in* $\mathfrak{v}_{[\sigma]}(N(S^2)^C)$ *is equal to* 2.

Proof. By virtue of Theorem 3 of [I], it is sufficient to show that for any $[\sigma]\in D(G;K,\rho^N)$ every eigenvalue of the operator S_σ is an algebraic number and that every eigenspace of S_σ is of dimension 2. Let W be the representation space of σ. Put

$$a=-(k-1)(k+2).$$

Let $\{\omega'_{-d}, \cdots, \omega'_{-1}, \omega'_0, \omega'_1, \cdots, \omega'_d\}$ be the basis of $(W^* \otimes V^C)_0$ in Lemma 12.3.2. Put for $i=0, 1, \cdots, d$

$$\begin{cases} b^{i-1}{}_i = -(k+i)(t+i)\,, \\ b^i{}_i = t(t+1)-2i^2\,, \\ b^{i+1}{}_i = -(k-i)(t-i)\,, \end{cases}$$

and

$$B = \frac{1}{c}\begin{pmatrix} b^2{}_2+a & b^2{}_3 & & & 0 \\ b^3{}_2 & b^3{}_3 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & b^{d-1}{}_d \\ 0 & & & b^d{}_{d-1} & b^d{}_d \end{pmatrix}.$$

Let B' be the matrix expression of the linear mapping S_σ of $(W^* \otimes (V^N)^C)_0$ with respect to the basis $\{\omega'_{-2}, \cdots, \omega'_{-d}, \omega'_2, \cdots, \omega'_d\}$. Then we have by (12.3.1), Lemma 12.3.1, Lemma 12.3.2 and (5.2.3) of [I]

$$B' = \begin{pmatrix} B & 0 \\ 0 & B \end{pmatrix}.$$

Therefore all eigenvalues of S_σ are algebraic numbers. Since $b^{i+1}{}_i \neq 0$, $i=2, 3, \cdots, d-1$, each eigenspace of S_σ is of dimension 2. Q.E.D.

12.4. We have

Theorem 12.4.1. *Let* $F\colon (S^2, c\langle\ ,\ \rangle) \to S$, $F(xK) = \rho_k(x)F(o)$, *be the full equivariant minimal isometric immersion induced from* $\rho = \rho_k$, $k=2, 3, \cdots$. Put

$$\Pi_0 = \{[\sigma] \in D(G; K, \rho^N);\ S_\sigma \textit{ has an eigenvalue } 0\}.$$

Then we have

$$\Pi_0 = \{[\sigma] \in D(G; K, \rho^N);\ \Lambda_\sigma = 3\phi_1, 5\phi_1, \cdots, (2k-1)\phi_1\}.$$

Theorefore the nullity of F is equal to $2(k-1)(2k+3)$.

Proof. Let $b^{i+1}{}_i$, $b^i{}_i$, $b^{i+1}{}_i$ and B denote the same ones as in the proof of Theorem 12.3.3. Put for $i=2, 3, \cdots, d$

$$\begin{cases} c(h) = \dfrac{k(k+2h-1)}{2h}\,, \\[2ex] a(h) = -\dfrac{2(k-1)(k+2h)}{h+1}\,, \\[2ex] b^{i-1}{}_i(h) = -\dfrac{(k+2h+i-2)(t+2h+i-2)(i-1)(2h+i-2)}{i(h+i-2)}\,, \\[2ex] b^i{}_i(h) = 2h-2+t(t+2h-1)-2i(2h+i-2)\,, \end{cases}$$

$$\left| b^{i+1}{}_i(h) = -\frac{(i+1)(k-i)(t-i)}{h+i}, \right.$$

$$B(h) = \frac{1}{c(h)}\begin{pmatrix} b^2{}_2(h)+a(h) & b^2{}_3(h) & & & 0 \\ b^3{}_2(h) & b^3{}_3(h) & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & b^{d-1}{}_d(h) \\ 0 & & & b^d{}_{d-1}(h) & b^d{}_d(h) \end{pmatrix}.$$

