Ōshima, H.
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ON C-PROJECTIVE 8 AND 10 STEMS

HIDEAKI ŌSHIMA*)

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In [9] the author computed the stable C-projective homotopy of spheres through the 13-stem but left the 8-stem $\pi_{2n+7}^{SC}(S^{2n-1})$ for $n\equiv 0,6 \bmod (8)$ and the 10-stem $\pi_{2n+9}^{SC}(S^{2n-1})$ for $n\equiv 2 \bmod (3)$ unsolved. The purpose of this note is to determine these stems. These stems are also dealt with by Snow [11] and Gilbert and Zvengrwoski [3] which is a part of a master's thesis by Gilbert [7]. Our method is rather more elementary than the methods of [3] and [11] in the sense that their main tools are the spectral sequences but we do not use any spectral sequences.

Our results are the followings.

Theorem.

$$\text{(I)} \qquad \pi_{2n+7}^{SC}(S^{2n-1}) = \begin{cases} G_8 & \text{if } n\equiv 6 \bmod (8) \\ Z_2\{\eta\sigma\} & \text{if } n\equiv 0 \bmod (8)\,, \end{cases}$$

$$\text{(II)} \qquad \pi_{2n+9}^{SC}(S^{2n-1}) = \begin{cases} Z_2\{\eta\mu\} & \text{if } n\equiv 5 \bmod (6) \\ 0 & \text{if } n\equiv 2 \bmod (6)\,. \end{cases}$$

We use the notations in [8], [9] and [12] freely.

By Atiyah's S-duality theorem we have

Lemma 1.

$$\pi_{2n+2l-1}^{SC}(S^{2n-k}) = \operatorname{Ker}\,[i_{l*}\colon \pi_{2m+2l+k-1}^{s}(S^{2m}) \to \pi_{2m+2l+k-1}^{s}(P_{m+l+1,l+1})]$$

where $m=jM_{l+1}(C)-n-l-1$ for large j, and $k=0$ or 1. In particular,

$$\pi_{2n+7}^{SC}(S^{2n-1}) = \operatorname{Ker}\,[i_{4*}\colon \pi_{2m+8}^{s}(S^{2m}) \to \pi_{2m+8}^{s}(P_{m+5,5})],\ m = jM_5(C)-n-5$$

and

$$\pi_{2n+9}^{SC}(S^{2n-1}) = \operatorname{Ker}\,[i_{5*}\colon \pi_{2m+10}^{s}(S^{2m}) \to \pi_{2m+10}^{s}(P_{m+6,6})],\ m = jM_6(C)-n-6$$

for large j.

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Proof.

$$\begin{aligned}
&\pi^{SC}_{2n+2l-1}(S^{2n-k})\\
&\quad = \mathrm{Im}\,[(Ep_{n+l,l})^*\colon \pi_s^{2n-k+1}(EP_{n+l,l}) \to \pi_s^{2n-k+1}(S^{2(n+l)})]\\
&\quad = \mathrm{Ker}\,[q_{l*}\colon \pi_s^{2n-k+1}(S^{2(n+l)}) \to \pi_s^{2n-k+1}(P_{n+l+1,l+1})]\\
&\quad = \mathrm{Ker}\,[i_{l*}\colon \pi^s_{2m+2l+k-1}(S^{2m}) \to \pi^s_{2m+2l+k-1}(P_{m+l+1,l+1})]
\end{aligned}$$

for large j and $m=jM_{l+1}(C)-n-l-1$. Here the first equality follows from (1.3) of [9], the second one from the stable cohomotopy exact sequence associated with the cofibration $S^{2(n+l)-1}\xrightarrow{p_{n+l,l}}P_{n+l,l}\xrightarrow{i_1}P_{n+l+1,l+1}$, and the last one does from Atiyah's S-duality theorem [1] (or see (4.4) and (4.5) of [8]).

This lemma says that for our purpose it is enough to calculate some stable homotopy groups of the stunted complex projective spaces. Those we need are easily computed by the methods of [10], but they have been done in [2], [4], [5] and [6] so that we do not give proofs of (i), (ii), $\cdots$, (xiii) except (iv) below.

We explain some notations used below. We denote an element of $\pi^s_{2m+i}(P_{m+l,l})$ by $[\alpha]_l$ which is mapped to $\alpha\in G_{i-2l+2}$ by $q_{l-1*}\colon \pi^s_{2m+i}(P_{m+l,l})\to \pi^s_{2m+i}(S^{2(m+l-1)})=G_{i-2l+2}$. For a prime number p, ${}^p\pi^s_i(\ )$ denotes the p-primary component of $\pi^s_i(\ )$.

