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QUASICONFORMAL MAPPINGS OF SUBMANIFOLDS IN R^n WITH THEIR APPLICATION TO A PROBLEM OF MINIMAL SURFACES

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0. Introduction

R. Courant conjectured in his eminent monograph [6] that a minimal surface could be analytically extended as a minimal surface beyond any analytic subarc γ of the boundary curve. In comparison with the case where γ is a straight segment he remarked: "The difficulty of the problem will be appreciated if one notes that the analytic boundary γ may conceivably be represented by a vector whose components are non-analytic functions of the arc-length on the contour of its parameter domain." This question, already elucidated by Hildebrandt [11] in the affirmative, has undoubtedly motivated the present investigation.

We shall be aware that we often encounter the typical theorems in numerous text books, monographs and papers on the complex analysis of one or several variables whose assumptions involve *analytic arcs* or *analytic curves*, for example:

Let γ (resp. γ') be a non-singular analytic boundary subarc of a plane region B (resp. B'). If a univalent holomorphic function $f(z)$ maps the region B conformally onto B' and further $B\cup\gamma$ homeomorphically onto $B'\cup\gamma'$, then $f(z)$ is continued analytically up to $B\cup\gamma$.

In my previous paper [17] I pointed out the fact that the analyticity assumption in all such statements can be weakened up to the regular smoothness as a corollary to a general theorem on the Teichmüller mapping.

Curiously enough, intensive studies concerning the analytic arcs immersed in the general position of $\boldsymbol{R}^n$ ($n\geq 3$) seems very rare within the knowledge of this author. According to his opinion, a kind of obscurities against the commonness of the term *analytic arcs* subsisted even in the Courant's conjecture.

The present memoir has been written from an attempt to clarify those questions and answer the aforesaid conjecture through a quasiconformal approach under the much less restrictive situation that γ has only to satisfy some non-singular smoothness. As a matter of fact, a conditioned non-singular thrice continuous differentiability of γ is sufficient.

1. Notations and terminologies

In this paper $n\in \boldsymbol{Z}^+$ is always not smaller than 3 unless otherwise stated explicitly and i is the index running from 1 to n. Let A, A' be subsets of $\boldsymbol{R}^n$. The difference of the sets A, A', i.e. the set of elements belonging to A but not to A' is denoted by $A \setminus A'$. The symbols $\operatorname{int} A$ and $\operatorname{clo} A$ stand for the set of interior points of A and the closure of A respectively in reference to the neighbourhoods of dimension considered. The term *region* shall always mean a connected open set, while *domain* need not even be open. The notations t, u, v are used as real variables and $w=u+\sqrt{-1}\,v$, $\overline{w}=u-\sqrt{-1}v\in \boldsymbol{C}$. Furthermore the followings are employed consistently:

$I=[-1, 1]$: the 1-dimensional unit closed interval;

$]a, b[=\{x\in \boldsymbol{R}\,|\,a<x<b\}$, everytime $a<b$;

$\left.\begin{array}{l} B^+=\{(u, v)\,|\,-1<u<1,\ 0<v<1\} \\ B^-=\{(u, v)\,|\,-1<u<1,\ -\delta'<v<0\} \end{array}\right\}$: the 2-dimensional open intervals;

$\boldsymbol{x}={}^t(x^1, x^2, \cdots, x^n)$: the real n-vector with the i-th component x^i, or equivalently a point of $\boldsymbol{R}^n$ with the i-th coordinate x^i $(i=1,2,\cdots,n)$.

$C^r=C^r[\cdot]$ denotes, as usual, the class of functions with the r-th continuous derivatives on the point set $\cdot$. Similarly $C^r_0[B]$ is the subclass of $C^r[B]$ with a support comprised in the region B.

When $\boldsymbol{x}$ varies in a continuous manner depending on one real variable t, one will obtain an *arc* γ defined by the equation $\boldsymbol{x}=\boldsymbol{x}(t)$. Here we introduce the three classes of arcs for later use:

$\mathcal{A}=\mathcal{A}^0$: the collection of all simple open continuous arcs whose loci lie in $\boldsymbol{R}^n$;

$\mathcal{A}^r$ $(r\in \boldsymbol{Z}^+)$: the collection of all non-singular simple open C^r-arcs embedded in $\boldsymbol{R}^n$;

$\mathcal{A}^\omega$: the collection of all non-singular simple open analytic arcs embedded in $\boldsymbol{R}^n$.

If $\boldsymbol{x}$ depends, on the other hand, on two independent real variables, say u and v, or equivalently on one complex variable $w=u+\sqrt{-1}\,v$ ranging over a subregion of $\boldsymbol{R}^2=\boldsymbol{C}$, one has a surface S as a 2-dimensional submanifold of $\boldsymbol{R}^n$.

In both cases we need sometimes regard those submanifolds merely as subsets of $\boldsymbol{R}^n$ discarding their parametrizations, which is the so-called *locus* of the arcs or of the surfaces, denoted by $\operatorname{loc}\gamma$ or $\operatorname{loc}S$ etc. henceforth.

The inner product of real n-vectors $\boldsymbol{x}$, $\boldsymbol{x}'$ is written as $\langle \boldsymbol{x}, \boldsymbol{x}'\rangle$, whereas $|\boldsymbol{x}|=\sqrt{\langle \boldsymbol{x}, \boldsymbol{x}\rangle}$.

In reference to a differentiable surface S: $\boldsymbol{x}=\boldsymbol{x}(u,v)$

$$g_{11}(u, v) = |\partial \boldsymbol{x}/\partial u|^2, \qquad g_{22}(u, v) = |\partial \boldsymbol{x}/\partial v|^2$$
$$g_{12}(u, v) = \langle \partial \boldsymbol{x}/\partial u,\ \partial \boldsymbol{x}/\partial v\rangle$$

are designated as the components of the first fundamental form of S. The dilatation-quotient of the mapping $\boldsymbol{x}=\boldsymbol{x}(u,v)$ is defined in terms of them as

$$D(w;\boldsymbol{x})=\frac{g_{11}+g_{22}+\sqrt{(g_{11}-g_{22})^2+4g_{12}^2}}{2\sqrt{|g_{11}g_{22}-g_{12}^2|}}$$

at points where $g_{11}g_{22}-g_{12}^2$ does not vanish.

When we take a closed Jordan region $\operatorname{clo} B$ for a quadrilateral by marking four points on ∂B, we sometimes write $\dot{B}$ instead of $\operatorname{clo} B$. The *modulus* of a quadrilateral Ω can be defined, regardless of whether lying on a plane or on a surface, directly by means of the path-families or through conformal mappings onto a rectangle, which shall be denoted by $\operatorname{Mod}\Omega$.

2. One and two-dimensional submanifolds in $\boldsymbol{R}^n$

Having examined and compared as various defining statements for arcs or curves included in prevalent monographs on analysis as our eyes could reach (e.g., Ahlfors-Sario [3], Fleming [7], Nitsche [12], Osgood [13], Radó [15], Sasaki [16], Springer [19], Väisälä [20], etc.) we finally come to be convinced that the followings are the fittest for our current purpose.

Let $x^i=x^i(t)\,(i=1,2,\cdots,n)$ be an n-tuple of real-valued continuous functions in a real variable t ranging over the open interval $\operatorname{int} I=]-1,1[$ such that

$$(1)\qquad -1<t_1\neq t_2<1 \ \text{ implies } \sum_{i=1}^{n}|x^i(t_1)-x^i(t_2)|\neq 0.$$

Then we understand that a parametric representation (or equation)

$$(2)\qquad x^i=x^i(t),\quad i=1,2,\cdots,n$$

of a simple open continuous arc γ has been set up, calling the point set $\{\boldsymbol{x}=\boldsymbol{x}(t)\,|\,-1<t<1\}$ the *locus* of γ and denoting it by the symbol $\operatorname{loc}\gamma$. Let $\mathfrak{T}$ denote the collection of all orientation-preserving homeomorphisms $\tau(t)$ of the 1-simplex I. $\mathfrak{T}$ is non-void, since we have a function

$$(3)\qquad t'=\tau(t)=\begin{cases}[((1-a)t+1)^r-1]/(1-a^r), & (-1\leq t<0)\\ [1-(1-(1-a)t)^r]/(1-a^r), & (0\leq t\leq 1)\end{cases}$$

with any $r\in\boldsymbol{Z}^+$ and a constant $a\in]0,1[$, which is of class C^∞, strictly monotone-increasing for $-1\leq t\leq 1$ and satisfies $\tau(-1)=-1$, $\tau(1)=1$, $\tau^{(\nu)}(-1)=\tau^{(\nu)}(1)$ $(\nu=1,2,\cdots)$.

DEFINITION 1. Under the term *simple open continuous arc* lying in $\boldsymbol{R}^n$ we mean the equivalence class of all homeomorphisms of $\operatorname{int} I$ onto $\operatorname{loc}\gamma$ factored by modulo $\mathfrak{T}$. If, in particular, the equation (2) are defined on the closed interval I and fulfill the subsidiary condition $x^i(-1)=x^i(1)$ for all $i=1,2,\cdots,n$, γ is a Jordan curve in $\boldsymbol{R}^n$. As an immediate consequence we have

Theorem 1. *The concept of a simple open continuous arc or of a Jordan curve is equivalent to the real* 1*-dimensional topological submanifold of* $\boldsymbol{R}^n$.

Next suppose that the representative (2) of γ fulfills the conditions below not necessarily including (1):

1° $\boldsymbol{x}(t)$ is of class C^r in int I $(r\in\boldsymbol{Z}^+)$;

2° $\lim_{t\to\pm1} d^s\boldsymbol{x}/dt^s$ exists finitely for every $s=1,2,\cdots,r$;

3° $d\boldsymbol{x}/dt \neq 0$.

Then we say (under the additional hypothesis $\lim_{t\to-1} d^s\boldsymbol{x}/dt^s=\lim_{t\to1} d^s\boldsymbol{x}/dt^s$ $(s=1,2,\cdots,r)$ in case clo (loc γ) has no extremities) that a C^r-diffeomorphism $\boldsymbol{x}=\boldsymbol{x}(t)$ of I onto clo (loc γ) is defined. By analogy with $\mathcal{I}$, $\mathcal{I}^r$ denotes the collection of all orientation-preserving C^r-automorphisms $\tau(t)$ of I such that $d^s\tau/dt^s|_{t=-1} =d^s\tau/dt^s|_{t=1}$ for every $s=1,2,\cdots,r$. $\mathcal{I}^r$ is non-void owing to the actual presence of (3) and we have naturally

Definition 2. Under the term *open* (resp. *closed*) *non-singular* C^r*-arc* γ immersed in $\boldsymbol{R}^n$ we mean the equivalence class of all C^r-diffeomorphisms of int I (resp. I) onto loc γ (resp. clo (loc γ)) (modulo $\mathcal{I}^r$). If (2) satisfies (1) in addition, γ is a non-singular simple C^r-arc.

Theorem 2. *The non-singular open* C^r*-arc or the Jordan* C^r*-curve in the above sense is a real* 1*-dimensional* C^r*-submanifold of* $\boldsymbol{R}^n$.

