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ON CHARACTERISTIC CLASSES OF KÄHLER FOLIATIONS

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0. Introduction

The purpose of this note is to study Kähler foliations, which are defined by requiring the transition functions to be holomorphic isometries of a Kähler manifold (see Definition 1.1), by adopting the method of [6] [7] [10]. In some sense Kähler foliations are the holomorphic analogue of Riemannian foliations and characteristic classes of the latter have been profoundly investigated by Lazarov and Pasternack (cf.[9], also see [6]). However from the view point of characteristic classes, the situations are completely different. Namely the vanishing phenomenon of the Pontrjagin classes of the normal bundles in the Riemannian case is much stronger than that in the smooth case (cf. strong vanishing theorem of Pasternack [12] and the Bott's vanishing theorem [1]). By contrast, we do not have any strong vanishing phenomenon in the Kähler foliations. This fact reflects in the secondary characteristic classes. For example, all the secondary classes of smooth foliations are zero on Riemannian foliations, but some of the secondary classes of holomorphic foliations may be non-zero on Kähler foliations. A new ingredient of our context is the Kähler form which is a closed 2-form defined for any Kähler foliation.

In § 1 we define Kähler foliations and construct characteristic classes of them and in § 2 we compute the cohomology of certain truncated Weil algebra. In §§ 3 and 4, we study the relationships of our characteristic classes with those of Riemannian and holomorphic foliations. Finally in § 5 we consider deformations of Kähler foliations.

1 Construction of the characteristic classes

In this section we define the notion of Kähler foliations and construct characteristic classes of them.

DEFINITION 1.1. A codimension n Kähler foliation F on a smooth manifold M is a maximal family of submersions $f_\alpha\colon U_\alpha \to (C^n, g_\alpha)$, where U_α is an

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open set in M and g_α is a Kähler metric on $\boldsymbol{C}^n$, satisfying the condition: for every $x \in U_\alpha \cap U_\beta$ there exists a local holomorphic isometry $\gamma_{\alpha\beta}$ such that $f_\beta = \gamma_{\beta\alpha} \circ f_\alpha$ near x.

This is a holomorphic version of the notion of Riemannian foliations. Since $\gamma_{\beta\alpha}$ is a holomorphic isometry, patching together the pull backs of unitary frame bundle of $(\boldsymbol{C}^n, g_\alpha)$ by the map f_α, we obtain a principal $U(n)$-bundle $\pi: U(F) \to M$. We call it the unitary frame bundle of the foliation F. Let θ_0^α and θ_1^α be the canonical form and the unique torsionfree Hermitian connection form of $(\boldsymbol{C}^n, g_\alpha)$. Since holomorphic isometries preserve these forms, we can define global 1-forms θ_0 and θ_1 on $U(F)$ such that $\theta_0|_{\pi^{-1}(U_\alpha)}$, $\theta_1|_{\pi^{-1}(U_\alpha)}$ are the pull backs of θ_0^α, θ_1^α. Let $E_C(n)$ be the group generated by parallel transformations and unitary transformations on $\boldsymbol{C}^n$ (which is a semi-direct product of $\boldsymbol{C}^n$ and $U(n)$). The pair (θ_0, θ_1) is an $\mathfrak{e}_C(n)$-valued 1-form, where $\mathfrak{e}_C(n)$ is the Lie algebra of $E_C(n)$. (θ_0, θ_1) defines a d.g.a. map

$$\phi: W(\mathfrak{e}_C(n)) \to \Omega^*(U(F))$$

where $W(\mathfrak{e}_C(n))$ is the Weil algebra of $\mathfrak{e}_C(n)$ and $\Omega^*(U(F))$ is the de Rham complex of $U(F)$. Let $\omega^i, \omega^i_j, \Omega^i, \Omega^i_j \in W(\mathfrak{e}_C(n))$ be the universal connection and curvature forms corresponding to the usual basis (over $\boldsymbol{R}$) of $\mathfrak{e}_C(n) = \boldsymbol{C}^n + \mathfrak{u}(n) \subset \boldsymbol{R}^{2n} + \mathfrak{so}(2n)$. If we denote $\theta^i, \theta^i_j, \Theta^i, \Theta^i_j$ for the ϕ-images of $\omega^i, \omega^i_j, \Omega^i, \Omega^i_j$ respectively, then they satisfy the following equations (cf. [8])

$$(2.1)\qquad \begin{aligned} &\text{(i)} && \Theta^i = d\theta^i + \theta^i_k \wedge \theta^k_i = 0 \qquad \text{(torsionfree-ness)} \\ &\text{(ii)} && d\theta^i_j = -\theta^i_k \wedge \theta^k_j + \Theta^i_j \\ &\text{(iii)} && \Theta^i_j \wedge \theta^j = 0 \qquad \text{(the first Bianchi's identity).} \end{aligned}$$

Therefore Ker ϕ contains an ideal I of $W(\mathfrak{e}_C(n))$ generated by the following elements.

(2.2)

(i) Ω^i

(ii) elements whose "length" l is greater than n, where l is defined by the conditions : $l(\omega^i_j) = l(\Omega^i) = 0$, $l(\omega^i) = 1$ and $l(\Omega^i_j) = 2$.

(iii) $\Omega^i_j \wedge \omega^j$.

