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S^1-INVARIANT HOMOTOPY OF SPHERES

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The main purpose of this note is to present tables of S^1-invariant (or "complex projective") homotopy of spheres through the stable 13-stem and unstable 6-stem. Randall [8] has previously computed these through the stable 6-stem, while various cases in the unstable 1, 2 and 3-stems have been computed by Rees [10], and Ōshima [5]. It has recently come to our attention that Ōshima [6] has also computed the S^1-invariant stable homotopy through the 13-stem (by quite different methods) and obtained some partial results in higher stems. His results, however, require substantial arithmetical work to reduce them to the explicit form we give here. We use his results to settle one question concerning the 3-primary part of the 11-stem which our methods did not cover. In all common cases, after a good deal of calculation, our results and Ōshima's can be seen to agree.

Another purpose of this note is to attempt to introduce a consistent and suggestive terminology for what has been referred to in the past as "symmetric," "projective," or "F-projective" homotopy. The more general notion of G-invariant homotopy is defined in §1, and some basic properties are given. In §2 the methods used for computing the stable stems are outlined, and in §3 the non-stable cases are dealt with. Tables summarzing the results appear in §4. This work was part of a Master's Thesis by the first named author [4].

1. *G*-invariant homotopy

Let G be a topological group acting on S^n such that the orbit space $Y=S^n/G$ is a CW-complex (necessarily of dimension less than or equal to n). We usually write Y_r for the quotient $Y/Y^{(r-1)}$, and γ for any of the quotient maps $S^n \twoheadrightarrow Y$ or $S^n \twoheadrightarrow Y_r$.

DEFINITION. 1.1 A map $S^n \xrightarrow{f} X$ is called strictly G-invariant if $f=f_1\gamma$ for some $f_1: Y \to X$, and G-invariant if $f \simeq f_1\gamma$ for some $f_1: Y \to X$. A class $\alpha \in \pi_n(X)$ is called G-invariant if it is represented by a strictly G-invariant map $S^n \xrightarrow{f} X$ (in which case any representative of α is G-invariant).

We write $\pi_n^G(X)$ for the subset of G-invariant homotopy classes of $\pi_n(X)$ (or $\pi_n^\mu(X)$ in case the group action $\mu: G\times S^n \to S^n$ is not obvious).

The following proposition gives the most elementary properties of G-invaiiant homotopy, the proofs being trivial.

Proposition. 1.2. (a) $\pi_n^G(X) \neq \phi$, *indeed* $0 \in \pi_n^G(X)$.
(b) *If* $g: X \to X'$ *then* $g_* \pi_n^G(X) \subset \pi_n^G(X')$.
(c) *If* $H \subset G$ *acts on* S^n *by restriction of the G-action, then* $\pi_n^G(X) \subset \pi_n^H(X)$.

The cases $G=Z_2, S^1, S^3$ acting in the standard way on spheres S^n, S^{2n+1}, S^{4n+3} respectively, have been subjected to the most study and have been called real projective, complex projective, and quaternionic projective respectively (also real symmetric, etc.). In general $\pi_n^G(X)$ is not a subgroup, as we see in the next example.

EXAMPLE 1.3. $\pi_3^{S^1}(S^2) = \{k^2\eta : k \in Z\}$.

Proof. See [4], [5], or [10].

Proposition 1.4. *If X is an H-space or n is in the stable range (i.e.* $n<2r-1$ *and X is* $(r-1)$*-connected), then* $\pi_n^G(X)$ *is a subgroup of* $\pi_n(X)$.

Proof. See [4] or [11].

An alternative definition for $\pi_n^G(X)$ is:

$$\pi_n^G(X) = Im[\gamma^\#: [Y,X] \to [S^n,X] = \pi_n(X)]\,.$$

The following useful lemma is an easy consequence of the homotopy extension property and cellular approximation.

Lemma 1.5. *If X is* $(r-1)$*-connected then*

$$\pi_n^G(X) = Im[\gamma^\#: [Y_r,X] \to \pi_n(X)]\,.$$

Proof. See [4].

2. Computation of stable G-invariant homotopy

The following theorem can be used for computing any stable G-invariant homotopy subgroup:

Theorem 2.1. *Take X to be* $(r-1)$*-connected and* $n<2r-1$. *Then*

$$\pi_n^G(X) \approx \mathrm{Ker}\,[(Dp\wedge 1)_*: \pi_{M-2}(S^{M-n-2}\wedge X) \to \pi_{M-2}(D(C_\gamma)\wedge X)]\,,$$

where M is sufficiently large, $D=D_{M-1}$ *is the Spanier-Whitehead* $(M$-$1)$*-dual, and*

p: $C_\gamma \to S^{n+1}$ *is the map induced by the cofibration sequence of* $S^n \xrightarrow{\gamma} Y_r$.

Proof. Applying the functor $\{-, \Sigma X\}$ to the cofibration sequence results in an exact sequence:

$$\{C_\gamma, \Sigma X\} \xleftarrow{p^\#} \{S^{n+1}, \Sigma X\} \xleftarrow{(\Sigma\gamma)^\#} \{Y_r, \Sigma X\} .$$

Hence, $\pi_n^G(X) = \mathrm{Im}\ \gamma^\# \approx \mathrm{Im}(\Sigma\gamma)^\# = \mathrm{Ker}\ p^\#$.