Then we have

$$\begin{cases} c(1) = c, \ a(1) = a, \ b^{i-1}{}_i(1) = b^{i-1}{}_i, \\ b^i{}_i(1) = b^i{}_i, \ b^{i+1}{}_i(1) = b^{i+1}{}_i, \ B(1) = B. \end{cases}$$

We have by (11.6.3)

$$\det(c(h)B(h)) = \frac{2k+t+2h-2}{t+2h}\prod_{i=1}^{d-1}(t-2i-1)(2k+t+2h-2i-2).$$

Since this equality holds for infinitely many $h \geqq 2$, the equality is valid for $h=1$. Hence

$$\det(cB) = \frac{2k+t}{t+2}\prod_{i=1}^{d-1}(t-2i-1)(2k+t-2i). \tag{12.4.1}$$

If $t>2k-1$, we have $t-2i-1>0$ for $i=1,2,\cdots,d-1$. Since $2k+t-2i \geqq t+2>0$ for $i=1,2,\cdots,d-1$, we obtain the first assertion by (12.4.1). If W is an irreducible G-module with the highest weight $i\phi_1$, then we have $\dim W=2i+1$. Therefore we obtain the second assertion by (2) of Theorem 12.3.3. Q.E.D.

Let $\tilde{U}$ be the space of Killing vector fields on the unit sphere S. Then the Lie group G acts on $\tilde{U}$ in the following manner:

$$(\sigma(x)\tilde{f})(p) = d(\rho(x))\tilde{f}(\rho(x^{-1})p) \qquad \text{for } x \in G, \ \tilde{f} \in \tilde{U} \text{ and } p \in S,$$

where $d(\rho(x))$ denotes the differential of the isometry $\rho(x)$ of S. Let $L(V)$ be the space of linear mappings of V. Put

$$\mathfrak{so}(V) = \{A \in L(V);\ A^* = -A\},$$

where A^* denotes the adjoint linear mapping of A. Then $\mathfrak{so}(V)$ is a G-module with the following action:

$$\omega\colon G \to GL(\mathfrak{so}(V)), \ \omega(x)X = \rho(x)X\rho(x^{-1}) \quad \text{for } x \in G \text{ and } X \in \mathfrak{so}(V).$$

Then $\tilde{U}$ is canonically G-isomorphic to $\mathfrak{so}(V)$. Put

$$\tilde{U}_{|S^2} = \{\tilde{f}_{|S^2};\ \tilde{f} \in \tilde{U}\}.$$

where $\tilde{f}_{|S^2}$ denotes the section of $T(S)_{|S^2}$ induced from $\tilde{f}$. Then $\tilde{U}_{|S^2}$ is a G-module. Since the immersion $F\colon S^2\to S$ is full, $\tilde{U}_{|S^2}$ is G-isomorphic to $\tilde{U}$. Put

$$\tilde{J}_0=\{\tilde{f}^N;\ \tilde{f}\in\tilde{U}_{|S^2}\}\ .$$

An element of $\tilde{J}_0$ is called a *Killing-Jacobi field*. Put

$$\tilde{T}=\{\tilde{f}\in\tilde{U}_{|S^2};\ \tilde{f}^N=0\}\ .$$

Then $\tilde{J}_0$ is a G-module and $\tilde{T}$ is a G-submodule of $\tilde{U}_{|S^2}$ which is G-isomorphic to the G-module $\mathfrak{g}$ with the adjoint action. Therefore $\tilde{U}_{|S^2}$ is G-isomorphic to the direct sum $\tilde{J}_0+\mathfrak{g}$ of $\tilde{J}_0$ and $\mathfrak{g}$. We denote by $\tilde{J}_0{}^C$ the complexification of $\tilde{J}_0$. Let W_i be a complex irreducible G-module with the highest weight $i\phi_1$. Then we have