If we denote $\tilde{W}(\mathfrak{e}_C(n)) = W(\mathfrak{e}_C(n))/I$, then ϕ induces a d.g.a. map

$$\phi: \tilde{W}(\mathfrak{e}_C(n)) \to \Omega^*(U(F)).$$

Now suppose that the normal bundle of F is trivialized by a cross section $s: M \to U(F)$, then we obtain

$$H^*(\tilde{W}(\mathfrak{e}_C(n))) \to H^*_{DR}(U(F)) \xrightarrow{s^*} H^*_{DR}(M).$$

We denote $BK\bar{\Gamma}_n$ for the classifying space for codimension n Kählerian Haefliger structures with trivial normal bundles which is defined similarly as the ordinary Haefliger structures. Since the above construction is functorial, we obtain a homomorphism

$$\phi\colon H^*(\tilde{W}(\mathfrak{e}_C(n))) \to H^*(BK\bar{\Gamma}_n; \boldsymbol{R}). \tag{1.3}$$

Considering $U(n)$-basic elements of $\tilde{W}(\mathfrak{e}_C(n))$ we can also define a d.g.a. map $\phi\colon \tilde{W}(\mathfrak{e}_C(n))_{U(n)} \to \Omega^*(M)$ and this yields a homomorphism

$$\phi\colon H^*(\tilde{W}(\mathfrak{e}_C(n))_{U(n)}) \to H^*(BK\Gamma_n; \boldsymbol{R}), \tag{1.4}$$

where $BK\Gamma_n$ denotes the classifying space for codimension n Kählerian Haefliger structures.

The above is our construction of characteristic classes of Kähler foliations.

2. Cohomology of $\tilde{W}(\mathfrak{e}_C(n))$

Here we compute the cohomology of $\tilde{W}(\mathfrak{e}_C(n))$. $\tilde{W}(\mathfrak{e}_C(n))$ has a decreasing filtration F^p defined by $F^p = \{x \in \tilde{W}(\mathfrak{e}_C(n));\ l(x) \geqq p\}$ where l is the function on $\tilde{W}(\mathfrak{e}_C(n))$ induced by the length on $W(\mathfrak{e}_C(n))$. Let $\{E_r^{p,q}, d_r\}$ be the spectral sequence associated with this filtration. If we define $M_p = \{x \in \tilde{W}(\mathfrak{e}_C(n));\ l(x) = p$ and x contains no $\omega_j^i\}$, then $\mathfrak{u}(n)$ acts on M_p by the Lie derivative. Thus M_p is a $\mathfrak{u}(n)$-module. Let $C^q(\mathfrak{u}(n); M_p)$ be the set of q-cochains on $\mathfrak{u}(n)$ with coefficients in M_p. Then

$$E_0^{p,q} = \wedge^q(\mathfrak{u}(n)) \otimes M_p \cong C^q(\mathfrak{u}(n); M_p) \tag{2.1}$$

and this identification is compatible with the differentials. Thus

$$E_1^{p,q} \cong H^q(\mathfrak{u}(n); M_p). \tag{2.2}$$

Let $M_p^{\mathfrak{u}(n)}$ be the $\mathfrak{u}(n)$-invariant subspace of M_p.

Lemma 2.1. $H^*(\mathfrak{u}(n); M_p) \cong H^*(\mathfrak{u}(n)) \otimes M_p^{\mathfrak{u}(n)}$.

Proof. Let Z be the center of $\mathfrak{u}(n)$. Then we have $\mathfrak{u}(n) = \mathfrak{su}(n) \oplus Z$. Z-action on M_p is obtained by differentiating the action of $S^1 \subset U(n)$ on M_p. Since S^1 is compact, M_p breaks up as a sum of S^1-invariant ($=L_I$-invariant, I is the generator of Z) subspaces $M_p = M_p^Z \oplus W$. Then the assertion follows from an argument in Corollary *IV* 2.2 [11]. q.e.d.

Now let $N_p = \{x \in W(\mathfrak{e}_C(n));\ l(x) = p,\ x$ contains no $\omega_j^i\}$. Then we show

Lemma 2.2. $N_p^{\mathfrak{u}(n)}/I \cap N_p^{\mathfrak{u}(n)} \cong M_p^{\mathfrak{u}(n)}$.

Proof. We have a short exact sequence of $\mathfrak{u}(n)$-modules:

$$0 \to I \to W(\mathfrak{e}_C(n)) \to \tilde{W}(\mathfrak{e}_C(n)) \to 0\,.$$

This induces a long exact sequence

$$\begin{aligned}0 &\to H^0(\mathfrak{u}(n);\, I) \to H^0(\mathfrak{u}(n);\, W(\mathfrak{e}_C(n))) \to H^0(\mathfrak{u}(n);\, \tilde{W}(\mathfrak{e}_C(n)))\\ &\to H^1(\mathfrak{u}(n);\, I) \to H^1(\mathfrak{u}(n);\, W(\mathfrak{e}_C(n))) \to \cdots\,,\end{aligned}$$

where we have $H^0(\mathfrak{u}(n);\, I)=I^{\mathfrak{u}(n)}$, $H^0(\mathfrak{u}(n);\, W(\mathfrak{e}_C(n)))=N^{\mathfrak{u}(n)}(=\bigoplus_p N_p^{\mathfrak{u}(n)})$ and $H^0(\mathfrak{u}(n);\, \tilde{W}(\mathfrak{e}_C(n)))=M^{\mathfrak{u}(n)}\,(=\bigoplus_p M_p^{\mathfrak{u}(n)})$. Now by the argument of Lemma 2.1,

$$\begin{aligned}&H^1(\mathfrak{u}(n);\, I)\cong H^1(\mathfrak{u}(n))\oplus I^{\mathfrak{u}(n)} \text{ and}\\ &H^1(\mathfrak{u}(n);\, W(\mathfrak{e}_C(n)))\cong H^1(\mathfrak{u}(n))\otimes N^{\mathfrak{u}(n)}\,.\end{aligned}$$

Therefore the map $H^1(\mathfrak{u}(n);\, I)\to H^1(\mathfrak{u}(n); W(\mathfrak{e}_C(n)))$ is injective, which implies that the homomorphism $N^{\mathfrak{u}(n)}\to M^{\mathfrak{u}(n)}$ is surjective. This completes the proof. q.e.d.