Next, consider the following diagram:

$$\begin{array}{ccc} \{S^{n+1}, \Sigma X\} & \xrightarrow{p^\#} & \{C_\gamma, \Sigma X\} \\ \downarrow\approx & & \downarrow\approx \\ \{S^{M-1}, S^{M-n-2}\wedge\Sigma X\} & \xrightarrow{(Dp\wedge 1)_\#} & \{S^{M-1}, D(C_\gamma)\wedge\Sigma X\} \\ \| & & \| \\ \{S^{M-2}, S^{M-n-2}\wedge X\} & \xrightarrow{(Dp\wedge 1)_\#} & \{S^{M-2}, D(C_\gamma)\wedge X\} \\ \uparrow\approx & & \uparrow\approx \\ \pi_{M-2}(S^{M-n-2}\wedge X) & \xrightarrow{(Dp\wedge 1)_*} & \pi_{M-2}(D(C_\gamma)\wedge X) \end{array}$$

Since the vertical maps are given respectively by $(M-1)$-duality, desuspension and stability this diagram commutes and these maps are isomorphisms. Hence $\pi_n^G(X) \approx \mathrm{Ker}\ p^\# \approx \mathrm{Ker}(Dp\wedge 1)_*$.

In the case of S^1-invariant homotopy, say $\pi_{2r+2k-1}(X)$ with X $(2r-2)$ or $(2r-1)$-connected, we have $Y_{2r} = Y_{2r-1} = CP_r^{r+k-1}$, where $CP_q^p = CP^p/CP^{q-1}$, $CP_q = CP^\infty/CP^{q-1}$ as usual. Hence $C_\gamma = CP_r^{r+k}$. Furthermore, taking M divisible by any sufficiently large power of 2, $D_{2M-1}CP_r^{r+k} = \Sigma CP_{M-r-k-1}^{M-r-1}$ ([2]), and $D_{2M-1}(p) = \Sigma J$ where $J: S^{2(M-r-k-1)} \to CP_{M-r-k-1}^{M-r-1}$ is the inclusion into the bottom cell. For X a sphere the theorem now gives:

Lemma 2.2. (a) *For* $k < r-1$

$$\pi_{2r+2k-1}^{S^1}(S^{2r-1}) \approx \mathrm{Ker}[J_*: \pi_{2M-2r-2}(S^{2M-2r-2k-2}) \to \pi_{2M-2r-2}(CP_{M-r-k-1})] .$$

(b) *For* $k < r$

$$\pi_{2r+2k-1}^{S^1}(S^{2r}) \approx \mathrm{Ker}[J_*: {}_{2M-2r-3}(S^{2M-2r-2k-2}) \to \pi_{2M-2r-3}(CP_{M-r-k-1})] .$$

These lemmas refer to the entire groups but are also true when the groups are replaced by their p-primary components. In our computations we deal with the 2, 3, 5 and 7-primary components individually. All others are trivial in the range being considered.

For the 2-primary component Mosher's results [3] can be applied to Lemma 2.2 directly, giving $\pi_{n+k}^{S^1}(S^n)$ for $k \leq 11$ (2-primary part) in the stable range. However, at any prime p, the following method is applicable.

The first step is to compute the stable homotopy groups of CP_m and S^{2m} through the appropriate stem. This is done by means of the Adams Spectral Sequence. For any space X the E_2 term can be computed by constructing a minimal resolution $\{M_s\}$ for $H^*(X)$ and using the equality

$$\mathrm{Ext}_A^{s,t}(H^*(X;Z_p), Z_p) = \mathrm{Hom}^t(M_s, Z_p)$$

where A is the mod p Steenrod algebra. Let $\{M_s\}$ and $\{N_s\}$ represent minimal resolutions of $H^*(CP_m)$ and $H^*(S^{2m})$, respectively.

Having computed the E_2 terms, it is routine to find

$$\mathrm{Ker}[J_{\sharp,2}\colon \mathrm{Ext}_A(H^*(S^{2m}); Z_p), Z_p) \to \mathrm{Ext}_A(H^*(CP_m; Z_p), Z_p)]$$

by the following procedure. Starting with $J^*\colon H^*(CP_m; Z_p) \to H^*(S^{2m}; Z_p)$, we can choose $\bar{J}^*\colon \{M_s\} \to \{N_s\}$ to be an A-chain map which covers J^*. $J_{\sharp,2}$ is defined to be $\mathrm{Hom}_A(\bar{J}^*, Z_p)$.

At this point we must consider the effect of any possible Adams differentials as well as those cases where an element in the kernel of $J_{\sharp,2}$ gives a homotopy class which can map under J_* to a non-zero class of higher filtration. Using this information it is possible to compute J_*.