Lemma 12.4.2. *The G-module $\tilde{J}_0{}^C$ is G-isomorphic to the direct sum $\sum_{i=2}^{k} W_{2i-1}$ of the G-modules W_{2i-1}, $i=2,3,\cdots,k$.*

Proof. By the above argument the direct sum $\tilde{J}_0{}^C+\mathfrak{g}^C$ of $\tilde{J}_0{}^C$ and $\mathfrak{g}^C$ is is G-isomorphic to $\mathfrak{so}(V)^C$, the complexification of $\mathfrak{so}(V)$. Therefore it is sufficient to show that $\mathfrak{so}(V)^C$ is G-isomorphic to the direct sum $\sum_{i=1}^{k} W_{2i-1}$ of the G-modules W_{2i-1}, $i=1,2,\cdots,k$. Since Cartan subalgebras of $\mathfrak{so}(V)$ are conjugate (cf. Helgason [4] p. 211), we may choose an orthonormal basis of V with the following property: For $H=a\phi_1\in\mathfrak{t}$, the matrix expression of $d\rho(H)$ with respect to this orthonormal basis of V is given by

$$\begin{pmatrix} 0 & & & & \\ & \boxed{\begin{matrix} 0 & -a \\ a & 0 \end{matrix}} & & & \\ & & \ddots & & \\ & & & \boxed{\begin{matrix} 0 & -ka \\ ka & 0 \end{matrix}} \\ & 0 & & & \end{pmatrix}.$$

Therefore we have by straightforward calculations

$$Tr(\omega(\exp H))=k+2\sum_{i=1}^{k}\cos ia+4\sum_{1\leqq i<j\leqq k}\cos ia\cos ja\ .$$

Hence the character χ_ω of ω is given by

$$\begin{aligned}\chi_\omega=k&+\sum_{i=1}^{k}\{e(i\phi_1)+e(-i\phi_1)\}\\ &+\sum_{1\leqq i<j\leqq k}\{e((i+j)\phi_1)+e(-(i+j)\phi_1)+e((j-i)\phi_1)+e((i-j)\phi_1)\}\end{aligned}$$

on $\mathfrak{t}$.

Therefore we have by straightforward calculations

$$\begin{aligned}\chi_\omega \xi_{\delta_G} &= \chi_\omega \{e(\frac{1}{2}\phi_1) - e(-\frac{1}{2}\phi_1)\} \\ &= \sum_{j=1}^{k} \xi_{(2j-1)\phi_1+\delta_G}\,.\end{aligned}$$

This proves the lemma. Q.E.D.

Now, recalling that dim $\mathcal{J}_0$=the Killing nullity of F, we have the following theorem by the above lemma, (2) of Theorem 12.3.3 and Theorem 12.4.1.

Theorem 12.4.3. *Let F be as in Theorem* 12.4.1. *Then the nullity of F is equal to twice its Killing nullity.*

REMARK 12.4.1. We may also compute the Killing nullity of F by applying Proposition 3.4.2 of [I]. Note that Lemma 12.4.2 gives the G-module structure of the space $\mathcal{J}_0$ of Killing-Jacobi fields.

REMARK 12.4.2. A cross-section of $f\in\Gamma(N(S^2))$ is called a *Jacobi field,* if it satisfies $\tilde{S}\tilde{f}=0$. A full minimal isometric immersion of $(S^2, c\langle\ ,\ \rangle)$ into a unit sphere S is rigid, and induced from some ρ_k in the way described in Remark 3.2.1 of [I] (Calabi [1] p. 123, Do Carmo-Wallach [3] p. 103). Therefore Theorem 12.4.3 shows the followings: Let $F\colon (S^2, c\langle\ ,\ \rangle)\to S$ be a full minimal isometric immersion. Then there exists a Jacobi field which does not arise from any one-parameter families of minimal isometric immersions.

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