By virtue of Lemma 2.2, it is enough to determine $N_p^{\mathfrak{u}(n)}/I\cap N_p^{\mathfrak{u}(n)}$ instead of $M_p^{\mathfrak{u}(n)}$. To simplify the computation, we consider the complexification of N_p. If we put $\phi^i=\omega^i+\sqrt{-1}\omega^{n+i}$, $\psi_j^i=\Omega_j^i+\sqrt{-1}\Omega_j^{n+i}$ $(i, j=1, \cdots, n)$, then $\bigoplus_p(N_p\otimes \boldsymbol{C})$ is multiplicatively generated by ϕ^i, $\bar{\phi}^i$, ψ_j^i where $\bar{\phi}^i=\omega^i-\sqrt{-1}\omega^{n+i}$, ($\bar{\psi}_j^i$ is not necessary because $\bar{\psi}_j^i=-\psi_i^j$). The action of an element $A=(a_j^i)\in U(n)$ is given as follows;

$$\begin{aligned}A\cdot\phi^i &= \sum_k a_i^k\phi^k, \quad A\cdot\bar{\phi}^i = \sum_k \bar{a}_i^k\bar{\phi}^k\,, \quad \text{and}\\ A\cdot\psi_j^i &= \sum_k a_i^k\psi_j^k\,.\end{aligned} \tag{2.3}$$

Let $\langle\phi\rangle, \langle\bar{\phi}\rangle, \langle\psi\rangle$ be the complex vector spaces with bases $\{\phi^1, \cdots, \phi^n\}$, $\{\bar{\phi}^1, \cdots, \bar{\phi}^n\}$, $\{\psi_j^i\}$ respectively and let $\langle\phi\rangle^m, \langle\bar{\phi}\rangle^m, \langle\psi\rangle^m$ be the tensor products of m-copies of them. $U(n)$ acts on these spaces by the diagnoal action. Now we define an action of $S_{p,q,r}=S_p\times S_q\times S_r$ (the product of the symmetric groups of degrees p, q, r) on $\langle\phi\rangle^p\otimes\langle\bar{\phi}\rangle^q\otimes\langle\psi\rangle^r$ by

$$(\sigma_1, \sigma_2, \sigma_3)\cdot\phi^I\bar{\phi}^J\psi_L^K = \phi^{\sigma_1(I)}\bar{\phi}^{\sigma_2(J)}\psi_{\sigma_3(L)}^{\sigma_3(K)}$$

where $(\sigma_1, \sigma_2, \sigma_3)\in S_{p,q,r}$ and ϕ^I, $\phi^{\sigma(I)}$ denote $\phi^{i_1}\cdots\phi^{i_p}$, $\phi^{\sigma(i_1)}\cdots\phi^{\sigma(i_p)}$ respectively, etc.. If we use the same letter $S_{p,q,r}$ for a linear endomorphism on $\langle\phi\rangle^p\otimes\langle\bar{\phi}\rangle^q\otimes\langle\psi\rangle^r$ given by

$$S_{p,q,r}(\phi^I\bar{\phi}^J\psi_L^K) = \sum sgn(\sigma_p)sgn(\sigma_q)(\sigma_p, \sigma_q, \sigma_r)\cdot\phi^I\bar{\phi}^J\psi_L^K$$

where $(\sigma_p, \sigma_q, \sigma_r)$ ranges over all elements of $S_{p,q,r}$, then the $S_{p,q,r}$-invariant subspaces ($=\operatorname{Im} S_{p,q,r}$) is equal to $\Lambda^p\langle\phi\rangle\otimes\Lambda^q\langle\bar{\phi}\rangle\otimes S^r\langle\psi\rangle\subset N_{p+q+2r}\otimes\boldsymbol{C}$. We denote this subspace by $N_{p,q,r}$. Since $S_{p,q,r}$ is $U(n)$-equivariant,

$$N_{p,q,r}^{\mathfrak{u}(n)}\subset(\langle\phi\rangle^p\otimes\langle\bar{\phi}\rangle^q\otimes\langle\psi\rangle^r)^{\mathfrak{u}(n)}\,.$$

We can consider $\langle\psi\rangle^p\otimes\langle\bar{\psi}\rangle^q\otimes\langle\psi\rangle^r\cong\langle\psi\rangle^{p+r}\otimes\langle\bar{\psi}\rangle^{q+r}$ by the map $\psi^I\bar{\psi}^J\psi^K_L\to\psi^{I,K}\bar{\psi}^{J,L}$. Since this map is $U(n)$-quivariant,

$$(\langle\psi\rangle^p\otimes\langle\bar{\psi}\rangle^q\otimes\langle\psi\rangle^r)^{\mathfrak{u}(n)}=(\langle\psi\rangle^{p+r}\otimes\langle\bar{\psi}\rangle^{q+r})^{\mathfrak{u}(n)}\,.$$

Lemma 2.3. *The vector space $(\langle\psi\rangle^l\otimes\langle\bar{\psi}\rangle^m)^{\mathfrak{u}(n)}$ is non trivial only for $l=m$ and in that case it has a basis of the following tensors:* $\sum_I\psi^I\bar{\psi}^{\sigma(I)}$ *where* $\sigma\in S_l$.