There are many cases where no differentials can exist because of dimensional reasons, and other places where their existence can be decided by naturality. In certain cases the exact homotopy sequence of the cofibration

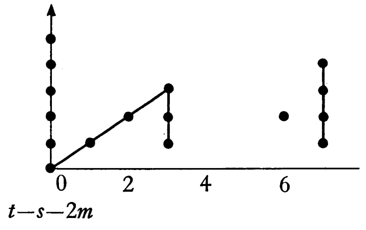
$$S^{2m} \xrightarrow{J} CP_m \to CP_{m+1}$$

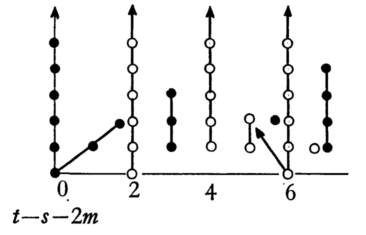
can also be used to help determine the existence of differentials.

DIAGRAMS 2.4. The Adams Spectral Sequences for the 2-primary parts of $\pi_*(S^{2m})$ and $\pi_*(CP_m)$ are illustrated through total degree $2m+7$ by the diagrams below (for CP_m not all relations are shown as it would be confusing). The action of J_* is indicated as follows. When elements $\hat{e}_{i,j}$ and $\hat{f}_{i,j}$ are represented by disks it implies that $J_{\sharp,2}(\hat{e}_{i,j})=\hat{f}_{i,j}$. $J_{\sharp,2}$ is zero on those represented by circles. For example, in the $m\equiv 1(8)$ diagram, $J_{\sharp,2}(\hat{e}_{2,8})=\hat{f}_{2,8}$, $J_{\sharp,2}(\hat{e}_{3,6})=0$. All differentials are indicated (by a remark or an arrow $\nwarrow$), and if any class is mapped to one of a higher filtration it is noted. The corresponding diagrams at the primes 3, 5, 7 are generally much simpler and are not given here.

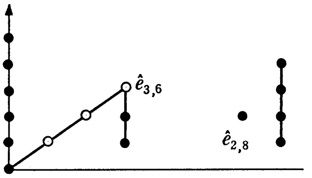
$$\pi_*(S^{2m}) \xrightarrow{J_*} \pi_*(CP_m)$$

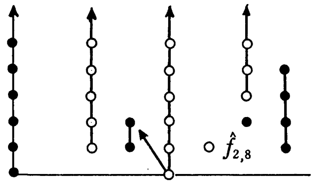
$m \equiv 0 \pmod 8$





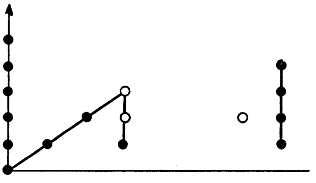
$m \equiv 1 \pmod 8$

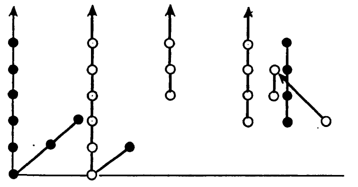




$^*\pi_{2m+7}(CP_m) = Z_8$; $m \equiv 1 \pmod{16}$
Z_{16}; $m \equiv 10 \pmod{16}$

$m \equiv 2 \pmod 8$





$^*\pi_{2m+7}(CP_m) = Z_8$; $m \equiv 2 \pmod{16}$
Z_{16}; $m \equiv 10 \pmod{16}$

$m \equiv 3 \pmod 8$

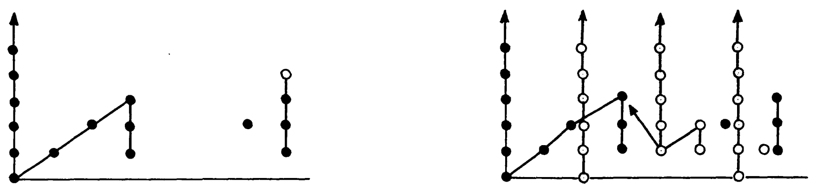
${}^*\pi_{2m+7}(CP_m) = Z_2 + Z_2$; $m \equiv 3 \pmod{16}$

$Z_2 + Z_4$; $m \equiv 11 \pmod{32}$

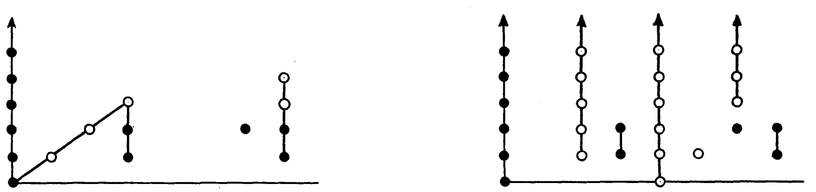
$Z_2 + Z_8$; $m \equiv 27 \pmod{64}$

$Z_2 + Z_{16}$; $m \equiv 59 \pmod{64}$

$m \equiv 4 \pmod 8$



$m \equiv 5 \pmod 8$



by cellular approximation.