Let us impose a far stronger restriction than the non-singular C^r-differentiability on the representative $\boldsymbol{x}=\boldsymbol{x}(t)$ of γ. To any $t_0\in$ int I there shall be some $\delta^i=\delta^i(t_0)>0$ put into correspondence in such a way that each component $x^i(t)$ admits a power series expansion in the real variable t with real coefficients convergent in the interval $t_0-\delta^i<t<t_0+\delta^i$ $(i=1,2,\cdots,n)$. We obtain a pair of statements:

Definition 3. Under the term open *analytic arc* γ we understand the equivalence class of all real analytic mappings $\boldsymbol{x}=\boldsymbol{x}(t)$ of int I onto loc γ factored by modulo $\mathcal{I}^\omega$, where $\mathcal{I}^\omega$ denotes the group of all non-singular real analytic automorphisms of int I preserving the orientation. Of course it is meaningful to talk about $\mathcal{I}^\omega$ in view of the presence of one element $\tau(t)=\sin(\pi t/2)$. Similarly to the above two cases we have a simple open analytic arc by adding the subsidiary condition (1).

Theorem 3. *A non-singular open analytic arc is a* 1*-dimensional analytic submanifold of* $\boldsymbol{R}^n$.

The non-singular C^r-differentiability (resp. analyticity) of an arc implies no more than the fact that it admits at least one parametrization which enjoys the non-singular C^r-differentiability (resp. analyticity). Hence we have trivially

$$\mathcal{A}^{\omega}\subseteq\mathcal{A}^{r}\subseteq\mathcal{A}^{0} \qquad (1\leq r\leq+\infty),$$

but it is not so clear whether or not the inclusions are strict. Indeed we have shown

Theorem 4 (cf. Shibata [17], p. 100). *If $n=2$, then $\mathcal{A}^2=\mathcal{A}^{\omega}$.*

The above definition for an analytic arc immersed in $\boldsymbol{R}^n$ is substantially the same as the one included in the classical book by Osgood ([13], II_1, pp. 1–2), and in case $n=2$, it turns out to coincide with the one appearing in most of text books on the theory of functions of a complex variable or on the Riemann surfaces through the customary identification of the real (x^1, x^2)-space with the complex $(x^1+\sqrt{-1}\,x^2)$-space.

Before passing to the 2-dimensional submanifolds let us recall an important lemma in the theory of plane quasiconformal mappings which remains to hold even for non-quasiconformal mappings:

Lemma 1 (Gehring-Lehto [8], cf. Ahlfors [2], pp. 24–27 too). *If an open mapping $\phi(w)$ of a subregion B of $\boldsymbol{C}$ into $\boldsymbol{C}$ has partial derivatives $\partial\phi/\partial w$, $\partial\phi/\partial\overline{w}$ almost everywhere on B, $\phi(w)$ is totally differentiable almost everywhere on B.*

It seems difficult to give a complete definition for a *surface* which applies to all uses. We shall be obliged to content ourselves with the one that suits our current purpose.

DEFINITION 6. Suppose that a real n-vector-valued continuous function $\boldsymbol{x}=\boldsymbol{x}(w)$ is L^2-derivable (cf. Bers [5]), absolutely continuous in 2-dimensional sense in a Jordan subregion B of $\boldsymbol{C}$ with sufficiently smooth boundary and that $g_{11}g_{22}-g_{12}^2>0$ almost everywhere on B. Assume further that $\omega=\phi(w)$ is a L^2-derivable, measurable orientation-preserving homeomorphism of B onto the open unit disk $\Delta=\{\omega \mid |\omega|<1\}$ together with its inverse $\phi^{-1}(\omega)$. Then we call the equivalence class S of $\{\boldsymbol{x}(w)\}$ divided modulo the collection $\{\phi(w)\}$ to be a real *differentiable surface-portion.*

REMARK 1. $\omega=\phi(w)$ is totally differentiable almost everywhere on B (Lemma 1).

Let Γ be a Jordan curve in $\boldsymbol{R}^n$ such that $\Gamma\mapsto\partial B$ is a homeomorphism. If $\boldsymbol{x}(w)$ and $\phi(w)$ in the above definition of differentiable surface-portion satisfy the following boundary conditions, we call the similar equivalence class $S=\{\boldsymbol{x}(w)\}$ to be a *differentiable surface with the contour* Γ:

1° $\boldsymbol{x}(w)$ is continuous up to $\operatorname{clo} B$;

2° $\phi(w)$ is a homeomorphism of $\operatorname{clo} B$ onto $\operatorname{clo} \Delta$.

By replacing the L^2-derivability of $\boldsymbol{x}(w)$ and $\phi(w)$ by C^r-smoothness ($1 \leq r \leq +\infty$) we can obtain the definition for C^r-smooth surface-portions or surfaces with boundaries. Further by taking a complex n-vector-valued mapping $\boldsymbol{z}=\boldsymbol{z}(w)$ instead of $\boldsymbol{x}=\boldsymbol{x}(w)$, we can define the differentiable or C^r-smooth complex surface likewise.

Theorem 5. *Let γ be a simple closed C^r-arc embedded in $\boldsymbol{R}^n$ ($r \in \boldsymbol{Z}^+$) with the parameter interval I. Then, for the open arc, which is the restriction of γ to* int *I to be a simple analytic arc it is necessary and sufficient that there exists a certain surface-portion S in the complex n-space $\boldsymbol{C}^n$ of complexification of $\boldsymbol{R}^n=\{x\}$, satisfying the conditions:*

1° *loc S comprises loc γ;*

2° *the parametrization that makes S C^r-smooth in a region comprising I also induces a parametrization that makes γ C^r-smooth on I;*

3° *when S is mapped conformally into the plane $\boldsymbol{C}$, loc γ goes to a straight segment.*

Proof. Since the defining i-th coordinate $x^i(u)$ of γ is of class C^1 in the open u-interval int $I=\{u \mid -1<u<1\}$ and the finite limits $\lim \partial x^i/\partial u$ exist as $u \to \pm 1 + \mp 0$, $x^i(u)$ is considered to be continuous on I ($i=1,2,\cdots,n$). It is easy to construct real-valued functions $\phi^i(u,v)$ which are of class $C^r[\boldsymbol{R}^2]$ and coincide with $x^i(u)$ on I ($i=1,2,\cdots,n$). The complex-valued function

$$z^i(w)=\phi^i(u,v)+\sqrt{-1}\,[\phi^i(u,v)-x^i(u)]$$

is of class $C^r[\boldsymbol{C}]$ and coincides with $x^i(u)$ on the real segment I. Recommended is the function $z^i(w)$ as the i-th complex coordinate $z^i=x^i+\sqrt{-1}\,y^i$ of the smooth surface-portion S subject to the requirements 1°, 2°. A simple calculation

$$\begin{aligned}
&\frac{\partial z^i}{\partial w}\overline{\left(\frac{\partial z^i}{\partial \overline{w}}\right)} \\
(4)\quad &=\frac{1}{4}\left\{\frac{\partial x^i}{\partial u}+\frac{\partial y^i}{\partial v}+\sqrt{-1}\left(\frac{\partial y^i}{\partial u}-\frac{\partial x^i}{\partial v}\right)\right\}\left\{\left(\frac{\partial x^i}{\partial u}-\frac{\partial y^i}{\partial v}-\sqrt{-1}\left(\frac{\partial y^i}{\partial u}+\frac{\partial x^i}{\partial v}\right)\right\}\right. \\
&=\frac{1}{4}\left[\left(\frac{\partial x^i}{\partial u}\right)^2+\left(\frac{\partial y^i}{\partial u}\right)^2-\left(\frac{\partial x^i}{\partial v}\right)^2-\left(\frac{\partial y^i}{\partial v}\right)^2-2\sqrt{-1}\left(\frac{\partial x^i}{\partial u}\frac{\partial x^i}{\partial v}+\frac{\partial y^i}{\partial u}\frac{\partial y^i}{\partial v}\right)\right]
\end{aligned}$$

yields

$$(5)\qquad \sum_{i=1}^{n}\frac{\partial z^i}{\partial w}\overline{\left(\frac{\partial z^i}{\partial \overline{w}}\right)}=\frac{1}{4}(g_{11}(u,v)-g_{22}(u,v)-2\sqrt{-1}\,g_{12}(u,v))$$

with the components of fundamental tensors referred to the coordinate $(x^1, y^1, \cdots, x^n, y^n)$ in $\boldsymbol{R}^{2n}$.

Now suppose that the restriction of γ to $\operatorname{int} I$ is an analytic arc in the sense of Definition 3. Then the defining power series for $\gamma|_{\operatorname{int} I}$ extends naturally to a system of holomorphic functions $z^i=z^i(w)$ defined in some region B containing $\operatorname{int} I\,(i=1,2,\cdots,n)$, which determines a surface S immersed in $\boldsymbol{C}^n$ and subject to the requirements $1°, 2°$ of the theorem. Since this surface S makes the left-hand side of (5) vanish, we must have $g_{11}(u,v)-g_{22}(u,v)=g_{12}(u,v)=0$ in B, which proves the necessity.

On the other hand from (4) it follows that

$$
\begin{aligned}
&16\sum_{i=1}^{n}\left|\frac{\partial z^i}{\partial w}\right|^2\left|\frac{\partial z^i}{\partial \overline{w}}\right|^2\\
&=\sum_{i=1}^{n}\left|\left\{\left(\frac{\partial x^i}{\partial u}\right)^2+\left(\frac{\partial y^i}{\partial u}\right)^2+\left(\frac{\partial x^i}{\partial v}\right)^2+\left(\frac{\partial y^i}{\partial v}\right)^2\right\}^2-4\left(\frac{\partial x^i}{\partial u}\frac{\partial y^i}{\partial v}-\frac{\partial x^i}{\partial v}\frac{\partial y^i}{\partial u}\right)^2\right|\\
(6)\quad&\leq\left|\sum_{i=1}^{n}\left\{\left(\frac{\partial x^i}{\partial u}\right)^2+\left(\frac{\partial y^i}{\partial u}\right)^2+\left(\frac{\partial x^i}{\partial v}\right)^2+\left(\frac{\partial y^i}{\partial v}\right)^2\right\}\right|^2-4\sum_{i=1}^{n}\left(\frac{\partial x^i}{\partial u}\frac{\partial y^i}{\partial v}-\frac{\partial x^i}{\partial v}\frac{\partial y^i}{\partial u}\right)^2\\
&=(g_{11}(u,v)+g_{22}(u,v))^2-4(g_{11}(u,v)g_{22}(u,v)-g_{12}(u,v)^2)\\
&=(g_{11}(u,v)-g_{22}(u,v))^2+4g_{12}(u,v)^2\,.
\end{aligned}
$$

If the correspondence of (u,v) onto S satisfying $1°, 2°$ is conformal, the right-hand side of (6) vanishes. Hence we must have

$$\frac{\partial z^i}{\partial w}=0 \quad\text{or}\quad \frac{\partial z^i}{\partial \overline{w}}=0, \qquad i=1,2,\cdots,n\,.$$

But the first case cannot occur on account of the orientation-preserving property of the mapping $(u,v)\mapsto S$ (cf. Remark 2 below) and $z^i(w)\,(i=1,2,\cdots,n)$ are holomorphic in a neighbourhood of $\operatorname{int} I$, i.e. γ is an analytic arc. q.e.d.