Proof. Let $f=\sum_{I,J}\alpha^{I,J}\psi^I\bar{\psi}^J\in(\langle\psi\rangle^l\otimes\langle\bar{\psi}\rangle^m)^{\mathfrak{u}(n)}$ wherea $\alpha^{I,J}$ are coefficients in $\boldsymbol{C}$. f is $\mathfrak{u}(n)$-invariant if and only if $\alpha^{I,J}=a^I_K\bar{a}^J_L\alpha^{K,L}$, for any $(a^i_j)\in U(n)$. Taking (a^i_j) to be the diagonal matrix with entries $a^i_i=1$ for $i\neq k$, $a^k_k=\lambda$ ($|\lambda|=1$), $a^i_j=0$ for $i\neq j$, we can show that the entries of I and J coincide. Letting $I_0=(1,\cdots,l)$ we prove the following

$$\alpha^{I,J}=\sum_{\sigma(I)=J}\alpha^{I_0,\sigma(I_0)} \tag{2.4}$$

by the decreasing induction on r=the number of distinct elements in I. For $r=m$ (2.4) clearly holds. Suppose (2.4) is true for $r=s+1$. We consider the special case when $I=J=(\underbrace{1,\cdots,1}_{a_1},\underbrace{2,\cdots,2}_{a_2},\cdots,\underbrace{s,\cdots,s}_{a_s})$. The other cases can be treated similarly. We can regard f as an $\boldsymbol{R}$-multilinear function: $(\boldsymbol{C}^n)^{2l}\to\boldsymbol{C}$ by considering ψ^i (resp. $\bar{\psi}^i$) to be the mapping: $\boldsymbol{C}^n\to\boldsymbol{C}$ given by

$$\psi^i(z_1,\cdots,z_n)=z_i\ (\text{resp. }\bar{\psi}^i(z_1,\cdots,z_n)=\bar{z}_i)\,.$$

We denote $e_i=(0,\cdots,\underset{\hat{i}}{1},\cdots,0)$. Since $\alpha^{K,L}=0$ unless K coincides with L as sets,

$$f(e_1{}^{a_1},e_2{}^{a_2},\cdots,e_{s-1}{}^{a_{s-1}},(e_s+e_{s+1})^{a_s-1},e_s,e_1{}^{a_1},\cdots,e_{s-1}{}^{a_{s-1}},e_{s+1}{}^{a_s})=0$$

where $e_1{}^{a_1}$ denotes $\underbrace{e_1,\cdots,e_1}_{a_1}$ and the same rule for other letters. Since the transformation A of $\boldsymbol{C}^n$ given by

$$\begin{aligned}&A\cdot e_s=1/\sqrt{2}(e_s-e_{s+1}),\quad A\cdot e_{s+1}=1/\sqrt{2}(e_s+e_{s+1})\\&A\cdot e_t=e_t\ (t\neq s)\end{aligned}$$

is a unitary transformation, we have

$$1/2(f(e_1{}^{a_1},\cdots,e_s{}^{a_s-1},e_s-e_{s-1},e_1{}^{a_1},\cdots,e_{s-1}{}^{a_{s-1}},(e_s+e_{s+1})^{a_s}))=0\,.$$

Calculation shows

$$\begin{aligned}&f(e_1{}^{a_1},\cdots,e_s{}^{a_s-1},e_s-e_{s-1},e_1{}^{a_1},\cdots,e_{s-1}{}^{a_{s-1}},(e_s+e_{s+1})^{a_s})\\&=\alpha^{I,I}-\sum_u\alpha^{1^{a_1},\cdots,s^{a_s-1},s+1,1^{a_1},\cdots,s^{a_s-1},s,\cdots,\underset{\hat{u}}{s+1},\cdots,s}\end{aligned}$$

where 1^{a_1} denotes $\underbrace{1,\cdots,1}_{a_1}$ etc.. Therefore by the induction assumption (2.4) holds for $r=s$. This completes the proof. q.e.d.

Now we define elements s_k and Φ of $M^{\mathfrak{u}(n)}$ by

$$
(2.5)\qquad \begin{aligned} s_k &= \operatorname{Trace}(\psi_j^i)^k \qquad \text{for } k \text{ even},\\ &= \sqrt{-1}\operatorname{Trace}(\psi_j^i)^k \qquad \text{for } k \text{ odd},\\ \Phi &= \sqrt{-1}\sum \psi^i\bar{\psi}^i = \sum \omega^i\omega^{n+i}\,. \end{aligned}
$$

These forms are real because ψ_j^i is skew-Hermitian.

Proposition 2.4. $M^{\mathfrak{u}(n)}=\boldsymbol{R}[s_1, \cdots, s_n, \Phi]/\{degree >2n\}$.

Proof. By Lemma 2.3 a basis for $(\langle\psi\rangle^p\otimes\langle\bar{\psi}\rangle^q\otimes\langle\psi\rangle^r)^{\mathfrak{u}(n)}$ is given by the tensors

$$
\omega(\sigma) = \sum_{I,J} \psi^I\bar{\psi}^{\sigma(I,J)_1}\psi^J_{\sigma(I,J)_2}
$$

where $(\sigma(I, J)_1, \sigma(I, J)_2)$ are defined by $\sigma(I, J)=(\sigma(I, J)_1, \sigma(I, J)_2)$. Therefore $N^{\mathfrak{u}(n)}_{p,q,r}=S_{p,q,r}(\langle\psi\rangle^p\otimes\langle\bar{\psi}\rangle^q\otimes\langle\psi\rangle^r)^{\mathfrak{u}(n)}$ is spanned by the tensors $\omega(\sigma)$: $\sigma\in S_{p+r}$. If we denote $\omega(\sigma)$ for the image of the projection $N^{\mathfrak{u}(n)}_{p,q,r}\to N^{\mathfrak{u}(n)}_{p,q,r}/N^{\mathfrak{u}(n)}_{p,q,r}\cap(I\otimes\boldsymbol{C})$, then $\omega(\sigma)=0$ for σ such that $\sigma(1, \cdots, p)\not\subset(1, \cdots, p)$ by the Bianchi's identity. Therefore

$$
\begin{aligned} (N^{\mathfrak{u}(n)}/N^{\mathfrak{u}(n)}\cap I)\otimes\boldsymbol{C} &= \bigoplus_{p,q,r} N^{\mathfrak{u}(n)}_{p,q,r}/N^{\mathfrak{u}(n)}_{p,q,r}\cap(I\otimes\boldsymbol{C})\\ &= \boldsymbol{C}[s_1, \cdots, s_n, \Phi]/\{\text{degree} >2n\}\,. \end{aligned}
$$