Lemma 3.4. *For $n>1$ the Hopf map $\eta\colon S^3\to S^2$ induces isomorphisms $\eta_*\colon \pi^{S1}_{2n+1}(S^3)\xrightarrow{\approx}\pi^{S1}_{2n+1}(S^2)$.*

Proof. Consider the Hopf fibration $S^1\to S^3\xrightarrow{\eta}S^2$. For $n>1$, this induces the fibre mapping sequence:

$$\cdots\to 0=[CP^n, S^1]\to[CP^n, S^3]\xrightarrow{\eta_\#}[CP^n, S^2].$$

Because of cup products, for $n>1$, any $f\colon CP^n\to S^2$ is such that $f^*\colon H^2(S^2;Z)\to H^2(CP^n;Z)$ is trivial, and hence $0=f_*\colon H_2(CP^n;Z)\to H_2(S^2;Z)$. By [1], p. 68, f lifts to S^3 and $\eta_\#$ is therefore an epimorphism. Thus $\eta_\#$ is an isomorphism. By the following commutative diagram

$$\begin{array}{ccc}
[CP^n, S^3] & \xrightarrow[\approx]{\eta_\#} & [CP^n, S^2] \\
\downarrow{\scriptstyle \gamma_1^\#} & & \downarrow{\scriptstyle \gamma^\#} \\
\pi_{2n+1}(S^3) & \xrightarrow[\approx]{\eta_*} & \pi_{2n+1}(S^2)
\end{array}$$

η_* maps $\pi^{S1}_{2n+1}(S^3)=\mathrm{Im}\ \gamma_{1\#}$ isomorphically onto $\pi^{S1}_{2n+1}(S^2)=\mathrm{Im}\ \gamma_\#$.

Proposition 3.5. $\pi^{S1}_{2n+1}(S^3)=0\subset\pi_5(S^3)=Z_2$.

Proof. Consider the factorization

$$\begin{array}{ccc}
S^5 & \longrightarrow & S^3 \\
& {\scriptstyle\gamma}\searrow & \nearrow \\
& CP^2_2=S^4 &
\end{array}$$

Since $\pi^{S^1}_5(S^4)=0$ (from stable case), $\gamma\simeq 0$ and the result follows.

Corollary. $\pi^{S^1}_5(S^2)=0$ (by 3.4 and 3.5)

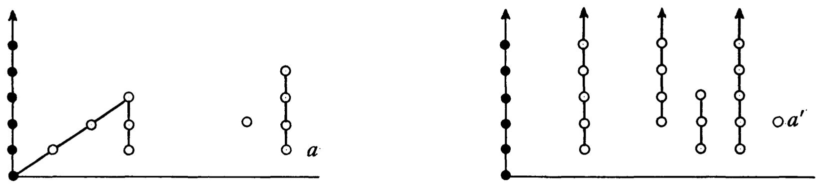
This result, along with previous non-stable computations by Ōshima and Rees, completes the nustable 1,2,3 stems. The next proposition gives the results for the 4-stem.

Proposition 3.6. (a) $\pi^{S^1}_9(S^5)=0\subset\pi_9(S^5)\approx Z_2$.
(b) $\pi^{S^1}_7(S^3)=\pi_7(S^3)\approx Z_2$.

Proof. (a) $\pi^{S^1}_9(S^5)\subset\pi^{Z_2}_9(S^5)=0$ [7].
(b) $\pi_7(S^3)=Z_2[\zeta\circ\eta_6]$. Now $\eta_6\in\pi^{S^1}_7(S^6)$ (§2), so $\zeta\circ\eta_6\in\pi^{S^1}_7(S^3)$ by naturality.

$m \equiv 6 \pmod 8$

$m \equiv 7 \pmod 8$



$^*J_*$ takes the class of a to the class of a'. However, on the E_2 level, $J_\sharp(a)=0$.

3. Unstable computations

Calculating $\pi^{S^1}_{n+k}(S^n)$ in the unstable range $k \geq n-1$ presents extra difficulties since in general this is not a subgroup. Although no general technique such as that of §2 is available, one useful method is to write

$$\pi^{S^1}_{2m+1}(X) = \mathrm{Im}[\gamma^\sharp\colon [CP^m_r, X] \to \pi_{2m+1}(X)]$$

and then pick the truncation number r as large as possible (cf. Lemma 3.1 below). A second method is to apply the stable computations of §2 when possible.

Lemma 3.1. *For* $k > n-1$

$$\pi^{S^1}_{2n+2k+1}(S^{2n}) = \mathrm{Im}[\gamma^\sharp\colon [CP^{n+k}_{n+1}, S^{2n}] \to \pi_{2n+2k+1}(S^{2n})]\,.$$

Proof. By cellular approximation, truncation at CP^{n+k}_n is possible. Let ζ_n generate $H^{2n}(CP^{n+k}_n; Z)$. Since $k \geq n$, $\zeta_n^2 \neq 0$. It follows that any map $f\colon CP^{n+k}_n \to S^{2n}$ induces $f^*=0\colon H^{2n}(S^{2n}; Z) \to H^{2n}(CP^{n+k}_n; Z)$, and therefore f factors through CP^{n+k}_{n+1}.

REMARK 3.2. In the remaining non-stable case $k=n-1$ the Hopf invariant $h\colon \pi_{4n-1}(S^{2n}) \to Z$ will yield useful information, at least on the infinite cyclic subgroup of $\pi_{4n-1}(S^{2n})$.

REMARK 3.3. In $\pi^{S^1}_{2n+2k+1}(S^{2n+1})$ truncation occurs automatically at CP^{n+k}_{n+1},

Corollary. $\pi_7^{S^1}(S^2)=\pi_7(S^2)\approx Z_2$ (by 3.4).

We now turn to the two remaining cases in the 5-stem.

Theorem 3.7. $\pi_9^{S^1}(S^4)=0\subset\pi_9(S^4)\approx Z_2$.