REMARK 2. Of course the simple C^r-smoothness or analyticity of arcs are local properties and the context of the theorem remains invariant under the positive orthogonal transformation of $\boldsymbol{R}^n$. So we may assume without losing the generality that every projection $z^i(\operatorname{loc}\gamma)$ $(i=1,2,\cdots,n)$ is a simple arc, that the sufficiently narrow surface-portion S is an embedding into $\boldsymbol{C}^n=\boldsymbol{R}^{2n}$ and that the complex coordinate $z^i(w)$ $(i=1,2,\cdots,n)$ has a Jacobian $|\partial z^i/\partial w|^2-|\partial z^i/\partial\overline{w}|^2$ of definite sign in some neighbourhood of $\operatorname{int} I$.

Within a similar circle of ideas we can show

Theorem 6. *Let S be an arbitrary differentiable surface-portion in $\boldsymbol{R}^n$. Then there exists a suitable complex surface-portion $\hat{S}$ with the following properties:*

$1°$ *$\hat{S}$ lies in $\boldsymbol{C}^n$;*

$2°$ *the projection of $\hat{S}$ into $\boldsymbol{R}^n$ just coincides with S;*

$3°$ *S admits an isothermal coordinate $w=u+\sqrt{-1}\,v$ at the point where it is totally differentiable if and only if the mapping $w\mapsto\hat{S}$ is derivable either in w or* *in $\overline{w}$ on its definition domain;*

4° *$w \mapsto S$ is injective if and only if $w \mapsto \hat{S}$ is injective.*

Proof. Suppose that S is represented by an equation $\boldsymbol{x}={}^t(x^1(u,v), x^2(u,v), \cdots, x^n(u,v))$ on a region B in the $w=u+\sqrt{-1}v$-plane. Of possibly various complexifications of our real surface S we are merely concerned with the simplest one at present, which serves us fairly well. Denoting by $(y^1, y^2, \cdots, y^n)$ a permutation of $(x^1, x^2, \cdots, x^n)$ such that $y^i \neq x^i$ for all $i=1,2,\cdots,n$, we set $z^i=x^i+\sqrt{-1}y^i$. Then we have a complex surface-portion $\hat{S}$ represented by the equation $z^i=z^i(u,v)$ $(i=1,2,\cdots,n)$ which satisfies the conditions 1°, 2°. At the point where $x^i(u,v)$ $(i=1,2,\cdots,n)$ are totally differentiable, we have

$$(7) \qquad 2\sum_{i=1}^{n}\frac{\partial z^i}{\partial w}\overline{\left(\frac{\partial z^i}{\partial \overline{w}}\right)} = g_{11}(u,v)-g_{22}(u,v)-2\sqrt{-1}\,g_{12}(u,v),$$

$$(8) \qquad 4\sum_{i=1}^{n}\left|\frac{\partial z^i}{\partial w}\right|^2\left|\frac{\partial z^i}{\partial \overline{w}}\right|^2 \leq (g_{11}(u,v)-g_{22}(u,v))^2+4g_{12}(u,v)^2$$

similarly to (5), (6), the components of fundamental forms at the right-hand side being referred to S. Assume the coordinate w to be isothermal for S. Then either $\partial z^i/\partial w$ or $\partial z^i/\partial \overline{w}$ must vanish identically for every $i=1,2,\cdots,n$ on account of (8). The converse is immediate from (7).

Next let ${}^t(x^1(w), x^2(w), \cdots, x^n(w))$ be injective but let $\boldsymbol{z}(w_1)=\boldsymbol{z}(w_2)$ for some parameter values w_1, w_2. Then we have $y^i(w_1)=y^i(w_2)$ $(i=1,2,\cdots,n)$ by definition and hence $x^i(w_1)=x^i(w_2)$ $(i=1,2,\cdots,n)$. Therefore $w_1=w_2$, showing the injectiveness of $\boldsymbol{z}(u,v)$, and vice versa. q.e.d.

Definition 7. If an isothermal coordinate w of the surface S satisfies $\partial \boldsymbol{z}/\partial \overline{w}=0$ for the coordinate $\boldsymbol{z}$ of some complexification $\hat{S}$ throughout the interior of its definition domain, we call $w \mapsto \hat{S}$ to be a *holomorphic extension* of S into $\boldsymbol{C}^n$.

3. Statement of the main theorem and quasiconformal continuation of a minimal surface

A vector-valued C^2-smooth function $\boldsymbol{x}(u,v)=\begin{pmatrix} x^1(u,v) \\ x^2(u,v) \\ \vdots \\ x^n(u,v)\end{pmatrix}$ is called to be a harmonic vector in a region B of (u,v)-plane, if each component $x^i(u,v)$ satisfies the Laplace differential equation

$$\frac{\partial^2 x^i}{\partial u^2}+\frac{\partial^2 x^i}{\partial v^2}=0, \quad i=1,2,\cdots,n$$

in B.

DEFINITION 8. (Courant [6], p. 100). A minimal surface spanning a given contour Γ is the differentiable surface with the boundary curve Γ, which is represented by a harmonic vector $\mathfrak{x}(u, v)$ and for which (u, v) is isothermal in the interior of its parameter domain.

In the present study we restrict ourselves to the minimal surfaces spanned by a Jordan contour. Consider a Jordan curve Γ: $\boldsymbol{x}=\mathfrak{x}(u, v)$ in $\boldsymbol{R}^n$, which shall be parametrized in such a way that, as the parameter w runs around the boundary of a domain, say B^+, counter-clockwise, the vector $\mathfrak{x}(w)$ describes monotonically the curve Γ exactly once: Γ is just the bounding frame of our minimal surface inquired. First we pose

Assumption I. The contour Γ spans at least one minimal surface S_0 without boundary branch points defined on B^+.

The above hypothesis should be taken for granted. Based on Assumption I, our reasoning will make full use of the results of the following statements henceforth:

1° there exists a vector-valued function $\boldsymbol{x}=\mathfrak{x}(w)$ which is harmonic in B^+ and is continuous on clo B^+;

2° the parameter w is isothermal with respect to the harmonic surface $S_0=\{\boldsymbol{x}=\mathfrak{x}(w) \mid w \in B^+\}$.

Now, we wish to notice below that it is not so hard but is rather natural to replace the *analyticity* of the subarc γ appearing in the aforesaid Courant's conjecture (or equivalently in our extension problem) by the C^3-regularity. So we pose

Assumption II. The contour Γ contains a non-singular simple open C^3-smooth subarc γ with the parameter interval int I.

Lemma 2 (Hildebrandt [11], Nitsche [12], pp. 306–312). *Under Assumptions I, II let $\mathfrak{x}(w)$ denote the harmonic vector in B^+ which spans the minimal surface S_0. Then $\mathfrak{x}(w)$ is of class C^2[int I].*

REMARK 3. Our stand-point is such that the word C^r-smoothness of arcs should be understood as the one we have fully discussed in introducing Definition 2. According to Gulliver-Spruck [10] (p. 331), however, the smoothness of Γ (accordingly, of γ) is originally meant by the smoothness induced from the minimal surface in question. If one adopts this definition, our assumption posed on γ in the following Theorem 7 can be weakened, at least formally, up to the regular C^2-smoothness.

Now let us announce our main result which we shall prove in the next section:

Theorem 7. *Under Assumptions I, II the contour Γ spans at least one* *minimal surface which is prolongable beyond the non-singular open subarc* γ *of* Γ *as a minimal surface.*

The main body of its proof will be preceded by some preparations concerning the boundary behaviour of harmonic functions as well as the dependence of solutions of an ordinary differential equations on the initial condition.

Lemma 3 (Zygmund [21], pp. 102–103). *Let* $U(w)$ $(w=\rho e^{\sqrt{-1}\theta})$ *be a real-valued function continuous on* $|w|\leq 1$ *and harmonic in* $|w|<1$ *such that* $\partial U(e^{\sqrt{-1}\theta})/\partial\theta$ *exists and is of class* $C^1[]\theta_1, \theta_2[]$ $(0\leq\theta_1<\theta_2<2\pi)$. *Then*

(a) $\lim_{\rho\to 1}\partial U(\rho e^{\sqrt{-1}\theta})/\partial\theta=\partial U(e^{\sqrt{-1}\theta})/\partial\theta$;

(b) $\lim_{\rho\to 1}\partial U(\rho e^{\sqrt{-1}\theta})/\partial\rho$ *exists and is continuous on* $]\theta_1, \theta_2[$.

The convergences are uniform on every closed subinterval of $]\theta_1, \theta_2[$.

Lemma 4 (Petrovski [14], pp. 96–97). *Let* $\boldsymbol{x}={}^t(x^1, \cdots, x^m)$ *be an m-vector and let an m-vector-valued function* $f(t,\boldsymbol{x})$ *together with its m partial derivatives* $\partial f(t,\boldsymbol{x})/\partial x^i$ $(i=1,2,\cdots,m)$ *be continuous in a product-subregion* $B=J\times B_0$ *of* $\boldsymbol{R}^{m+1}$, *where* B_0 *is a subregion of* $\boldsymbol{R}^m$ *and* $J=\{t\,|\,\alpha<t<\beta\}$. *Then to any point* $(\tau_0, \boldsymbol{a}_0)$ *of* B *there corresponds a constant* $q>0$ *such that for every* $(\tau,\boldsymbol{a})$ *in the open subinterval* $\{\tau\,|\,|\tau-\tau_0|<q\}\times\{\boldsymbol{a}\,|\,|\boldsymbol{a}-\boldsymbol{a}_0|<q\}$ *of* B *the solution* $\boldsymbol{x}=X(t;\tau,\boldsymbol{a})$ *of the differential equation*

$$\frac{d\boldsymbol{x}}{dt}=f(t, \boldsymbol{x}), \quad ((t, \boldsymbol{x})\in B) \tag{9}$$

satisfying the initial condition $X(\tau;\tau,\boldsymbol{a})=\boldsymbol{a}$, *known to exist uniquely, is differentiable in the variable* $\boldsymbol{a}={}^t(a^1, \cdots, a^m)$, *and further* $\partial X(t;\tau,\boldsymbol{a})/\partial a^i$ $(i=1,2,\cdots,m)$ *are continuous in* $J\times\{\tau\,|\,|\tau-\tau_0|<q\}\times\{\boldsymbol{a}\,|\,|\boldsymbol{a}-\boldsymbol{a}_0|<q\}$.

Taking an arbitrarily small positive number $\eta(<1)$, we fix it once and for all. Notice that the quantity $|\partial\mathfrak{x}/\partial u|$ has a positive minimum on the closed subinterval $I'=\{u\,|\,-1+\eta\leq u\leq 1-\eta\}$ of I. Let us denote by γ' the restriction of the arc γ to the parameter interval I'.

Theorem 8. *In a sufficiently small neighbourhood of* I' *the minimal surface* S_0 *is an embedding into* $\boldsymbol{R}^n$.

Proof. It suffices to show the following fact: there is a positive number $\delta(<1)$ such that the restriction of the mapping $w\mapsto\mathfrak{x}(w)$ to the subregion

$$B^{+\prime}=\{(u, v)\,|\,-1+\eta<u<1-\eta,\ 0<v<\delta\}$$

of B^+ is an injection.