Lemma 2. *For m odd we have*

(i) $\pi^s_{2m+5}(P_{m+2,2})=Z_8\{[\nu]_2\}\oplus Z_3\{[\alpha_1]_2\}$,

(ii) $\pi^s_{2m+8}(P_{m+2,2})=Z_4\{[\nu]_2\nu\}$,

(iii) $i_{1*}\bar{\nu}=i_{1*}\varepsilon=2[\nu]_2\nu$,

(iv) $$\pi^s_{2m+8}(P_{m+5,5})=Z\{[Q^s\{m+5,5\}\iota]_5\}\oplus\begin{cases} Z_4\{i_{3*}[\nu]_2\nu\} & \textit{if } m\equiv 3 \bmod (8)\\ Z_2\{i_{3*}[\nu]_2\nu\} & \textit{if } m\equiv 7 \bmod (8)\\ 0 & \textit{if } m\equiv 1 \bmod (4).\end{cases}$$

Proof of (iv). Put

$$e_m=\begin{cases} 1 & \text{if } m\equiv 1 \bmod (4)\\ 2 & \text{if } m\equiv 3 \bmod (4).\end{cases}$$

Then we have

(v) ${}^2\pi^s_{2m+7}(P_{m+,33})=Z_{16}\{i_{2*}\sigma\}\oplus Z_{8/e_m}\{[e_m\nu]_3\}$,

(vi) ${}^2\pi^s_{2m+7}(P_{m+4,4})=Z_{16}\{i_{3*}\sigma\}\oplus Z_{4/e_m}\{i_{1*}[e_m\nu]_3\}$,

(vii) $[e_m\nu]_3\eta=((m+3)e_m/2)i_{1*}[\nu]_2\nu$,

(viii) $$\pi^s_{2m+8}(P_{m+4,4})=\begin{cases} Z_4\{i_{2*}[\nu]_2\nu\} & \textit{if } m\equiv 3 \bmod (8)\\ Z_2\{i_{2*}[\nu]_2\nu\} & \textit{if } m\equiv 7 \bmod (8)\\ 0 & \textit{if } m\equiv 1 \bmod (4).\end{cases}$$

By (T_4) and $(T)'_4$ of [10] we can choose an element $p'\in\pi^s_{2m+7}(P_{m+3,3})$ with

$i_{1*}p'=p_{m+4,4}$ and $q_{2*}p'=((m+5)/2)(\nu+\alpha_1)$. We can put $p'=a_m i_{2*}\sigma+b_m[e_m\nu]_3+$ *odd torsions* for some integers a_m and b_m. Then

$$(1) \qquad p_{m+4,4}=a_m i_{3*}\sigma+b_m i_{1*}[e_m\nu]_3+\textit{odd torsions}$$

and

$$\begin{aligned}((m+5)/2)\nu+\textit{odd torsions} &= q_{2*}p' \\ &= b_m e_m \nu+\textit{odd torsions}\end{aligned}$$

so that

$$(2) \qquad b_m e_m \equiv (m+5)/2 \bmod (8)\,.$$

Consider the exact sequence (S_5) of [10]:

$$\cdots \to G_1=Z_2\{\eta\} \xrightarrow{p_{m+4,4*}} \pi^s_{2m+8}(P_{m+4,4}) \xrightarrow{i_{1*}} \pi^s_{2m+8}(P_{m+5,5}) \xrightarrow{q_{4*}} G_0=Z\{\iota\}\to\cdots.$$

We have

$$\begin{aligned} p_{m+4,4*}(\eta) &= a_m i_{3*}\sigma\eta+b_m i_{1*}[e_m\nu]_3\eta, \quad \text{by (1)} \\ &= a_m i_{3*}(\bar\nu+\varepsilon)+b_m i_{1*}[e_m\nu]_3\eta \\ &= b_m i_{1*}[e_m\nu]_3\eta, \quad \textit{by (iii) and (viii)} \\ &= b_m((m+3)/2)e_m i_{2*}[\nu]_2\nu, \quad \textit{by (vii)} \\ &= ((m+3)(m+5)/4)i_{2*}[\nu]_2\nu, \quad \textit{by (2) and (viii)} \\ &= 0, \quad \textit{by (viii)}\,. \end{aligned}$$

Then we have a short exact sequence:

$$0\to \pi^s_{2m+8}(P_{m+4,4}) \xrightarrow{i_{1*}} \pi^s_{2m+8}(P_{m+5,5}) \xrightarrow{q_{4*}} Z\{Q^s\{m+5,\,5\}\iota\}\to 0\,,$$

since $\#p_{m+4,4}=Q^s\{m+5,\,5\}$. Of course this splits. Thus the proof of (iv) is completed.

Now we prove (I). Since $i_{4*}=i_{3*}i_{1*}$, by (iii) and (iv) we have

$$\begin{aligned}&\operatorname{Ker}[i_{4*}\colon \pi^s_{2m+8}(S^{2m})\to\pi^s_{2m+8}(P_{m+5,5})] \\ &=\begin{cases} G_8 & \textit{if } m\equiv 1 \bmod (4) \textit{ or } 7 \bmod (8) \\ Z_2\{\bar\nu+\varepsilon\} & \textit{if } m\equiv 3 \bmod (8)\,. \end{cases}\end{aligned}$$

Then (I) follows from (iii) of (2.8) of [9], since $M_5(C)=2^6\cdot 3^2\cdot 5\equiv 0 \bmod (8)$, $m\equiv -n-5 \bmod (8)$, and $\eta\sigma=\bar\nu+\varepsilon$.