Taking the real part of this space we obtain our proposition. q.e.d.

Now we define a d.g.a. KW_n as follows. Let $Ts_k\in W(\mathfrak{e}_C(n))$ be the Chern-Simons' transgression form of s_k (cf. [3]) and u_k be $\pi(Ts_k)$, where π: $W(\mathfrak{e}_C(n))\to \tilde{W}(\mathfrak{e}_C(n))$ is the projection. Then clearly $du_k=s_k$. We define KW_n to be the subalgebra generated by u_k, s_k, Φ. It is easy to see that KW_n is isomorphic to

$$
E(u_1, \cdots, u_n)\otimes\hat{\boldsymbol{R}}[s_1, \cdots, s_n, \Phi]
$$

where E denotes the real exterior algebra and $\hat{\boldsymbol{R}}[\ \ ]$ is the real polynomial algebra truncated by the elements of degree$>2n$. Now recall that, in our spectral sequence computing the cohomology of $\tilde{W}(\mathfrak{e}_C(n))$, $E_1^{p,q}=H^q(\mathfrak{u}(n))\otimes M_p^{\mathfrak{u}(n)}$. The above results show that the inclusion of the subalgebra KW_n in $\tilde{W}(\mathfrak{e}_C(n))$ induces an isomorphism on the E_1-term. Therefore, by the spectral sequence comparison theorem, we obtain

Theorem 2.5. $H^*(\tilde{W}(\mathfrak{e}_C(n)))\cong H^*(KW_n)$.

Let $I=(i_1, \cdots, i_s)$ and $J=(j_1, \cdots, j_t)$ be s and t-tuples of positive integers with $i_1<\cdots<i_s$ and $j_1\leqq\cdots\leqq j_t$. We denote $u_Is_J\Phi^k$ for $u_{i_1}\cdots u_{i_s}s_{j_1}\cdots s_{j_t}\Phi^k\in KW_n$. Note that if $|J|+k>n$, then $u_Is_J\Phi^k=0$ where $|J|=j_1+\cdots+j_t$. Now the technique of Vey in [5] shows the following.

Proposition 2.6. *A basis for $H^*(KW_n)$ is given by the classes of the elements $u_I s_J \Phi^k$ with*

(i) $i_1+|J|+k>n$

(ii) $i_1 \leqq j_1$ *where we understand $i_1=\infty$ if $I=\phi$ and similarly for j_1.*

Since $\tilde{W}(\mathfrak{e}_C(n))^{\mathfrak{u}(n)}=M^{\mathfrak{u}(n)}$, we can determine $H^*(\tilde{W}(\mathfrak{e}_C(n))_{U(n)})$ as follows.

Proposition 2.7. *The classes of the elements $s_J\Phi^k$ with $|J|+k\leqq n$ form a basis of $H^*(\tilde{W}(\mathfrak{e}_C(n))_{U(n)})$.*

Now the class Φ has the following geometric meaning. On each Kähler manifold there is defined a 2-form called the Kählerian form and holomorphic isometries between Kähler manifolds preserve these forms. Therefore if F is a codimension n Kähler foliation on a smooth manifold M defined by submersions f_α: $U_\alpha \to (\boldsymbol{C}^n, g_\alpha)$ (see Definition 1.1), then the local forms $f_\alpha{}^*$ (Kähler form of g_α) on U_α define a global 2-form $\Phi(F)$ on M which is closed. We call $\Phi(F)$ the Kähler form of the foliation F. On the other hand, from the definition of our characteristic classes, we have a closed 2-form $\phi(\Phi)$ on M. We have

Proposition 2.8. $\phi(\Phi)=1/2\Phi(F)$.

Proof. Since $\Phi=\sum \omega^i\omega^{n+i}$, we have $\phi(\Phi)=\sum \theta^i\theta^{n+i}$. But it is easy to see that this form is the lift of $1/2\ \Phi(F)$ to $U(F)$. q.e.d.

3. Relation with Riemannian case

Let F be a codimension n Riemannian foliation on a smooth manifold M and let $O(F)$ be the orthonormal frame bundle of F. Let $E(n)$ be the group of Euclidean motions on $\boldsymbol{R}^n$, $\mathfrak{e}(n)$ the Lie algebra of $E(n)$ and $W(\mathfrak{e}(n))$ the Weil algebra of $\mathfrak{e}(n)$. Then in [6], [7], [10] a characteristic homomorphism

$$\phi\colon H^*(\tilde{W}(\mathfrak{e}(n)) \to H^*(BR\bar{\Gamma}_n;\ \boldsymbol{R})$$