Suppose, contrary to the assertion, there were a sequence $\{\delta_\nu\}_{\nu=1,2,\cdots}$ decreasing to zero such that the rectangle $B_\nu^{+\prime}=\{(u,v)\,|\,-1+\eta\leq u\leq 1-\eta,\ 0\leq v\leq\delta_\nu\}$

contains a pair of points w'_ν, w''_ν satisfying $\mathfrak{x}(w'_\nu)=\mathfrak{x}(w''_\nu)$. The formers cannot contain any respective subsequences with distinct limit points, since γ is simple. Hence they must cluster at a single point $w_0 \in I'$. On the other hand every i-th coordinate $x^i(w)$ of the vector $\mathfrak{x}(w)$ has at least one stationary point a^i on the segment connecting w'_ν with w''_ν ($i=1,2,\cdots,n$). For every i, a^i tends to w_0 as $\nu \to \infty$, since both w'_ν and w''_ν approach a single point w_0. It must follow that $\partial\mathfrak{x}/\partial u$ vanishes at w_0 (Lemma 3), contradicting the non-singularity assumption of $\mathfrak{x}$ on I'. q.e.d.

A. Let us denote by S_0^* the restriction of S_0 to the subregion $B^{+\prime}$ of B^+. Let the unit open disk in the $\rho e^{\sqrt{-1}\theta}$-plane ($0 \leq \rho \leq 1$, $\theta \in \boldsymbol{R}$) be a conformal image of B^+ and let the closed θ-interval $[\theta_1, \theta_2]$ come from I'. If we write $\mathfrak{y}(\rho, \theta)=\mathfrak{x}(w)$, both $\partial^2\mathfrak{y}(\rho, \theta)/\partial\theta^2$ and $\partial\mathfrak{y}(\rho, \theta)/\partial\rho$ are continuous for $\rho \in [0, 1]$, $\theta \in [\theta_1, \theta_2]$ (Lemma 3 (a), (b)), hence is $\partial^2\mathfrak{x}(\rho, \theta)/\partial\rho^2$ seen to tend to a finite limit as $\rho \to 1$ on $\theta_1 \leq \theta \leq \theta_2$ on account of the Laplace equation.

B. The normal directions to I' and to the closed circumferential arc $\widehat{e^{\sqrt{-1}\theta_1}, e^{\sqrt{-1}\theta_2}}$ corresponds to each other. Therefore $\partial^2\mathfrak{x}(u, v)/\partial v^2$ can be defined continuously up to $B^+ \cup I'$. Further $\partial^2\mathfrak{x}(u, v)/\partial v^2 \neq 0$ on I' for similar reasons. The non-vanishing continuous n-vector-valued function $\boldsymbol{b}(\boldsymbol{x})=[\partial^2\mathfrak{x}(u, v)/\partial v^2]_{w=\mathfrak{x}^{-1}(\boldsymbol{x})}$ is defined on some closed subset of $\operatorname{loc}\mathfrak{x}(\operatorname{clo} B^{+\prime})$ including $\operatorname{loc}\gamma'$ (Theorem 8). The Lebesgue's theorem allows us to extend $\boldsymbol{b}(\boldsymbol{x})$ continuously up to the whole space $\boldsymbol{R}^n$. There is an n-dimensional neighbourhood $N(\operatorname{loc}\gamma')$ of $\operatorname{loc}\gamma'$ in which the non-vanishing continuous vector field $\{\boldsymbol{b}(\boldsymbol{x})\}$ is defined. By solving the differential equation

$$\frac{d\boldsymbol{x}}{dv}=\boldsymbol{b}(\boldsymbol{x}), \quad (\boldsymbol{x} \in N(\operatorname{loc}\gamma'))$$

with the initial value $[\partial\mathfrak{x}(u, v)/\partial v]_{w=\mathfrak{x}^{-1}(\boldsymbol{x})}$ on $\operatorname{loc}\gamma'$, we have a vector field $\{\boldsymbol{p}(\boldsymbol{x})\}_{\boldsymbol{x}\in\operatorname{loc}\gamma'}$ in $N(\operatorname{loc}\gamma')$, which agrees with $[\partial\mathfrak{x}(u, v)/\partial v]_{w=\mathfrak{x}^{-1}(\boldsymbol{x})}$ on $\operatorname{loc} S_0 \cap N(\operatorname{loc}\gamma')$. It amounts to saying that the vector field $[\partial\mathfrak{x}(u, v)/\partial v]_{w=\mathfrak{x}^{-1}(\boldsymbol{x})}$ on $\operatorname{loc} S_0 \cap N(\operatorname{loc}\gamma')$ has been extended up to $N(\operatorname{loc}\gamma')$ in C^1-smooth manner.

C. There is an n-dimensional neighbourhood $N_1(\operatorname{loc}\gamma')$ of $\operatorname{loc}\gamma'$ in which $\boldsymbol{p}(\boldsymbol{x}) \neq 0$: for, $\boldsymbol{p}(\boldsymbol{x})=[\partial\mathfrak{x}(u, v)/\partial v]_{w=\mathfrak{x}^{-1}(\boldsymbol{x})} \neq 0$ for all $\boldsymbol{x} \in \operatorname{loc}\gamma'$, since otherwise $g_{11}(u, v)=g_{22}(u, v)=0$ somewhere on I', which contradicts the non-branching character of $\mathfrak{x}(w)$. Denoting the C^1-smooth components of the non-vanishing n-vector $\boldsymbol{p}(\boldsymbol{x})$ by $p^1(\boldsymbol{x}), p^2(\boldsymbol{x}), \cdots, p^n(\boldsymbol{x})$, we consider the quasilinear partial differential equation

$$\sum_{i=1}^{n-1} p^i(\boldsymbol{x}) \frac{\partial x^n}{\partial x^i}=p^n(\boldsymbol{x}) \tag{10}$$

as well as the system of ordinary differential equations

$$\frac{dx^1}{p^1(\boldsymbol{x})}=\frac{dx^2}{p^2(\boldsymbol{x})}=\cdots=\frac{dx^n}{p^n(\boldsymbol{x})}=dt\,. \tag{11}$$

We can solve (10) locally with the initial arc γ' to obtain a smooth surface S' comprised in the n-dimensional neighbourhood such that clo (loc S') $\cap$ clo (loc S_0)$=$loc γ'. Let $\boldsymbol{y}$ be an arbitrary point on loc γ' and let $C_v(\boldsymbol{y})$ the solution arc of (11) through $\boldsymbol{y}$. Then $C_v(\boldsymbol{y})$ intersects γ' orthogonally and is known to lie on S'. Let us denote by δ' the minimal length of the family of arcs $\{C_v(\boldsymbol{y})\}_{\boldsymbol{y}\in \mathrm{loc}\,\gamma'}$. It is evident that $\delta'>0$.

D. Now let us start from one end point $\boldsymbol{x}_1=\mathfrak{x}(-1+\eta)$ of γ' to proceed along the arc $C_v(\boldsymbol{x}_1)$ on loc S'. We stop just after the trip of length δ' at the determinate point $\boldsymbol{x}^*$. The subarc of $C_v(\boldsymbol{x}_1)$ with the extremities $\boldsymbol{x}_1$, $\boldsymbol{x}^*$ shall simply be denoted by C'.

Given any point $\boldsymbol{t}$ on loc C', a unique $(n-1)$-dimensional submanifold through $\boldsymbol{t}$ intersects the vector field $(p^1(\boldsymbol{x}), p^2(\boldsymbol{x}), \cdots, p^n(\boldsymbol{x}))$ orthogonally (cf. Shibata-Mohri [18], Theorem 3), whose meet with S' shall be denoted by $C_u(\boldsymbol{t})$. Denoting by $\boldsymbol{x}_2$ the other end-point $\mathfrak{x}(1-\eta)$ of γ', we write $\boldsymbol{x}^{**}=$loc $C_u(\boldsymbol{x}^*)\cap$ loc $C_v(\boldsymbol{x}_2)$. Thus we have obtained a subquadrilateral $\Omega(\boldsymbol{x}_1,\boldsymbol{x}^*,\boldsymbol{x}^{**},\boldsymbol{x}_2)$ of loc S' bounded by the four arcs γ', C', $C_u(\boldsymbol{x}^*)$ and $C_v(\boldsymbol{x}_2)$. We are going to represent the quadrilateral $\Omega(\boldsymbol{x}_1,\boldsymbol{x}^*,\boldsymbol{x}^{**},\boldsymbol{x}_2)$ by the parameter (u, v) ranging over the plane rectangle

$$B^{-\prime}=\{(u, v)\,|\,-1+\eta<u<1-\eta,\ -\delta'<v<0\}\,.$$

To this end, to any point $\boldsymbol{x}\in\Omega(\boldsymbol{x}_1, \boldsymbol{x}^*, \boldsymbol{x}^{**}, \boldsymbol{x}_2)$ we put the Cartesian coordinates (u, v) into correspondence in the following way:

1° u is the parameter value representing the point loc $C_v(\boldsymbol{x})\cap$ loc γ' as the one on the boundary subarc ∂S_0 of the original minimal surface S_0;

2° v is the length of the subarc of C' with the initial point $\boldsymbol{x}_1$ and the end-point loc $C_u(\boldsymbol{x})\cap$ loc C'.

Then the correspondence $\mathring{B}^{-\prime}\ni(u, v)\mapsto\boldsymbol{x}\in\Omega(\boldsymbol{x}_1,\boldsymbol{x}^*,\boldsymbol{x}^{**},\boldsymbol{x}_2)$, which we shall still denote by $\mathfrak{x}(u, v)$, is one-to-one. Lemmas 3, 4 and the compactness of clo $B^{-\prime}$, allows us to conclude that the components $g_{11}(u,v), g_{12}(u,v), g_{22}(u,v)$ of the fundamental form of the closed subsurface $\Omega(\boldsymbol{x}_1,\boldsymbol{x}^*,\boldsymbol{x}^{**},\boldsymbol{x}_2)$ with respect to those parameters (u,v) are bounded above on clo B'. From the construction of the parameters follows $g_{12}(u, v)=0$ and further we can see by interchanging the initial and variable points in Lemma 4 and Theorem 3 in [18] that $g_{11}(u, v)$, $g_{22}(u, v)$ have a positive lower bound on clo $B^{-\prime}$. Therefore the Jacobian $\sqrt{g_{11}(u, v)g_{22}(u, v)-g_{12}(u, v)^2}$ never vanishes and there exists a constant $K_0\geq 1$ satisfying

$$g_{11}(u, v)+g_{22}(u, v)\leq[(K_0^2+1)/K_0]\sqrt{g_{11}(u, v)g_{22}(u, v)-g_{12}(u, v)^2}$$

throughout $B^+ \cup I' \cup B^{-\prime}$. The extended surface $S_0 \cup \gamma' \cup S'$ has obviously a finite area, say V. We have proved

Proposition 1. *The minimal surface S_0 with the parameter domain B^+ is prolongable across the subarc γ' of γ up to $S_0 \cup \gamma' \cup S'$ in the following manner:*

1° *S' is represented injectively in the parameter domain $B^{-\prime}$ contiguous to B^+ on I';*