Proof. By Lemma 3.1 we must consider

$$\mathrm{Im}[\gamma^\#: [CP_3^4, S^4]\to\pi_9(S^4)]\,.$$

The cofibration sequence $S^7\xrightarrow{\gamma}S^6=CP_3^3\to CP_3^4\to S^8\xrightarrow{\Sigma\gamma}S^7\to\cdots$ gives rise to the exact sequence

$$\pi_7(S^4)\xleftarrow{\gamma^\#}\pi_6(S^4)\leftarrow[CP_3^4, S^4]\leftarrow\pi_8(S^4)\xleftarrow{(\Sigma\gamma)^\#}\pi_7(S^4)\leftarrow\cdots\,.$$

Now $\pi_7^{S^1}(S^6)=Z_2[\eta_6]$ (§2), hence $\gamma\simeq\eta_6$ and $\Sigma\gamma\simeq\eta_7$. It follows readily from known facts about compositions in the homotopy of spheres [12] that $\gamma^\#$ is monic and $(\Sigma\gamma)^\#$ epic. Hence $[CP_3^4, S^4]=0$.

Lemma 3.8. *$[CP_3^5, S^6]\approx Z$ with generator $[\beta]$ such that the composition $S^6\xrightarrow{J}CP_3^5\xrightarrow{\beta}S^6$ has degree 8 (note $[CP_3^5, S^6]$ is a group by stability).*

Proof. Consider the cofibration sequence

$$(1)\qquad S^7\xrightarrow{\gamma}CP_3^3=S^6\xrightarrow{J}CP_3^4\to S^8\xrightarrow{\Sigma\gamma}S^7\to\cdots,$$

where $\gamma\simeq\eta_6$ as in 3.7. This induces the exact sequence.

$$\begin{array}{ccccccccc}
[S^7, S^6] & \xleftarrow{\eta^\#} & [S^6, S^6] & \xleftarrow{j^\#} & [CP_3^4, S^6] & \leftarrow & [S^8, S^6] & \xleftarrow{\eta^\#} & [S^7, S^6] \leftarrow\cdots \\
\wr\wr & & \wr\wr & & & & \wr\wr & & \wr\wr \\
Z_2 & \twoheadleftarrow & Z & & & & Z_2[\eta^2] & \xleftarrow{\approx} & Z_2[\eta]
\end{array}$$

Hence $[CP_3^4, S^6]\approx Z$ with generator $[\alpha]$ such that $[\alpha j]=j^\#[\alpha]=2\iota_6$.

Now look at the Puppe sequence

$$(2)\qquad S^9\xrightarrow{\gamma}CP_3^4\xrightarrow{i}CP_3^5\to S^{10}\to\cdots\,.$$

In order to identify $[\gamma]\in\pi_9(CP_3^4)$ we consider the induced homotopy sequence, which is stable and hence exact:

$$\begin{array}{ccccccc}
\pi_9(S^9) & \xrightarrow{\gamma_*} & \pi_9(CP_3^4) & \to & \pi_9(CP_3^5) & \to & \pi_9(S^{10})\to\cdots \\
\wr\wr & & \wr\wr & & \wr\wr & & \| \\
Z & & Z_4+Z_3 & & Z_3 & & 0
\end{array}$$

It follows from exactness that $[\gamma]$ generates the Z_4 summand.

From (1) we get the following exact sequence:

$$\begin{array}{ccccccccc}
\pi_9(S^7) & \overset{\eta_*}{\to} & \pi_9(S^6) & \overset{j_*}{\to} & \pi_9(CP_3^4) & \to & \pi_9(S^8) & \overset{\eta_*}{\to} & \pi_9(S^7) \to \cdots \\
\wr\wr & & \wr\wr & & \wr\wr & & \wr\wr & & \wr\wr \\
Z_2 & \rightarrowtail & Z_8+Z_3 & & Z_4+Z_3 & & Z_2 & \overset{\approx}{\to} & Z_2
\end{array}$$

Again, using exactness we find $j_*[\nu]=[\gamma]$ where ν generates Z_8, i.e. $\gamma\simeq j\nu$. From this and (2) we get:

$$\begin{array}{ccccccc}
\pi_9(S^6) & \overset{\gamma^\sharp}{\longleftarrow} & [CP_3^4, S^6] & \overset{i^\sharp}{\leftarrow} & [CP_3^5, S^6] & \leftarrow \pi_{10}(S^6) = 0 \leftarrow \cdots \\
\wr\wr & \nwarrow \nu^\sharp & j^\sharp \swarrow & & \wr\wr & \\
Z_3+Z_8 & & \pi_6(S^6) & & Z &
\end{array}$$

Therefore $\gamma^\sharp[\alpha]=\nu^\sharp j^\sharp[\alpha]=\nu^\sharp(2\iota_6)=2\nu$, which implies by exactness that $[CP_3^5, S^6]\approx Z$ with generator $[\beta]$ such that $i^\sharp[\beta]=4[\alpha]$. Thus $[\beta j_1]=j_1^\sharp[\beta]=j^\sharp i^\sharp[\beta]=j^\sharp(4[\alpha])=8\iota_6$.