was constructed, where $BR\bar{\Gamma}_n$ is the classifying space for codimension n Riemannian Haefliger structures with trivial normal bundles and $\tilde{W}(\mathfrak{e}(n))$ is the quotient algebra of $W(\mathfrak{e}(n))$ by some ideal. Let $f_{2k}\in I(\mathfrak{so}(n))$ be defined by $f_{2k}(X)=\mathrm{Trace}(X^{2k})$ for $X\in\mathfrak{so}(n)$ and for even n let $\chi\in I(\mathfrak{so}(n))$ be the Euler form. We can consider f_{2k}, χ to be elements of $W(\mathfrak{e}(n))$. Let Tf_{2k}, $T\chi$ be the transgression forms of f_{2k}, χ respectively. If we set $c_{2k}=\pi(f_{2k})$, $h_{2k}=\pi(Tf_{2k})$, $\chi=\pi(\chi)$ and $h_\chi=\pi(T\chi)$ where $\pi\colon W(\mathfrak{e}(n))\to\tilde{W}(\mathfrak{e}(n))$ is the projection, then the subcomplex RW_n of $\tilde{W}(\mathfrak{e}(n))$ generated by h_{2k}, c_{2k} and if n is even also by h_χ, χ is a finite complex expressed as

$$\begin{aligned} RW_n &= E(h_2, h_4, \cdots, h_{n-1})\otimes\hat{\boldsymbol{R}}[c_2, c_4, \cdots, c_{n-1}] \qquad n \text{ odd},\\ &= E(h_2, h_4, \cdots, h_{n-2}, h_\chi)\oplus\hat{\boldsymbol{R}}[c_2, c_4, \cdots, c_{n-2}, \chi] \qquad n \text{ even}.\end{aligned}$$

Furthermore let $r_p \in W(\mathfrak{e}(n))$ be the "*p-th* scalar curvature" defined in [10]. If we denote

$$\begin{aligned} E_n &= E(h_2, h_4, \cdots, h_{n-1}) && n \text{ odd}, \\ &= E(h_2, h_4, \cdots, h_{n-2}, h_\chi) && n \text{ even}, \end{aligned}$$

then all the forms of $r_p E_n$ are closed and therefore $RW_n \oplus \sum_{\substack{p:\text{even} \\ 0 \le p < n}} r_p E_n$ is a subcomplex of $\tilde{W}(\mathfrak{e}(n))$. The inclusion induces an isomorphism on cohomology. Namely

$$H^*(\tilde{W}(\mathfrak{e}(n))) \cong H^*(RW_n) \oplus \sum_{\substack{p:\text{even} \\ 0 \le p < n}} r_p E_n \tag{3.1}$$

(see Theorem 3.1 of [10]).

Now we have the forgetful map

$$BK\bar{\Gamma}_n \to BR\bar{\Gamma}_{2n} \quad \text{and} \quad BK\Gamma_n \to BR\Gamma_{2n}\,.$$

Let i: $\mathfrak{e}_C(n) \to \mathfrak{e}(2n)$ be the natural inclusion.

Proposition 3.1. *The following diagrams are commutative.*

$$\begin{array}{ccc} H^*(\tilde{W}(\mathfrak{e}(2n))) & \longrightarrow & H^*(BR\bar{\Gamma}_{2n}; \boldsymbol{R}) \\ \downarrow & & \downarrow \\ H^*(\tilde{W}(\mathfrak{e}_C(n))) & \longrightarrow & H^*(BK\bar{\Gamma}_n; \boldsymbol{R})\,, \end{array}$$

$$\begin{array}{ccc} H^*(\tilde{W}(\mathfrak{e}(2n))_{O(2n)}) & \longrightarrow & H^*(BR\Gamma_{2n}; \boldsymbol{R}) \\ \downarrow & & \downarrow \\ H^*(\tilde{W}(\mathfrak{e}_C(n))_{U(n)}) & \longrightarrow & H^*(BK\Gamma_n; \boldsymbol{R})\,. \end{array}$$

The homomorphisms $i^*: H^*(\tilde{W}(\mathfrak{e}(2n))) \to H^*(\tilde{W}(\mathfrak{e}_C(n)))$, $i^*: H^*(\tilde{W}(\mathfrak{e}(2n))_{O(2n)}) \to H^*(\tilde{W}(\mathfrak{e}_C(n))_{U(n)})$ in terms of h_i, c_j, χ, r_p, u_i, s_j, etc. can be completely determined. For example, as is well known, the image under i^* of monomials on c_j (=the Pontrjagin classes) can be uniquely described as polynomials on s_j (=the Chern classes) and of course $i^*(\chi)$=the *n-th* Chern class. The formura for the image of h_i can be easily deduced from the definitions. We omit the detailed description of these formulas. Here we only mention the formula for the class r_p.

Proposition 3.2.

$$i^*[r_p] = (-1)^{(n(n-1)+p)/2} 2^{p/2} (2n-p)!/(n-p/2)! [\Phi^{n-p/2} \sum_{|I|=p/2} a_I s_I]$$

where $a_I = (-1)^{\sum_{k=1}^{s} [(i_k+1)/2]} n!/n(n-i_1)\cdots(n-i_1-\cdots-i_{t-1})$ *for* $I=(i_1, \cdots, i_t)$.

The proof of this proposition can be given by calculations using the Bianchi's identity and is left to the readers.

This proposition shows that the *p-th* scalar curvature r_p for a Kähler foliation can be expressed in terms of the Kähler form and the Chern forms.

4. Relation with holomorphic case

A Kähler foliation can be regarded as a holomorphic foliation by forgetting the Kähler structures. We recall the construction of the characteristic classes of holomorphic foliations given by Bott in [2].