2° *the parametrization $B^+ \cup I' \cup B^{-\prime} \mapsto S_0 \cup \gamma' \cup S'$ is K-quasiconformal for some K $(\geq K_0)$.*

Hereupon let us make mention of a slight modification of the familiar modulus-estimate for plane quadrilaterals which was originated by Grötzsch and generalized later by Ahlfors (cf. Ahlfors [2], pp. 6–7):

Lemma 5. *Let $\zeta = \psi(w)$ be an orientation-preserving L^2-derivable homeomorphism of a rectangle $R = \{w \mid 0 < \operatorname{Re} w < a,\ 0 < \operatorname{Im} w < b\}$ onto an arbitrary plane curvilinear quadrilateral Ω satisfying $|\partial\psi/\partial w|^2 - |\partial\psi/\partial\overline{w}|^2 > 0$ almost everywhere on R. If $\psi(w)$ is absolutely continuous in 2-dimensional sense on R, we have*

$$\max\left\{\frac{\operatorname{Mod}\Omega}{\operatorname{Mod}R},\ \frac{\operatorname{Mod}R}{\operatorname{Mod}\Omega}\right\} \leq \frac{1}{2\sqrt{-1}ab}\iint_R \frac{|\partial\psi/\partial w| + |\partial\psi/\partial\overline{w}|}{|\partial\psi/\partial w| - |\partial\psi/\partial\overline{w}|}\, dw \wedge d\overline{w}\,. \tag{12}$$

Proof. Since both sides of the inequality (12) is invariant under conformal mappings of $\operatorname{int}\Omega$, it suffices to show the validity of (12) for the case in which Ω is a rectangle $R' = \{\zeta \mid 0 < \operatorname{Re}\zeta < a',\ 0 < \operatorname{Im}\zeta < b'\}$. Further we may assume $a'/b' \geq a/b$, since otherwise we have only to interchange the order of those coordinates. The dilatation-quotient of $\psi(w)$ is well defined almost everywhere in R. If it is not integrable over R, the conclusion takes place trivially. So we may assume that the right-hand side integral is finite. For almost all $v_0 \in [0, b]$ we have

$$\begin{aligned} a' &\leq \int_{u=0}^{a} |d\psi(u, v_0)| \leq \int_0^a \left(\left|\frac{\partial\psi}{\partial w}\right| + \left|\frac{\partial\psi}{\partial\overline{w}}\right| \right) du \\ &= \int_0^a \sqrt{\frac{|\partial\psi/\partial w| + |\partial\psi/\partial\overline{w}|}{|\partial\psi/\partial w| - |\partial\psi/\partial\overline{w}|}}\ \sqrt{|\partial\psi/\partial w|^2 - |\partial\psi/\partial\overline{w}|^2}\, du\,. \end{aligned}$$

Integration of both end sides over $0 \leq v \leq b$ yields

$$a'b \leq \iint_R \sqrt{\frac{|\partial\psi/\partial w| + |\partial\psi/\partial\overline{w}|}{|\partial\psi/\partial w| - |\partial\psi/\partial\overline{w}|}}\ \sqrt{|\partial\psi/\partial w|^2 - |\partial\psi/\partial\overline{w}|^2}\, du \wedge dv$$

and it follows by Schwarz's inequality that

$$a'^2 b^2 \leq a'b' \iint_R \frac{|\partial\psi/\partial w| + |\partial\psi/\partial\overline{w}|}{|\partial\psi/\partial w| - |\partial\psi/\partial\overline{w}|}\, du \wedge dv\,,$$

that is,

$$\frac{1}{ab}\iint_R \frac{|\partial\psi/\partial w|+|\partial\psi/\partial \overline{w}|}{|\partial\psi/\partial w|-|\partial\psi/\partial \overline{w}|}\,du\wedge dv \geq \frac{a'}{b'}:\frac{a}{b}\,.$$

q.e.d.

4. Prolonging the minimal surface

From now on we shall write B' for the region $B^+\cup \mathrm{int}\,\gamma'\cup B^{-\prime}$ for shortness' sake. This is nothing but the definition domain for the Dirichlet's functionals we intend to minimize.

Consider the family $\mathcal{X}=\{X_\lambda(w)\}_{\lambda\in\Lambda}$ of real n-vector-valued continuous functions on the closed region clo B' satisfying the conditions $1°\sim 7°$ described below:

$1°$ the mapping $\boldsymbol{x}=X_\lambda(w)$ provides a representation for a differentiable surface-portion Σ_λ with the parameter domain B';

$2°$ $X_\lambda(w)$ sends ∂B^+ onto Γ homeomorphically in the same sense as $\mathfrak{x}(w)$ did;

$3°$ $X_\lambda(I')=\mathrm{loc}\,\gamma'$;

$4°$ $X_\lambda(w)$ is injective on $B^{+\prime}\cup I'\cup B^{-\prime}$;

$5°$ the range $X_\lambda(\mathrm{clo}\,B^+\cup \mathrm{int}\,B^{-\prime})$ comprises the range $X_\lambda(\mathrm{clo}\,B^+)$ as a proper subset;

$6°$ there is a finite constant M such that $|X_\lambda(w)|\leq M$ on B' for all $\lambda\in\Lambda$;

$7°$ there exists the coordinate transformation $\omega=\phi_\lambda(w)$ of clo B' onto the closure of the unit disk $\Delta=\{\omega\,|\,|\omega|<1\}$ such that ω is the isothermal parameter for the surface-portion Σ_λ in Δ with the normalization $\phi_\lambda(0)=0$, $\phi_\lambda(1)=1$ admitting the holomorphic extension and that $\phi_\lambda(w)$ is a mean K-quasiconformal homeomorphism, namely

$$\iint_{B'}[D(w;\phi_\lambda)+(1/D(w;\phi_\lambda))]\,du\wedge dv\leq (K^2+1)[1+(1-\eta)\delta']/K\,,$$

$$\iint_{B'}[|\partial\phi_\lambda/\partial w|^2+|\partial\phi_\lambda/\partial\overline{w}|^2]\,du\wedge dv\leq \pi(K^2+1)/2K\,.$$

REMARK 4. One might assume beforehand that no $m\in\boldsymbol{Z}^+$ smaller than n exists satisfying loc $\Gamma\subset\boldsymbol{R}^m$. For otherwise, the concurrence functions could be restricted only to the m-vector-valued ones, because the integrand of the energy functional to be minimized in Proposition 3 satisfies trivially

$$\sum_{i=1}^{m}|\partial x^i/\partial w|^2\leq\sum_{i=1}^{n}|\partial x^i/\partial w|^2\,.$$

Proposition 2. *The prolonged surface $S_0\cup \mathit{int}\,\gamma'\cup S'$ admits not only an isothermal parameter but also a holomorphic extension.*

Proof. A. We solve the Beltrami differential equation $\partial\omega/\partial\overline{w}=h(w)\partial\omega/\partial w$

with the coefficient

$$h(w)$$
$$=\begin{cases}\dfrac{g_{11}+g_{22}+\sqrt{(g_{11}-g_{22})^2+4g_{12}^2}-2\sqrt{g_{11}g_{22}-g_{12}^2}}{g_{11}+g_{22}+\sqrt{(g_{11}-g_{22})^2+4g_{12}^2}+2\sqrt{g_{11}g_{22}-g_{12}^2}}\exp\left[\dfrac{\sqrt{-1}}{2}\operatorname{Tan}^{-1}\dfrac{2g_{12}}{g_{11}-g_{22}}\right] & \text{on } B^{-\prime}, \\ 0 & \text{elsewhere on } B'\end{cases}$$

to obtain the unique homeomorphism $\omega=\phi(w)$ of B' onto Δ normalized by $\phi(0)=0$, $\phi(1)=1$. This is the desired coordinate transformation such that ω is an isothermal parameter on Δ for the surface $S_0\cup\operatorname{int}\gamma'\cup S'$.

B. We shall show that the surface $\mathfrak{x}\circ\phi^{-1}(\omega)$ represented by the isothermal parameter ω extends holomorphically on Δ in the following way.

(a) Since $\mathfrak{x}\circ\phi^{-1}(\omega)$ is harmonic on the simply connected region $\phi(B^+)$, there exists a real-valued harmonic function $y^i=y^i(\omega)$ conjugate to the i-th coordinate $x^i=x^i(\omega)$ of the surface considered, which is determined up to an additive constant. Obviously $z^i=x^i+\sqrt{-1}\,y^i$ satisfies $\partial z^i/\partial\bar{\omega}=0$ on $\phi(B^+)$ $(i=1,2,\cdots,n)$.

(b) On the other hand, however, we don't know yet about the harmonicity of $x^i(\omega)$ off $\phi(B^+)$. So we cannot but complexify the questioned real surface-portion by utilizing only the isothermal character of $x^i(\omega)$ in a neighbourhood of the border $\phi(I')$ $(i=1,2,\cdots,n)$ (Theorem 6). But in order to visualize those circumstances more vividly, we give first a little detailed illustrations for the simplest case $n=3$.

The individual complex coordinate z^i $(i=1,2,3)$ introduced in Theorem 6 amounts to nothing but the projection of the surface into the respective coordinate-plane regarded as $\boldsymbol{C}$ in the space $\boldsymbol{R}^3$. Keeping the simplicity of γ' in mind, let us concern ourselves only with a restriction of the surface to a 3-dimensional neighbourhood of an interior point to loc γ'. According to Theorem 6, $x^1(\omega)+\sqrt{-1}\,x^2(\omega)$ is either holomorphic or anti-holomorphic.

In the first case the projection map loc $(S_0\cup\operatorname{int}\gamma'\cup S')\mapsto z^1=x^1+\sqrt{-1}\,x^2$ is sense-preserving. Then $x^2+\sqrt{-1}\,x^3$ is not anti-holomorphic, since otherwise, $x^3+\sqrt{-1}\,x^2$ must be holomorphic, accordingly $x^3-x^1=$const. identically, which cannot occur in view of Remark 4. Hence $x^2+\sqrt{-1}\,x^3$ is holomorphic. Analogously $x^3+\sqrt{-1}\,x^1$ is holomorphic, because not anti-holomorphic.

If, on the contrary, $x^1(\omega)+\sqrt{-1}\,x^2(\omega)$ is anti-holomorphic, $x^2+\sqrt{-1}\,x^1$, $x^3+\sqrt{-1}\,x^2$, $x^1+\sqrt{-1}\,x^3$ are holomorphic.

(c) Let w_0 be an arbitrary point in I' and let a disk-neighbourhood, say ρ-neighbourhood, of w_0 be denoted by $N_\rho(w_0)$. Write ω_0 for $\phi(w_0)$. As was seen in (b), to any $x^i(\omega_0)$ $(i=1,2,\cdots,n)$ there corresponds a unique $y^{i'}(\omega_0)$ of a permutation $y^{1'},y^{2'},\cdots,y^{n'}$ of $x^1,x^2,\cdots,x^n$ anyhow, such that $x^i(\omega_0)+\sqrt{-1}\,y^{i'}(\omega_0)$ is holomorphic. The inverse map of the homomorphism $x^1,x^2,\cdots,x^n\mapsto y^{1'},y^{2'},\cdots,y^{n'}$ is also one-valued by the same reason, hence isomorphic. If $\rho>0$ is sufficiently small, the above circumstances occur for all ω of $\phi(N_\rho(w_0))$, because γ' is simple. The closed interval I' is covered by a finite number of such neighbourhoods $\{N_\rho(w_0)\}_{w_0\in I'}$. Therefore $z^{i\prime}(\omega)=x^i(\omega)+\sqrt{-1}\,y^{i\prime}(\omega)$ is holomorphic on $\phi(B^{-\prime}\cup\operatorname{int} I')$ so far as we choose $\delta'>0$ sufficiently small in advance.