For (II) we only consider the 3-primary component by (2.11) of [9].

Lemma 3.

(ix) $\quad {}^3\pi^s_{2m+7}(P_{m+3,3})=\begin{cases} Z_9\{[\alpha_1]_3\} & \text{if } m\not\equiv 0 \bmod (3) \\ Z_3\{i_{2*}\alpha_2\}\oplus Z_3\{[\alpha_1]_3\} & \text{if } m\equiv 0 \bmod (3), \end{cases}$

(x) $\quad {}^3\pi^s_{2m+7}(P_{m+4,4})=\begin{cases} Z_9\{i_{1*}[\alpha_1]_3\} & \text{if } m\not\equiv 0 \bmod (3) \\ Z_3\{i_{3*}\alpha_2\}\oplus Z_3\{i_{1*}[\alpha_1]_3\} & \text{if } m\equiv 0 \bmod (3), \end{cases}$

(xi) $\quad {}^3\pi^s_{2m+10}(P_{m+3,3})=Z_3\{i_{2*}\beta_1\}$

(xii) $\quad [\alpha_1]_3\alpha_1=\begin{cases} -((m+3)/2)i_{2*}\beta_1 & \text{if } m\equiv 1 \bmod (2) \\ -(m/2)i_{2*}\beta_1 & \text{if } m\equiv 0 \bmod (2), \end{cases}$

(xiii) $\quad {}^3\pi^s_{2m+10}(P_{m+4,4})=Z_3\{i_{3*}\beta_1\}$.

Note that $p_{m+4,4}$ is an element of $\pi^s_{2m+7}(P_{m+4,4})$. If $m\equiv 0 \bmod (3)$, then $[\alpha_1]_3\alpha_1=0$ by (xii) so that by (x)

$$(3) \qquad p_{m+4,4*}(\alpha_1)=0 \qquad \text{if } m\equiv 0 \bmod (3)$$

since $\alpha_2\alpha_1=0$.

As seen in [10]

$$\# p_{m+4,4}=Q^s\{m+5,5\}=C\{jM_5(C)-m-5,5\} \qquad \text{for a large } j,$$

so that by (3.1) of [8] we have

Lemma 4.

$\nu_3(\# p_{m+4,4})$	m mod ( )
2	2(3)
1	0(3), 1, 7(9)
0	4(9)

.

If $m\equiv 2 \bmod (3)$, then we may put $p_{m+4,4}=yi_{1*}[\alpha_1]_3+$*other terms* for some integer y with $y\not\equiv 0 \bmod (3)$ by (x) and Lemma 4, and hence

$$(4) \qquad p_{m+4,4*}(\alpha_1)\neq 0 \qquad \text{if } m\equiv 2 \bmod (3)$$

by (xii) and (xiii). If $m\equiv 1 \bmod (3)$, then (x) and Lemma 4 say that $p_{m+4,4}$ is divisible by 3 so that

$$(5) \qquad p_{m+4,4*}(\alpha_1)=0 \qquad \text{if } m\equiv 1 \bmod (3).$$

Now we prove (II). Consider the 3-primary part of the exact sequence $(S)_6$ of [10]:

$$\cdots \to 0={}^3\pi^s_{2m+10}(S^{2m+9}) \to {}^3\pi^s_{2m+10}(P_{m+5,5}) \xrightarrow{i_{1*}} {}^3\pi^s_{2m+10}(P_{m+6,6}) \to \cdots$$

and of $(S)_5$:

$$\cdots \to {}^3\pi^s_{2m+10}(S^{2m+7}) = Z_3\{\alpha_1\} \xrightarrow{p_{m+4,4*}} {}^3\pi^s_{2m+10}(P_{m+4,4}) = Z_3\{i_{3*}\beta_1\}$$
$$\xrightarrow{i_{1*}} {}^3\pi^s_{2m+10}(P_{m+5,5}) \to {}^3\pi^s_{2m+10}(S^{2m+8}) = 0 \to \cdots$$

The first sequence implies

$$\begin{aligned}&\mathrm{Ker}\,[i_{5*}\colon {}^3\pi^s_{2m+10}(S^{2m}) \to {}^3\pi^s_{2m+10}(P_{m+6,6})]\\ &\quad = \mathrm{Ker}\,[i_{4*}\colon {}^3\pi^s_{2m+10}(S^{2m}) \to {}^3\pi^s_{2m+10}(P_{m+5,5})]\,,\end{aligned}$$

and the second one, (xiii), (3), (4) and (5) imply that this group is

$$\begin{cases} Z_3\{\beta_1\} & \text{if } m\equiv 2 \bmod (3) \\ 0 & \text{if } m\not\equiv 2 \bmod (3)\,. \end{cases}$$

Then (II) follows from (v) and (vi) of (2.11) of [9], since $M_6(C)=M_5(C)\equiv 0 \bmod (3)$ and $m\equiv -n \bmod (3)$.

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Department of Mathematics
Osaka City University
Osaka 558, Japan