Let F be a holomorphic foliation on a smooth manifold M and let $J_C^2(F)$ be the bundle of holomorphic 2-jets of F. We have a d.g.a. map

$$\psi\colon W(\mathfrak{gl}(n;\boldsymbol{C})) \to \Omega^*(J_C^2(F))\otimes\boldsymbol{C}$$

defined by $\psi(\omega_j^i)=\theta_j^i$, $\psi(\Omega_j^i)=d\theta_j^i+\theta_k^i\wedge\theta_j^k$ where ω_j^i, Ω_j^i are the universal connection and curvature forms of $\mathfrak{gl}(n;\boldsymbol{C})$ in terms of the natural basis and θ_j^i is the second order canonical forms on $J_C^2(F)$. ψ has a kernel I generated by monomials on Ω_j^i with degree $>2n$. Therefore if we set $\tilde{W}(\mathfrak{gl}(n;\boldsymbol{C}))=W(\mathfrak{gl}(n;\boldsymbol{C}))/I$ and assume that the normal bundle of F is trivialized by a cross section $s\colon M\to J_C^2(F)$, then we obtain a homomorphism

$$\psi^*\colon H^*(\tilde{W}(\mathfrak{gl}(n;\boldsymbol{C}))) \to H^*(J_C^2(F);\boldsymbol{C}) \to H^*(M\colon\boldsymbol{C})\,.$$

Since this construction is functorial, we have

$$\psi\colon H^*(\tilde{W}(\mathfrak{gl}(n;\boldsymbol{C}))) \to H^*(B\bar{\Gamma}_n\boldsymbol{C};\boldsymbol{C})\,.$$

Let $s_i\in I(\mathfrak{gl}(n;\boldsymbol{C}))$ be given by $s_i(X)=\text{Trace}\,(\sqrt{-1}\,X)^i$ for $X\in\mathfrak{gl}(n;\boldsymbol{C})$ and let $u_i=Ts_i$: the transgression form of s_i. u_i and s_i can be considered as elements of $W(\mathfrak{gl}(n;\boldsymbol{C}))$ and we use the the same letters for their images in $\tilde{W}(\mathfrak{gl}(n;\boldsymbol{C}))$. Now let $W_n^{\boldsymbol{C}}$ be the subalgebra of $\tilde{W}(\mathfrak{gl}(n;\boldsymbol{C}))$ generated by the elements s_i, u_i. Then we may write $W_n^{\boldsymbol{C}}=E(u_1,\cdots,u_n)\otimes\hat{\boldsymbol{C}}[s_1,\cdots,s_n]$ as usual (see [2]) and the inclusion $i\colon W_n^{\boldsymbol{C}}\to\tilde{W}(\mathfrak{gl}(n;\boldsymbol{C}))$ induces an isomorphism on cohomology.

Theorem 4.1. *Let F be a codimension n Kähler foliation on a smooth manifold M with a trivialized normal bundle. Then the class $\psi(u_Is_J)$ is a real class and coincides with $\phi(u_Is_J)$.*

Proof. This follows from the definitions of the characteristic classes of Kähler and holomorphic foliations. The point here is the fact that the s_i-form of a complex vector bundle with a Hermitian connection is a real form.
q.e.d.

REMARK 4.2. Bott[2] has also defined characteristic classes of holomorphic foliations whose normal bundles are not necessarily trivial by comparing Bott and Hermitian connections. For a Kähler foliation these classes are all zero because the unique torsionfree Hermitian connection is also a Bott connection.

5. Continuous variation

In this section we study continuous variations of our characteristic classes.

DEFINITION 5.1 An element α of $H^*(KW_n)$ is called rigid if for any one parameter family F_t of codimension n Kähler foliations on a smooth manifold M, the classes $\phi(\alpha)(F_t)$ is constant with respect to t, namely $\frac{d}{dt}(\phi(\alpha)(F_t))=0$ holds.

By the same argument as in Heitsch [5] we obtain

Proposition 5.2. *The class $[u_I s_J]$ is rigid if $i_1+|J|>n+1$.*

We conjecture that these classes are the only rigid classes. Thus the classes of $u_I s_J \Phi^k$ would be non rigid if $k>0$ or $k=0$, $i_1+|J|=n+1$. We cannot prove this conjecture at the moment. In the following we prove partial solution to it.

Let (M, g) be a Kähler manifold of dimension n and π: $U(M)\rightarrow M$ the unitary frame bundle of M. We define a smooth family of codimension n Kähler foliations $F(M, t)$ on $U(M)$ as follows. Let $(M, t^2 g)$ be the Kähler manifold obtained from M by the scale change $g\rightarrow t^2 g$ $(t>0)$. Then $F(M, t)$ is a foliation on $U(M)$ defined by pulling back the Kähler structure of $(M, t^2 g)$ by the projection π.
The unitary frame bundle of this foliation $U(F(M, t))$ has a cross section

$$\begin{array}{ccc} s_t\colon U(M) & \rightarrow & U(F(M, t))\subset U(M)\times U(M, t^2 g) \\ \Cup & & \Cup \\ x & \longrightarrow & (x, x/t) \end{array}$$

where $U(M, t^2 g)$ is the unitary frame bundle of the Kähler manifold $(M, t^2 g)$. From the definition of characteristic classes, we obtain

Proposition 5.3.

$$[u_I s_J \Phi^k](F(M, t)) = t^{2k}[(Ts)_I(M)\pi^*(s_J(M))\pi^*(\Phi(M, g))]$$

where $s_J(M)$ is the characteristic form of M corresponding to s_J, $(Ts)_I(M)$ is the Chern-Simons' transgression form of s_I and $\Phi(M, g)$ is the Kähler form of (M, g).

Now we show the following result.