(d) Since $\operatorname{Re} z^i(\omega)=\operatorname{Re} z^{i\prime}(\omega)=x^i(\omega)$ for all $\omega\in\phi(N(I')\cup B^+)$, $z^{i\prime}(\omega)$ turns out the unique holomorphic continuation of $z^i(\omega)$ on Δ. q.e.d.

Proposition 3. *The family $\mathfrak{X}$ contains at least one $X(w)$ for which the energy integral*

$$E[X(w)] = 2\iint_{B'}\left|\frac{\partial X}{\partial w}\right|^2 du\wedge dv = \frac{1}{2}\iint_{B'}[(g_{11}(u, v)+g_{22}(u, v)]du\wedge dv$$

over B' is finite. Every minimizing sequence for this functional on $\mathfrak{X}$ constitutes a normal family on clo B' *and is compact in the topology of uniform convergence on* clo B'.

Proof. $\mathfrak{X}$ is non-void. In fact, the quasiconformal representation $\boldsymbol{x}=\mathfrak{x}(w)$ of B' onto $S_0\cup\operatorname{int}\Gamma'\cup S'$ whose existence has just been established by Proposition 1 fulfills evidently the above conditions $1°\sim6°$. As to the isothermal coordinate $\omega=\phi(w)$ of the surface $S_0\cup\gamma'\cup S'$, the composite mapping $Y(\omega)=\mathfrak{x}\circ\phi^{-1}(\omega)$ induces a holomorphic extension (Proposition 2). Furthermore $\boldsymbol{x}=\mathfrak{x}(w)$ fulfills $E[\mathfrak{x}(w)]\leq(K_0^2+1)V/K_0$.

(a) The sequence of holomorphic extensions induced from the minimizing sequence for $E[X(w)]$ is equicontinuous on clo Δ; suppose the contrary. It contains a sequence $\{Z_\nu(\omega)\}_{\nu=1,2,\dots}$ such that $|Z_\nu(\omega'_\nu)-Z_\nu(\omega''_\nu)|\geq c$ for some point-sequences $\{\omega'_\nu\}_{\nu=1,2,\cdots}$, $\{\omega''_\nu\}_{\nu=1,2,\cdots}$ on clo Δ and with a positive constant c. We lose no generality in assuming $\lim_{\nu\to\infty}\omega'_\nu=\lim_{\nu\to\infty}\omega''_\nu=\omega_0\in$ clo Δ. To any small $\varepsilon>0$ there corresponds a $\nu_0=\nu_0(\varepsilon)\in\boldsymbol{Z}^+$ such that $|\omega'_\nu-\omega''_\nu|<\varepsilon$ so far as $\nu\geq\nu_0$. Fix such a ν for a moment. At least one coordinate-index, say i, fulfills the inequality

$$(13)\qquad |z^i_\nu(\omega'_\nu)-z^i(\omega''_\nu)|\geq c/n\,.$$

Though the number i varies with $\nu\to\infty$ in general, we may assume, by choosing a suitable subsequence of $\nu=\nu_0,\nu_0+1,\cdots$ if necessary, that (13) holds good for all sufficiently large index ν. Let $\kappa(\rho)$ denote the circumferential subarc of $\{\omega\,|\,|\omega-\omega_0|=\rho\}$ comprised in Δ $(\varepsilon<\rho<1)$. Since

$$\int_{\kappa(\rho)}|dz^i_\nu(\omega)|\geq c/n\qquad\text{for every}\quad\rho\in[\varepsilon, 1]\,,$$

we have

$$(c/n)^2\leq\left(\int_0^{2\pi}\left(\left|\frac{\partial z^i_\nu}{\partial\omega}\right|+\left|\frac{\partial z^i_\nu}{\partial\overline{\omega}}\right|\right)\rho d\theta\right)^2\leq 4\pi\rho\int_0^{2\pi}\left(\left|\frac{\partial z^i_\nu}{\partial\omega}\right|^2+\left|\frac{\partial z^i_\nu}{\partial\overline{\omega}}\right|^2\right)\rho d\theta\,,$$

hence

$$(c^2/8n\pi)\log(1/\varepsilon)\leq V\,,$$

which is absurd.

(b) $\{\phi_\lambda(w)\}_{\lambda\in\Lambda}$ is equicontinuous on clo B'. Suppose, on the contrary, there were sequences $\{w'_\nu\}_{\nu=1,2,\dots}$, $\{w''_\nu\}_{\nu=1,2,\dots}\subset$ clo B' and $\{\phi_\nu(w)\}_{\nu=1,2,\dots}$ such that $\lim_{\nu\to\infty} w'_\nu=\lim_{\nu\to\infty} w''_\nu=w_0$, while $|\phi_\nu(w'_\nu)-\phi_\nu(w''_\nu)|\geq c>0$. Set $\omega'_\nu=\phi_\nu(w'_\nu)$, $\omega''_\nu=\phi_\nu(w''_\nu)$. To any (but smaller than $\min\{|w_0|, \operatorname{dist}(w_0, \partial B')\}$ if $w_0\neq 0$) positive ε there corresponds a $\nu_0\in \boldsymbol{Z}^+$ such that $|w'_\nu-w_0|<\varepsilon$, $|w''_\nu-w_0|<\varepsilon$ for all $\nu\geq\nu_0$. Let $\kappa(\rho)$ denote the subarc of the circumference $|w-w_0|=\rho$ $(>\varepsilon)$ lying in B'. Let ρ vary over the interval $[\varepsilon,\rho_0]$, where $\rho_0=\max\{|w_0|, \operatorname{dist}(w_0,\partial B^+)\}$. Then the diameter of $\phi_\nu(\kappa(\rho))$ is not smaller than c. From

$$c^2\leq\left(\int_{\kappa(\rho)}|d\phi_\nu(w)|\right)^2\leq 4\pi\rho\int_0^{2\pi}\left(\left|\frac{\partial\phi_\nu}{\partial w}\right|^2+\left|\frac{\partial\phi_\nu}{\partial\bar{w}}\right|^2\right)\rho d\theta$$

(Lemma 1) follows

$$\frac{c^2}{4\pi}\int_\varepsilon^{\rho_0}\frac{d\rho}{\rho}\leq\frac{\pi}{2K}(K^2+1)$$

(cond. 7°), which is absurd.

(c) The minimizing sequence for the functional $E[X(w)]$ defined on the space $\mathscr{X}$ is equicontinuous on clo B' ((a), (b)), hence a normal family (cond. 6°), namely it contains a subsequence $\{X_\nu(w)\}_{\nu=1,2,\dots}$ uniformly convergent on clo B'.

(d) Set $\Xi(w)=\lim_{\nu\to\infty} X_\nu(w)$ on clo B'. Then $\Xi(w)$ is one-valued continuous mapping of clo B' into $\boldsymbol{R}^n$ and has the properties 2°, 3° postulated in this proposition. We show the injectiveness of $\Xi(w)$ in the neighbourhood of I'. Denote the normalized isothermal coordinate of $X_\nu(w)$ by $\phi_\nu(w)$ and the holomorphic extension of $Y_\nu(\omega)=X_\nu\circ\phi_\nu^{-1}(\omega)$ by $Z_\nu(\omega)$. Let $\{\nu_k\}_{k=1,2,\dots}$ be a sequence of indices such that $\{\phi_{\nu_k}(w)\}_{k=1,2,\dots}$ and $\{Z_{\nu_k}(\omega)\}_{k=1,2,\dots}$ converges uniformly on clo B' and clo Δ respectively ((a), (b)). Set $\phi(w)=\lim_{k\to\infty}\phi_{\nu_k}(w)$, $Z(\omega)=\lim_{k\to\infty} Z_{\nu_k}(\omega)$. By virtue of uniform convergence of $\{\phi_{\nu_k}(w)\}_{k=1,2,\dots}$ on clo B' the limit $\phi(w)$ never sends $\partial B'$ to int Δ, i.e., ϕ is a surjection of clo B' onto clo Δ. Next suppose $\phi(w)$ takes the same value ω_0 at two distinct points $w', w''\in$ clo B'. Given any small $\varepsilon>0$, there is a ν_0 such that $|\phi_{\nu_k}(w')-\omega_0|<\varepsilon$, $|\phi_{\nu_k}(w'')-\omega_0|<\varepsilon$ if $k\geq\nu_0$. Describe the circle centred at ω_0 with radius ρ, where $\rho<|\omega_0|$ or $\rho<1$ according as $\omega_0\neq 0$ or $\omega_0=0$. In the same way as in (b) we have

$$|w'-w''|^2\leq\left(\int_{|\omega-\omega_0|=\rho}|d\phi_{\nu_k}^{-1}(\omega)|\right)^2$$

$$\leq\left(\int_{|\omega-\omega_0|=\rho}(|\partial\phi_{\nu_k}^{-1}/\partial\omega|+|\partial\phi_{\nu_k}^{-1}/\partial\bar{\omega}|)\rho d\theta\right)^2 \tag{14}$$

$$\leq 4\pi\rho\int_0^{2\pi}(|\partial\phi_{\nu_k}^{-1}/\partial\omega|^2+|\partial\phi_{\nu_k}^{-1}/\partial\overline{\omega}|^2)\rho d\theta \,.$$

Hence

$$\frac{|w'-w''|^2}{2\pi}\int_\varepsilon \frac{d\rho}{\rho} \leq \iint_{B'}[D(w;\,\phi_{\nu_k})+(1/D(w;\,\phi_{\nu_k}))]du\wedge dv \tag{15}$$
$$\leq (K^2+1)\,[1+(1-\eta)\delta']/K$$

(cond. 7°), which is absurd. Therefore $\phi(w)$ is a homeomorphism of $\operatorname{clo} B'$ onto $\operatorname{clo}\Delta$. Since $Z(\omega)$ is injective on $\phi(B^{+\prime})$ by Hurwitz's theorem, $\Xi(w)=\lim_{k\to\infty}X_{\nu_k}(w)$ $=\lim_{k\to\infty}Z_{\nu_k}\circ\phi_{\nu_k}(w)=Z\circ\phi(w)$ is also injective on $B^{+\prime}$.