Theorem 3.9. *$\pi_{11}^{S^1}(S^6)=\{32k^2w_6\colon k\in Z\}$, where $\pi_{11}(S^6)=Z[w_6]$.*

Proof. Consider any map $\alpha\colon CP_3^5\to S^6$. By the lemma $[\alpha]=k[\beta]$ for some integer k. It also follows from the lemma that $\alpha^*(u_6)=8kv_3$, where u_6 generates $H^6(S^6;Z)\approx Z$ and v_i generates $H^{2i}(CP_r^n;Z)\approx Z$, $r\le i\le n$. Setting $f=\alpha\gamma$, there is a homotopy commutative diagram of cofibration sequences

$$\begin{array}{ccccccc}
S^{11} & \overset{\gamma}{\to} & CP_3^5 & \overset{i}{\hookrightarrow} & CP_3^6 & \to & S^{12} \\
\| & & \downarrow \alpha & & \downarrow \varphi & & \| \\
S^{11} & \overset{f}{\to} & S^6 & \overset{j}{\hookrightarrow} & C_f & \to & S^{12},
\end{array}$$

for a suitable map φ.

Clearly $H^6(C_f;Z)=Z[u]$ with $j^*(u)=u_6$, and $H^{12}(C_f;Z)=Z[v]$ with $\varphi^*(v)=v_6$. Since i^* is an isomorphism in dimension 6, it is easy to see that $\varphi^*(u)=8kv_3$. Thus

$$\varphi^*(u^2)=(\varphi^*(u))^2=64k^2v_3^2=64k^2v_6=\varphi^*(64k^2v)\,,$$
$$u^2=64k^2v\,.$$

Letting h denote the Hopf invariant, this shows

$$h(f)=64k^2=h(32k^2w_6),\quad\text{since } h(w_6)=2\,.$$

But h is monic here, so $f\simeq 32k^2w_6$.

Theorem 3.10. $\pi_{11}^{S^1}(S^5)=0\subset\pi_{11}(S^5)=Z_2$.

Proof. We must consider

$$\begin{array}{ccc} S^{11} & \longrightarrow & S^5 \\ & \gamma\searrow & \nearrow \\ & & CP_3^5 \end{array}$$

By the usual cofibrations one readily finds

$$0=j^{\#}\colon [CP_3^4, S^5]\to\pi_6(S^5)\,,$$

where $j\colon S^6\hookrightarrow CP_3^4$. Letting j_1 equal the composition $S^6\xrightarrow{j}CP_3^4\xrightarrow{i}CP_3^5, j_1^{\#}=j^{\#}i^{\#}$ $=0$. Using this we can trucate to

$$\begin{array}{ccc} S^{11} & \longrightarrow & S^5 \\ & \gamma\searrow & \nearrow \\ & & CP_4^5\simeq S^8\vee S^{10} \end{array}$$

Now we must identify $[\gamma]$. From the Puppe sequence

$$S^{11}\xrightarrow{\gamma}CP_4^5\xrightarrow{i}CP_4^6\to S^{12}\to\cdots$$

and stability we get the following induced exact sequence:

$$\begin{array}{ccccccc} \pi_{11}(S^{11}) & \xrightarrow{\gamma_*} & \pi_{11}(CP_4^5) & \xrightarrow{i_*} & \pi_{11}(CP_4^6) & \to & \pi_{11}(S^{12})\to\cdots \\ \wr\wr & & \wr\wr & & \wr\wr & & \| \\ Z & & Z_2+Z_8 & & Z_4 & & 0 \end{array}$$

$[\gamma]$ generates $\mathrm{Ker}[i_*\colon\pi_{11}(CP_4^5)\to\pi_{11}(CP_4^6)]$ so $[\gamma]=(\eta_{10}, 2\nu_8)$.

We know $\pi_8(S^5)=Z_8(\nu_5)$ and $\pi_{10}(S^5)=Z_2[\beta]$ with $\beta=\nu_5\circ\eta_8^2$, [12], p. 44. Now $(0,\nu_5)\circ(\eta_{10},2\nu_8)=2\nu_5\circ\nu_8=0\in Z_2$ and $(\beta,0)\circ(\eta_{10},2\nu_8)=\beta\circ\eta_{10}=\nu_5\circ\eta_8^2\circ\eta_{10}=\nu_5\circ\eta_8^3$ $=\nu_5\circ(4\nu_8)=4(\nu_5^2)=0\in Z_2$. Since these classes generate any classes of $\pi_{11}^{S^1}(S^5)$ it follows that $\pi_{11}^{S^1}(S^5)=0$.

Theorem 3.11. $\pi_9^{S^1}(S^3)=0\subseteq\pi_9(S^3)\approx Z_3$.