Proposition 5.4. *Let $N(=2^{n-1}n)$ be the number of the bases in Proposition 2.6 with $i_1=1$, $J=(n-k)$, $k\geqq 1$. Then there is a surjective homomorphism*

$$H_*(BK\overline{\Gamma}_n; \boldsymbol{Z}) \to \boldsymbol{R}^N \to 0\,.$$

We prepare several lemmas.

Lemma 5.5. *Let $P^k(\boldsymbol{C})$ be the complex projective space with the standard Kähler metric. Then the classes of $Ts_1(M)(Ts)_I(M)\pi^*(s_k(M))$ are linearly independent in $H^*(U(P^k(\boldsymbol{C})); R)$.*

Proof. By a well known theorem (see [4] for example), we have an isomorphism.

$$\Delta\colon H^*(U(P^k(\boldsymbol{C}))\colon \boldsymbol{R}) \to H^*(\boldsymbol{R}[\alpha]/(\alpha^{k+1})\otimes E(u_1, \cdots, u_k))$$

where the right hand side is the cohomology of a differential complex $\boldsymbol{R}[\alpha]/(\alpha^{k+1})\otimes E(u_1, \cdots, u_k)$ with a differential d defined by $d\alpha=0$, $du_i=\alpha^i$. The isomorphism Δ satisfies

$$\Delta([(Ts)_I(M)s_J(M)]) = [\alpha^{|J|}u_I]\,.$$

Therefore for the proof of the lemma, it is enough to show that $[\alpha^k u_1 u_I]$ are linearly independent in $H^*(\boldsymbol{R}[\alpha]/(\alpha^{k+1})\otimes E(u_1, \cdots, u_k))$. But this can be easily checked by a spectral sequence argument. q.e.d.

Lemma 5.6. *Let T be the complex 1-dimensional torus with the standard Kähler metric and M be a disjoint union of porducts of $P^{n-k}(\boldsymbol{C})$ and T^k. Then $[u_1u_Is_{n-k}\Phi^k]$ $(F(M, 1))$ are linearly independent in $H^*(U(M); \boldsymbol{R})$.*

Proof. Similarly as in Lemma 5.5, we have isomorphisms

$$\begin{aligned}\Delta_k\colon & H^*(U(P^{n-k}(\boldsymbol{C})\times T^k); \boldsymbol{R}) \to \\ & \qquad H^*(U(P^{n-k}(\boldsymbol{C}); \boldsymbol{R})\otimes H^*(T^k; \boldsymbol{R})\otimes E(u_1, \cdots, u_k)\end{aligned}$$

for $0\leqq k\leqq n$.
Let $\pi_k\colon H^*(T^k; \boldsymbol{R})\to H^{2k}(T^k; \boldsymbol{R})$ be the projection onto the part with degree $2k$ and $i_k\colon P^{n-k}(\boldsymbol{C})\times T^k\to \bigcup_l P^{n-l}(\boldsymbol{C})\times T^l$ be the inclusion. Then one can easily show

$$\begin{aligned}&(1\otimes\pi_k\otimes 1)\circ\Delta_k\circ i_k^*([u_1u_Is_{n-l}\Phi^l](F(M, 1))\\ &=0 \qquad\qquad l\neq k\,,\\ &=[Ts_1(M)(Ts)_{I_1}(M)\pi^*(s_{n-k}(M))]\otimes[\Phi(T^k)]^k\otimes u_{I_2} \qquad l=k\,,\end{aligned}$$

where $I_1=I\cap\{1, \cdots, n-k\}$, $I_2=I\cap\{n-k+1, \cdots, n\}$.
Hence the image of $u_1u_Is_{n-k}\Phi^k$ by the map

$$\begin{aligned}(1\otimes\pi_k\otimes 1)\circ\Delta_k\circ i_k^*\colon & H^*(U(M); \boldsymbol{R}) \to \\ & H^*(U(P^{n-k}(\boldsymbol{C})); \boldsymbol{R})\otimes H^*(T^k; \boldsymbol{R})\otimes E(u_{n-k+1}, \cdots, u_n)\end{aligned}$$

is

$$[Ts_1(P^{n-k}(\boldsymbol{C}))(Ts)_{I_1}(P^{n-k}(\boldsymbol{C}))\pi^*(s_{n-k}(P^{n-k}(\boldsymbol{C})))]\otimes[\Phi(T^k)]^k\otimes u_{I_2}\,.$$

Therefore in view of Lemma 5.5 we complete the proof. q.e.d.

Proof of Proposition 5.2 Choose homology classes $x(I, k) \in H_*(U(M); \boldsymbol{Z})$ so that the matrix $(\langle x(I, k), [u_1 u_J s_{n-l} \Phi^l](F(M, 1))\rangle)$ is non-singular. We introduce an N-vector valued parameter $t=(t(I, k)) \in \boldsymbol{R}^N$ and put

$$x(t) = \sum f_{t(I,k)*} x(I, k) \in H_*(BK\Gamma_n; \boldsymbol{Z})$$

where $f_{t(I,k)}: U(M) \to BK\Gamma_n$ is the classifying map of the foliation $F(M, t(I, k))$. Then

$$\langle x(t), [u_1 u_I s_{n-k} \Phi^k]\rangle = \sum_{J,l} t(J, l)^k \langle x(J, l), [u_1 u_I s_{n-k} \Phi^k](F(M, 1)\rangle .$$

Therefore if we define a map $\lambda: \boldsymbol{R}^N \to \boldsymbol{R}^N$ by $\lambda(t)=(\langle x(t), [u_1 u_I s_{n-k}, \Phi^k]\rangle)$, then the Jacobian matrix of λ at $t(I, k)=k-1$ is $\langle x(I, k), [u_1 u_I s_{n-l} \Phi^l](F(M, 1))\rangle$ which is non-singular. This completes the proof. q.e.d.

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