(e)[1] Henceforth we write simply ν in place of the index ν_k. Recall that $\phi_\nu(w)$ shares the ACL-property with $X_\nu(w)$ (cf. Bers [5]). Take an arbitray C^∞-function $T(w)$ supported by B'. Then it follows from the definition of L^2-derivatives that

$$\iint_{B'}[\phi_\nu(w)-\phi_{\nu+k}(w)]\,\frac{\partial T}{\partial w}\,du\wedge dv$$
$$=-\iint_{B'}\left[\frac{\partial\phi_\nu}{\partial w}-\frac{\partial\phi_{\nu+k}}{\partial w}\right]T(w)\,du\wedge dv\,.$$

Since $\{\phi_\nu(w)\}_{\nu=1,2,\dots}$ is a Cauchy sequence in the topology of uniform convergence on $\operatorname{clo} B'$, the sequence $\{\iint_{B'}(\partial\phi_\nu/\partial w)\,T(w)\,du\wedge dv\}_{\nu=1,2,\dots}$ of linear functionals in T is fundamental in the space $C_0^\infty[B']$. The limit

$$\Psi[T]=\lim_{\nu\to\infty}\iint_{B'}\frac{\partial\phi_\nu}{\partial w}\,T(w)\,du\wedge dv$$

is bounded in the space $L^2[B']$. Riesz's theorem ensures an inner product representation for the limiting functional in T with some element of $L^2[B']$, which we denote by $\partial\phi/\partial w$, namely $\Psi[T]=\langle\partial\phi/\partial w,\,T\rangle$. Putting the identity $\langle\phi_\nu,\,\partial T/\partial w\rangle=-\langle\partial\phi_\nu/\partial w,\,T\rangle$ into the relations

$$\lim_{\nu\to\infty}\langle\phi-\phi_\nu,\,\partial T/\partial w\rangle=0=-\lim_{\nu\to\infty}\langle(\partial\phi/\partial w)-(\partial\phi_\nu/\partial w),\,T\rangle\,,$$

we get

$$\langle\phi,\,\partial T/\partial w\rangle=-\langle\partial\phi/\partial w,\,T\rangle\,.$$

In the same way we can see

$$\langle\phi,\,\partial T/\partial\overline{w}\rangle=-\langle\partial\phi/\partial\overline{w},\,T\rangle$$

1) In this subsection we quote from Ahlfors [1] but without any quasiconformality assumption at all.

to hold, hence ϕ is L^2-derivable and its L^2-derivatives are $\partial\phi/\partial w$, $\partial\phi/\partial\overline{w}$ above introduced.

Next given any $\varepsilon>0$, there is a $\nu_0=\nu_0(\varepsilon)$, such that $|\phi_{\nu+k}(w)-\phi_\nu(w)|<\varepsilon$ everywhere on $\operatorname{clo} B'$ for all $\nu\geq\nu_0$ and $k=1,2,\cdots$. Let $\kappa(\rho_0)$: $|w-w_0|\leq\rho_0$ be an arbitrary disk comprised in B'. Consider the curvilinear integral

$$J_{\nu,k}(\rho)=\int_{\partial\kappa(\rho)}[\phi_{\nu+k}(w)-\phi_\nu(w)]d[\overline{\phi_{\nu+k}(w)-\phi_\nu(w)}],\quad (0\leq\rho\leq\rho_0)$$

along the circumference, which is well defined by virtue of Lemma 1. Applying the Schwarz's inequality to the estimate

$$|J_{\nu,k}(\rho)|\leq\varepsilon\int_{\partial\kappa(\rho)}\left(\left|\frac{\partial\phi_\nu}{\partial w}\right|+\left|\frac{\partial\phi_\nu}{\partial\overline{w}}\right|+\left|\frac{\partial\phi_{\nu+k}}{\partial w}\right|+\left|\frac{\partial\phi_{\nu+k}}{\partial\overline{w}}\right|\right)\rho\, d\arg w,$$

we see

$$\int_0^{\rho_0}[J_{\nu,k}(\rho)]^2\,d\rho\leq 4\varepsilon^2\pi^2\rho_0^2(K^2+1)/K\,,$$

hence

$$\lim_{\nu\to\infty}[\lim_{k\to\infty}J_{\nu,k}(\rho)]=0 \tag{16}$$

for almost all $\rho\in[0,\rho_0]$. Fix such a ρ at will. On the other hand the Green's theorem yields

$$\begin{aligned}2\sqrt{-1}\,J_{\nu,k}(\rho)&=\iint_{\kappa(\rho)}\left(\left|\frac{\partial\phi_\nu}{\partial w}\right|^2-\left|\frac{\partial\phi_\nu}{\partial\overline{w}}\right|^2\right)du\wedge dv\\&+\iint_{\kappa(\rho)}\left(\left|\frac{\partial\phi_{\nu+k}}{\partial w}\right|^2-\left|\frac{\partial\phi_{\nu+k}}{\partial\overline{w}}\right|^2\right)du\wedge dv-2\mathrm{Re}\iint_{\kappa(\rho)}\left[\frac{\partial\phi_\nu}{\partial w}\left(\overline{\frac{\partial\phi_{\nu+k}}{\partial w}}\right)-\frac{\partial\phi_\nu}{\partial\overline{w}}\left(\overline{\frac{\partial\phi_{\nu+k}}{\partial\overline{w}}}\right)\right]du\wedge dv\,.\end{aligned}\tag{17}$$

Let $k\to\infty$ first then let $\nu\to\infty$. The first and the second terms in the right-hand side of (17) are equal to $\operatorname{mes}\phi_\nu(\kappa(\rho))$ and $\operatorname{mes}\phi_{\nu+k}(\kappa(\rho))$ respectively, both tending to $\operatorname{mes}\phi(\kappa(\rho))$, while the third approaches twice of the double integral of $|\partial\phi/\partial w|^2-|\partial\phi/\partial\overline{w}|^2$ over $\kappa(\rho)$ owing to the weak convergence of those derivatives considered. We conclude from (16) that

$$\operatorname{mes}\phi(\kappa(\rho))=\iint_{\kappa(\rho)}\left(\left|\frac{\partial\phi}{\partial w}\right|^2-\left|\frac{\partial\phi}{\partial\overline{w}}\right|^2\right)du\wedge dv\,,$$

which is valid for every $\rho\in[0,\rho_0]$ by continuity of both sides. It follows that every Borel subset e of Δ has the measure expressed by the integral of the Jacobian of $\phi(w)$ over $\phi^{-1}(e)$, which shows the absolute continuity of ϕ.

(f) We can assert that $\{\phi_\nu^{-1}(\omega)\}_{\nu=1,2,\cdots}$ is the normal family on $\operatorname{clo}\Delta$ referring to the absurdities (14), (15) in (d). Moreover, $\{\phi_\nu^{-1}(\omega)\}_{\nu=1,,2,\cdots}$ itself is seen to converge, without mentioning any subsequences, uniformly on clo Δ towards $\phi^{-1}(\omega)$ by virtue of their injectiveness. The partial derivatives $\partial\phi_\nu^{-1}/\partial\overline{\omega}=-(\partial\phi_\nu/\partial\overline{w})/(|\partial\phi_\nu/\partial w|^2-|\partial\phi_\nu/\partial\overline{w}|^2)$ and $\partial\phi_\nu^{-1}/\partial\omega=\overline{(\partial\phi_\nu/\partial w)}/(|\partial\phi_\nu/\partial w|^2-|\partial\phi_\nu/\partial\overline{w}|^2)$ ($\nu=1,2,\cdots$), defined almost everywhere on Δ, satisfies together with an arbitrary function $T(\omega)\in C_0^\infty[\Delta]$ the relation

$$\lim_{\nu\to\infty}\langle(\partial(\phi_\nu^{-1}-\phi_{\nu+k}^{-1})/\partial\omega),\ T\rangle=0 \qquad (k=1,2,\cdots).$$

Hence follows the L^2-derivability, absolute continuity of $\phi^{-1}(\omega)$ and the weak convergence $\partial\phi_\nu^{-1}/\partial\omega\to\partial\phi^{-1}/\partial\omega$, $\partial\phi_\nu^{-1}/\partial\overline{\omega}\to\partial\phi^{-1}/\partial\overline{\omega}$ ($\nu\to\infty$) in quite the same way as in (e). Therefore $\phi(w)$ turns out an isothermal parameter of the differentiable surface represented by $\boldsymbol{x}=\Xi(w)$.

(g) Notice that Lemma 5 holds good in the fashion

$$\text{(18)} \qquad \max\left\{\frac{\operatorname{Mod} X_\lambda(\dot{B}')}{\operatorname{Mod}\dot{B}'},\ \frac{\operatorname{Mod}\dot{B}'}{\operatorname{Mod} X_\lambda(\dot{B}')}\right\}\leq\frac{1}{\operatorname{mes} B'}\iint_{B'}D(w;X_\lambda)\,du\wedge dv$$

for all $X_\lambda\in\mathfrak{X}$, since both the moduli of quadrilaterals and the dilatations of mappings are conformally invariant.

For shortness' sake we prefer hereby the abbreviations below for some specific points in $\boldsymbol{R}^n$, namely, the image-points of the corners of $B^{-\prime}$:

$$X_\lambda(-1+\eta)=\mathrm{O}_1,\ X_\lambda(1-\eta)=\mathrm{O}_2,$$
$$X_\lambda(-1+\eta-\sqrt{-1}\,\delta')=\mathrm{Q}_1,\ X_\lambda(1-\eta-\sqrt{-1}\,\delta')=\mathrm{Q}_2.$$

Further we mean by saying simply 'distance' the one on the subsurface loc $X_\lambda(B')$ for a moment. Any two of the four vertices $\mathrm{O}_j, \mathrm{Q}_j$ ($j=1,2$) of the quadrilateral loc $X_\lambda(\dot{B}^{-\prime})$ stand away at a distance with some positive lower bound for every $\lambda\in\Lambda$. For otherwise, suppose that only one pair of them could approach each other, say, O_1 and Q_1 for example, while the others not. Regarding the Jordan region B' as a quadrilateral $\dot{B}'$ with vertices $-1+\eta$, $-1+\eta-\sqrt{-1}\,\delta'$, $1-\eta$, $1+\sqrt{-1}$ and applying (18) to $\dot{B}'$, we arrive at an absurdity contrary to the condition 7°. On the other hand under the assumption that three of those four vertices might happen to approach simultaneously, say $\mathrm{O}_1, \mathrm{Q}_1, \mathrm{Q}_2$ for example, application of the same lemma to the quadrilateral $\dot{B}'$ with vertices at $-1+\eta$, $1-\eta-\sqrt{-1}\,\delta'$, $1+\sqrt{-1}$, $-1+\sqrt{-1}$ leads to a similar contradiction.

For any fixed λ, the shortest distance d between a pair of opposite sides $\widehat{\mathrm{O}_1, \mathrm{O}_2}$ and $\widehat{\mathrm{Q}_1, \mathrm{Q}_2}$ of the quadrilateral $X_\lambda(\dot{B}^{-\prime})$ is attained by the one of some point $\mathrm{P}_1\in\widehat{\mathrm{O}_1, \mathrm{O}_2}$ to some point $\mathrm{P}_2\in\widehat{\mathrm{Q}_1, \mathrm{Q}_2}$. When λ varies, we see first that P_1 approaches neither O_1 nor O_2 and that P_2 approaches neither Q_1 nor Q_2 with the aid of the super-additivity of modulus and the condition 7°. Therefore $\inf d=0$ would yield again the same contradiction.