Proof.

$$\begin{array}{ccc} S^{9} & \longrightarrow & S^3 \\ & \gamma\searrow & \nearrow \\ & & CP_2^4 \end{array}$$

Consider the mod 3 Postnikov system for S^3

$$\begin{array}{ccccc} & & \vdots & & \\ & & \downarrow & & \\ K(Z_3, 9) & \to & X_9 & \xrightarrow{x_{11}} & \\ & & \downarrow & & \\ K(Z_3, 6) & \xrightarrow{i} & X_6 & \xrightarrow{x_{10}} & K(Z_3, 10) \\ & & \downarrow & & \\ & & K(Z, 3) & \xrightarrow{\mathcal{P}^1\iota_3} & K(Z_3, 7) \end{array}$$

where $i^*(x_{10})=\mathcal{P}^1\iota_6$.

$$[CP_2^4, S^3]\approx[CP_2^4, X_6]\approx[CP_2^5, X] .$$

We have the fibre mapping exact sequence:

$$\begin{array}{ccccccc} [CP_2^5, K(Z, 2)] & \to & [CP_2^5, K(Z_3, 6)] & \overset{\approx}{\to} & [CP_2^5, X_6] & \to & [CP_2^5, K(Z, 3)] \\ \wr\wr & & \wr\wr & & & & \wr\wr \\ 0 & & H^6(CP_2^5, Z_3) = Z_3[\zeta_3] & & & & 0 \end{array}$$

For any $f: CP_2^4 \to X_6$ there is an extension to some $\bar{f}: CP_2^5 \to X_6$. The map $\bar{f}$ will lift to $CP_2^5 \to X_9$ if and only if $\bar{f}^*(x_{10})=0$. But

$$\bar{f}^*(x_{10}) = \zeta_3^* i^*(x_{10}) = \zeta_3^*(\mathcal{P}^1\iota_6) = \mathcal{P}^1\zeta_3 = 0 .$$

Therefore, $\bar{f}$ lifts to $g: CP_2^5 \to X_9$. The map g lifts to $CP_2^5 \to S^3$ since all other obstructions are in dimensions ≥ 11 so they they vanish. Therefore, any map $f: CP_2^4 \to S^3$ will factor through the inclusion μ as

$$\begin{array}{ccc} CP_2^4 & \xrightarrow{f} & S^3 \\ & {\scriptstyle\mu}\searrow \quad \nearrow{\scriptstyle g} & \\ & CP_2^5 & \end{array}$$

Since $S^9 \xrightarrow{\gamma} CP_2^4 \xrightarrow{\mu} CP_2^5$ represents two stages of a cofibration sequence it follows that $f\gamma \simeq g\mu\gamma = g(\mu\gamma) \simeq 0$.

Proposition 3.12. $\pi_{13}^{S^1}(S^7)=0\subseteq\pi_{13}(S^7)\approx Z_2$.

Proof.

$$\begin{array}{ccc} S^{13} & \longrightarrow & S^7 \\ & {\scriptstyle\gamma}\searrow \quad \nearrow & \\ & CP_4^6 & \end{array}$$

Consider $S^{13} \xrightarrow{\gamma} CP_4^6 \to CP_4^7 \to S^{14} \to \cdots$. By stability this induces the exact sequence

$$\pi_{13}(S^{13}) \xrightarrow{\gamma_*} \pi_{13}(CP_4^6) \to \pi_{13}(CP_4^7) \to \pi_{13}(S^{14}) = 0 \to \cdots,$$

hence $[\gamma]$ generates $\mathrm{Ker}[\pi_{13}(CP_4^6) \to \pi_{13}(CP_4^7)]$. By computing the 2-primary components of these groups we see that $[\gamma] \equiv 0 \pmod{C_2}$, hence $\pi_{13}^{S^1}(S^7)=0$.

Remark. In [9] Randall shows that the Whitehead square w_n is S^1-invariant only if $n=2^m-1$ for some m. From the results in this section we see that w_3, w_7 (also w_1 trivially) are S^1-invariant. It seems plausible to conjecture that w_n is otherwise never S^1-invariant.

4. Tables

Tables I and II below give the 2- and 3-primary subgroups of the S^1-invariant homotopy of spheres through the stable 13-stem. In this range all other p-primary parts are trivial, with the exceptions $\pi_{n+7}^{S^1}(S^n)=Z_5$ when $n \not\equiv 0 \pmod 5$ and $\pi_{n+11}^{S^1}(S^n)=Z_7$ when $n \not\equiv 0 \pmod 7$. Table III gives the unstable S^1-invariant subsets through the 6-stem.

Table I. $\pi_{n+k}^{S^1}(S^n)$, $k<n-1$, 2-primary part
(A, B, ... explained in Table Ia)

n modulo 16	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$k=0$		0		0		0		0		0		0		0		0
1	0		Z_2 η		0		Z_2 η		0		Z_2 η		0		Z_2 η	
2		Z_2 η^2		0		Z_2 η^2		0		Z_2 η^2		0		Z_2 η^2		0
3	Z_2 4ν		Z_2 4ν		Z_8 ν		Z_4 2ν		Z_4 2ν		0		Z_8 ν		Z_4 2ν	
4		0		0		0		0		0		0		0		0
5	0		0		0		0		0		0		0		0	
6		Z_2 ν^2		Z_2 ν^2		0		0		Z_2 ν^2		Z_2 ν^2		0		0
7	A_7		B_7		C_7		0		Z_8 2σ		Z_8 2σ		Z_4 4σ		Z_2 8σ	
8		0		Z_2^2 $\bar{\nu}, \eta\sigma$		0		Z_2^2 $\bar{\nu}, \eta\sigma$		0		Z_2^2 $\bar{\nu}, \eta\sigma$		0		Z_2 $\eta\sigma$
9	Z_2 $\eta\bar{\nu}$		B_9		0		D_9		Z_2 $\eta\bar{\nu}$		Z_3^3 $\eta\bar{\nu}, \eta^2\sigma, \mu$		0		Z_2^3 $\eta\bar{\nu}, \eta^2\sigma, \mu$	
10		Z_2 $\eta\mu$		0		Z_2 $\eta\mu$		0		Z_2 $\eta\mu$		0		Z_2 $\eta\mu$		0
11	A_{11}		B_{11}		C_{11}		D_{11}		Z_2 4ζ		Z_2 4ζ		Z_8 ζ		Z_4 2ζ	
12		0		0		0		0		0		0		0		0
13	0		0		0		0		0		0		0		0	