(h) The space of L^2-derivatives is weakly compact (cf. Akhiezer-Glazman

[4], pp. 46–47):

$$\iint_{B'}(|\partial\phi/\partial w|^2+|\partial\phi/\partial\overline{w}|^2)\,du\wedge dv$$
$$\leq \liminf_{\nu\to\infty}\iint_{B'}(|\partial\phi_\nu/\partial w|^2+|\partial\phi_\nu/\partial\overline{w}|^2)\,du\wedge dv\,,$$

(19)
$$\iint_{B'}[D(w;\phi)+(1/D(w;\phi))]\,du\wedge dv$$
$$=\frac{\sqrt{-1}}{2}\iint_{B'}(|\partial\phi^{-1}/\partial\omega|^2+|\partial\phi^{-1}/\partial\overline{\omega}|^2)\,d\omega\wedge d\overline{\omega}$$
$$\leq \liminf_{\nu\to\infty}\frac{\sqrt{-1}}{2}\iint_{B'}(|\partial\phi_\nu^{-1}/\partial\omega|^2+|\partial\phi_\nu^{-1}/\partial\overline{\omega}|^2)\,d\omega\wedge d\overline{\omega}$$
$$=\liminf_{\nu\to\infty}\iint_{B'}[D(w;\phi_\nu)+(1/D(w;\phi_\nu))]\,du\wedge dv\,,$$

hence the isothermal coordinate $\phi(w)$ of $\Xi(w)$ also satisfies the condition 7°.

Since $Z(\omega)$ was holomorphic in Δ ((d)), $\boldsymbol{x}=\Xi(w)=Z\circ\phi(w)$ turns out to be one of the representations of a certain differentiable surface with contour defined on clo B' satisfying the conditions 1°$\sim$7° and the proof of the proposition is completed.

If we denote by Σ_0' the differentiable surface represented by $\boldsymbol{x}=\Xi(w)$ on clo B', we have

Proposition 4. *The family $\mathfrak{X}$ contains at least one mapping $\boldsymbol{x}=\Xi(w)$ which minimizes the functional $E[X(w)]$ within $\mathfrak{X}$. $\boldsymbol{x}=\Xi(w)$ provides one of the parametric representations of a certain differentiable surface Σ_0' with contour.*

The proof is immediate in view of the weak compactness of $L^2[B']$ again.

Now, in broad terms, a minimal surface is characterized by the harmonicity of the surface with respect to an isothermal parameter. The thing well known but of some interest hereof is that the limiting surface above constructed, the solution to the minimum problem for the Dirichlet integral, reveals a kind of holomorphy of the parametrization automatically (cf. e.g., Courant [6], pp. 105–107). It seems to come from the simple connectivity of the parameter domain and we propose an alternative process showing the holomorphy before the harmonicity in a series of propositions as follows:

Proposition 5. *The dilatation-quotient $D(w;\phi)$ of the coordinate-transformation $\omega=\phi(w)$ of the limiting map $\boldsymbol{x}=\Xi(w)$ is not only finite but also equal to 1 almost everywhere on B'.*

Proof. The derivatives of the function $\omega=\phi(w)$ are finite almost everywhere on B' and $\phi^{-1}(\omega)$ is a measurable mapping (Proposition 3 (e), (f)). Hence $D(w; \phi) < +\infty$ almost everywhere on B'.

Next suppose impossibly there were a subset e of B' of positive measure on which $D(w; \phi) \geq 1+\varepsilon$ holds almost everywhere with some constant $\varepsilon > 0$. The Beltrami coefficient $\mu(w) = (\partial\phi/\partial\overline{w})/(\partial\phi/\partial w)$ vanishes nowhere on e. Let $\operatorname{Im}\zeta > 0$ be a uniquely determined conformal map of B' by means of the holomorphic injection $\zeta = F(w)$ with the normalization $F(-1)=0$, $F(-1+\eta)=1$, $F(1-\eta)=\infty$. Let $h(\zeta)$ be a complex-valued measurable function in $\operatorname{Im}\zeta > 0$ such that

$$\begin{cases} \left.\begin{aligned} &\arg h(\zeta) = \arg \mu\circ F^{-1}(\zeta) + (\pi/2), \quad (\mathrm{mod}\ 2\pi) \\ &0 < |h(\zeta)| < |\mu\circ F^{-1}(\zeta)| \end{aligned}\right\} & (\zeta \in F(e)), \\ h(\zeta) = 0 & \text{elsewhere.} \end{cases}$$

There exists a unique quasiconformal homeomoprhism $G(\zeta)$ of $\operatorname{Im}\zeta > 0$ onto itself satisfying $\partial G/\partial\overline{\zeta} = h(\zeta)\partial G/\partial\zeta$ and leaving the three points $0, 1, \infty$ fixed. If we set $\tilde{\Xi}(w) = \Xi\circ F^{-1}\circ G\circ F(w)$, we have

$$\begin{aligned} &D(w; \tilde{\Xi}) < D(w; \Xi), \quad (w \in e) \\ &D(w; \tilde{\Xi}) = D(w; \Xi) \quad \text{elsewhere on } B', \end{aligned}$$

or equivalently

$$\begin{aligned} &D(\boldsymbol{x}; \tilde{\Xi}^{-1}) < D(\boldsymbol{x}; \Xi^{-1}), \quad (\boldsymbol{x} \in \Xi(e)) \\ &D(\boldsymbol{x}; \tilde{\Xi}^{-1}) = D(\boldsymbol{x}; \Xi^{-1}) \quad \text{elsewhere on } \operatorname{loc}\Sigma_0'. \end{aligned}$$

Hence

$$\begin{aligned} 2E[\tilde{\Xi}] &= \int_{\Sigma_0'} [D(\boldsymbol{x}; \tilde{\Xi}^{-1}) + (1/D(\boldsymbol{x}; \tilde{\Xi}^{-1}))]\, d\sigma \\ &< \int_{\Sigma_0'} [D(\boldsymbol{x}; \Xi^{-1}) + (1/D(\boldsymbol{x}; \Xi^{-1}))]\, d\sigma = 2E[\Xi] \end{aligned}$$

($d\sigma$ being the area-element of Σ_0'), which contradicts the minimality of $E[\Xi]$. q.e.d.

Proposition 6. *The original parameter w itself is isothermal for the limiting surface Σ_0' on B'.*

Proof. Since $\partial\phi/\partial\overline{w} = 0$ almost everywhere on B' (Proposition 5), $\phi(w)$ is holomorphic on B' by Morera's theorem. Therefore $Z\circ\phi(w)$ is holomorphic on B' too. The conclusion follows immediately from Theorem 6.

Corollary 1.

$$\iint_{B'} [D(w; \phi) + (1/D(w; \phi))]\, du \wedge dv < (K^2+1)\,[1+(1-\eta)\delta']/K,$$

$$\iint_{B'}[|\partial\phi/\partial w|^2+|\partial\phi/\partial\overline{w}|^2]du\wedge dv<\pi(K^2+1)/2K.$$

Corollary 2. *$x=\Xi(w)$ belongs to $C^\infty[B']$.*

Proposition 7[2]**.** *$x=\Xi(w)$ is harmonic on B'.*

Proof. Let $h(w)$ denote an arbitrary complex-valued C_0^∞-function supported by a compact set $\kappa\subset B'$. From the $h(w)$ we make a deformation

$$\tilde{\Xi}(w)=\Xi(w+\alpha h(w))$$

of $\Xi(w)$ with an arbitrary complex constant α such that $\alpha h(w)$ is real on I'. If $|\alpha|$ is sufficiently small, $\tilde{\Xi}(w)$ enters $\mathscr{X}$ (Corollary 1). Hence

$$E[\tilde{\Xi}]\geq E[\Xi]. \tag{20}$$

Comparing $|d\tilde{\Xi}|^2$ with $|d\Xi|^2$ in regard to the direction-independent term $|dw|^2$ and taking account of arbitrariness of α, we see, after a rather lengthy but routine computation, that (20) implies

$$\mathrm{Re}\iint_{B'}[(g_{11}-g_{22})-2\sqrt{-1}\,g_{12}]\,(\partial h/\partial\overline{w})du\wedge dv=0$$

(Lemma 1), or equivalently

$$\iint_{B'}[(g_{11}-g_{22})-2\sqrt{-1}\,g_{12}]\,(\partial h/\partial\overline{w})du\wedge dv=0, \tag{21}$$

where $g_{11}=|\partial\Xi/\partial u|^2$, $g_{12}=\langle\partial\Xi/\partial u, \partial\Xi/\partial v\rangle$, $g_{22}=|\partial\Xi/\partial v|^2$. Applying the Green's theorem to (21), we get

$$\iint_{B'}[\partial(g_{11}-g_{22}-2\sqrt{-1}\,g_{12})/\partial\overline{w}]h(w)\;du\wedge dv=0$$

for any $h\in C_0^\infty[B']$ (Corollary 2), whence $\partial(g_{11}-g_{22}-2\sqrt{-1}\,g_{12})/\partial\overline{w}$ must vanish identically. It amounts to saying that $\partial^2\Xi/\partial w\partial\overline{w}=0$ holds everywhere on $\mathrm{int}\,B'$. q.e.d.

In consequence of Propositions 6 and 7 Σ_0' has turned out a minimal surface with the parameter domain $\mathrm{clo}\,B'$. Further $\partial(\mathrm{loc}\,\Sigma_0')$ contains $\mathrm{loc}\,\Gamma\setminus\mathrm{loc}\,\gamma'$ whereas $\mathrm{loc}\,\Sigma_0'$ comprises $\mathrm{int}(\mathrm{loc}\,\gamma')$ in its interior. The restriction Σ_0 of Σ_0' to B^+ is of course a minimal surface bounded by Γ. Since $\eta>0$ could be taken as small as one wanted, one has a true prolongation of Σ_0 across γ defined on the parameter domain $\mathrm{clo}(B^+\cup B^-)$. Thus Theorem 7 is proved.

Corollary 3. *Assume that the contour Γ contains a sufficiently smooth non-*

2) This treatment probably originated in Gerstenhaber-Rauch [9].

singular arc and that Γ *bounds a unique non-branching minimal surface* Σ_0. *Then* Σ_0 *can be continued beyond* γ *as a minimal surface.*

Corollary 4 (Extension of Theorem 4). *Suppose a non-singular simple open* C^3*-arc* γ *admits at least one polygonal extension which bounds a minimal surface without boundary branching. Then* γ *is an analytic arc.*

Proof. Let γ be represented with the parameter interval int I and let γ' the restriction of γ to $I'=[-1+\eta, 1-\eta]$. There is a simple polygon Π connecting the both extremities of loc γ', such that $\gamma' \cup \Pi$ is a homeomorphic image of ∂B^+ bounding a minimal surface S_0 without boundary branch points. Among all the minimal surfaces bounded by loc $(\gamma' \cup \Pi)$ there is at least one, say Σ_0, which is prolongable beyond int γ' up to a minimal surface Σ_0' with an isothermal parameter domain $B^+ \cup$ int $I' \cup B^{-\prime}$ (Theorem 7). Therefore γ' (except the extremities) is an analytic arc (Theorem 5), so is γ too.

Remark 5. In contrast to the familiarity with the fact that every compact smooth surface is made into a Riemann surface with the aid of a suitable change of local parameters, explicit mentions about the context of Corollary 4 have hitherto escaped the author's attention regrettedly.

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