Table Ia

$$A_7=\begin{cases}0 & ; n\equiv 0 \pmod{128}\\ Z_2[8\sigma]; & n\equiv 64 \pmod{128}\\ Z_4[4\sigma]; & n\equiv 32 \pmod{64}\\ Z_8[2\sigma]; & \text{otherwise}\end{cases}\qquad B_7=\begin{cases}0 & ; n\equiv 2 \pmod{32}\\ Z_2[8\sigma]; & n\equiv 18 \pmod{32}\end{cases}$$

$$C_7=\begin{cases}0 & ; n\equiv 4 \pmod{32}\\ Z_2[8\sigma]; & n\equiv 20 \pmod{32}\end{cases}$$

$$B_9=\begin{cases}Z_2[\eta\bar{\nu}]+Z_2[\eta^2\sigma] & ; n\equiv 2 \pmod{64}\\ Z_2[\eta\bar{\nu}]+Z_2[\eta^2\sigma]+Z_2[\mu]; & n\equiv 18 \pmod{32}\\ Z_2[\eta\bar{\nu}]+Z_2[\eta^2\sigma]+Z_2[\mu]; & n\equiv 34 \pmod{64}\end{cases}$$

$$D_9=\begin{cases}Z_2[\eta\bar{\nu}]+Z_2[\eta^2\sigma] & ; n\equiv 22 \pmod{32}\\ Z_2[\eta\bar{\nu}]+Z_2[\eta^2\sigma]+Z_2[\mu]; & n\equiv 6 \pmod{32}\end{cases}$$

$$A_{11}=\begin{cases}Z_2[4\zeta]; & n\equiv 0 \pmod{256}\\ Z_4[2\zeta]; & \text{otherwise}\end{cases}\qquad B_{11}=\begin{cases}Z_2[4\zeta]; & n\equiv 2 \pmod{64}\\ 0 & ; \text{othersiwise}\end{cases}$$

$$C_{11}=\begin{cases}Z_2[4\zeta]; & n\equiv 100 \pmod{128}\\ Z_4[2\zeta]; & n\equiv 36 \pmod{128}\\ Z_8[\zeta] & ; \text{otherwise}\end{cases}\qquad D_{11}=\begin{cases}0 & ; n\equiv 38 \pmod{64}\\ Z_2[4\zeta]; & n\equiv 6 \pmod{64}\\ Z_8[\zeta] & ; \text{otherswise}\end{cases}$$

Table II

$\pi^{S^1}_{n+k}(S^n)$, $k<n-1$

(3-primary part)

$n\equiv 0(9)$		1	2	3	4	5	6	7	8
$k=0$	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0
3	0	Z_3	Z_3	0	Z_3	Z_3	0	Z_3	Z_3
4	0	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0	0	0
6	0	0	0	0	0	0	0	0	0
7	0	Z_3	0	Z_3	Z_3	0	Z_3	Z_3	0
8	0	0	0	0	0	0	0	0	0
9	0	0	0	0	0	0	0	0	0
10	0	Z_3	0	0	Z_3	0	0	Z_3	0
11	P_{11}	Z_3	Q_{11}	Z_9	0	Z_9	Z_9	Z_3	Z_9
12	0	0	0	0	0	0	0	0	0
13	Z_3	Z_3	0	Z_3	Z_3	0	Z_3	Z_3	0

$$P_{11}=\begin{cases}0\ ; & n\equiv 0 \pmod{81}\\ Z_3; & n\equiv 27,\ 54 \pmod{81}\\ Z_9; & n\equiv 9,\ 18 \pmod{27}\end{cases}$$

$$Q_{11}=\begin{cases}0\ ; & n\equiv 2 \pmod{27}\\ Z_3; & n\equiv 11 \pmod{27}\\ Z_9; & n\equiv 20 \pmod{27}\end{cases}$$

Table III

$\pi^{S^1}_{n+k}(S^n)$, $k \geqq n-1$

	$n=2$	3	4	5	6	7
$k=1$	A_1	—	—	—	—	—
2	/	0	—	—	—	—
3	0	/	A_3	—	—	—
4	/	Z_2	/	0	—	—
5	Z_2	/	0	/	A_5	—
6	/	0	/	0	/	0

$A_1=\{k^2\eta : k \in Z\}$

$A_3=\left\{k^2\nu_4+\frac{k(k-1)}{2}\delta+6t\cdot\delta : k\in Z,\ t=0, 1\right\}$

$A_5=\{32k^2\omega_6 : k\in Z\}$

In Table I the subgroups are given with their corresponding generators, as these are not always obvious. In all other cases the subsets automatically determine the generators.

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