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A NOTE ON COHOMOLOGY WITH LOCAL COEFFICIENTS

YASUMASA HIRASHIMA

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For the ordinary cohomology groups of a good space X we have the homotopy classification theorem, $H^q(X; G) \cong [X, K(G, q)]$. It has been discussed by many authors (Olum [9], Gitler [5], Siegel [12] and McClendon [8]) that a similar theorem is valid for the cohomology with local coefficients. In this paper we give an elementary reasonable proof of the classfication theorem by means of direct generalizations of the results in May [6] and Cartan [1].

We frequently use the notations introduced in [6] without notice.

1. Definitions and main theorems

Let S denote the category of simplicial sets, in which we shall work. Let π be an abstract group, and let G be an abelian group. We fix a group homomorphism $\phi: \pi \to AutG$, where $AutG$ is the automorphism group of G. Let $(X, Y; \tau)$ be a simplicial pair (X, Y) (Y may be empty) with a twisting function (see [6]) $\tau: X \to \pi$. We define the group of cochains $C^n_\phi(X, Y; \tau)$ to be

$$\{f: X_n \to G \mid f(x) = 0 \text{ if } x = s_i y \text{ or } x \in Y_n\},$$

the coboundary δ by

$$\delta f(x) = \tau(x)^{-1} f(\partial_0 x) + \sum_{i=1}^{n+1} (-1)^i f(\partial_i x),\ x \in X_{n+1}, f \in C^n_\phi(X, Y; \tau).$$

$H^n_\phi(X, Y; \tau) = H^n_\phi(C^*_\phi(X, Y; \tau), \delta)$ is called the twisted cohomology group of $(X, Y; \tau)$ by ϕ.

Let L be a local system on X, i.e. a contravariant functor from the fundamental groupoid (see [4]) πX to the category of abelian groups. Suppose X is connected. We fix a vertex $x_0 \in X_0$ and $u_x \in \pi X(x_0, x)$, $x \in X_0$, in particular we choose $u_{x_0} = 1_{x_0}$. Then we have the twisting function $F(x_0, (u_x)): X \to \pi_1 X$ by

$$F(x_0, (u_x))(y) = u_{\partial_0 \partial_2 \cdots \partial_n y}^{-1} \partial_2 \cdots \partial_n y\, u_{\partial_1 \partial_2 \cdots \partial_n y},\ y \in X_n,$$

a group homomorphism $\phi(L): \pi_1 X \to AutL(x_0)$ by

$$ag = \phi(L)(a)(g) = L(a)^{-1}(g),\ a \in \pi_1 X = \pi X(x_0, x_0),\ g \in L(x_0)\,.$$

We remark that the multiplication of the fundamental group $\pi_1 X$ coincides with the morphism composition of πX (see [4], [6]) contrary to the usual topological definition. We define the local coefficient cohomology group of (X, Y), $H^n(X, Y; L)$ $(=H^n(X, Y; L, x_0, (u_x)))$, to be $H^n_{\phi(L)}(X, Y; F(x_0, (u_x)))$ (see [4; Appendix II], [9] and [13; Part III])

Let A^* be a $Aut\, G$-equivariant simplicial DG abelian group, i.e. ∂_i, s_i and δ commute with the $Aut\, G$-action. We suppose that A^* satisfies the following axioms (see [1]).

Axiom (a): *the sequence* $A^0 \xrightarrow{\delta} A^1 \xrightarrow{\delta} \cdots \xrightarrow{\delta} A^n \xrightarrow{\delta} \cdots$ *is exact, and* $Z^0 A = Ker\,\{\delta\colon A^0 \to A^1\}$ *is Aut G-isomorphic to the trivial simplicial abelian group* G.

Axiom (b): A^n *is* $Aut\, G$*-cotractible relative* $\{0\}$ $(n \geq 0)$.

EXAMPLE 1. Let C^* be the simplicial DG abelian group which is obtained by applying the normalized cochain complex functor $C^*(\ ; G)$ to the cosimplicial simplicial set

$$\Delta[0] \rightrightarrows \Delta[1] \mathrel{\substack{\rightarrow\\\rightarrow\\\rightarrow}} \cdots \mathrel{\substack{\rightarrow\\\vdots\\\rightarrow}} \Delta[n] \mathrel{\substack{\rightarrow\\\vdots\\\rightarrow}} \cdots .$$

EXAMPLE 2. Let A^*_{PL} be the simplicial DG algebra which is obtained by applying the PL de Rham functor (see [3]) to the cosimplicial simplicial complex

$$\Delta^0 \rightrightarrows \Delta^1 \mathrel{\substack{\rightarrow\\\rightarrow\\\rightarrow}} \cdots \mathrel{\substack{\rightarrow\\\vdots\\\rightarrow}} \Delta^n \mathrel{\substack{\rightarrow\\\vdots\\\rightarrow}} \cdots .$$

These two examples C^*, A^*_{PL} satisfy the axioms (see [1], [6]). For example Axiom (b) of A^n_{PL} is proved by the contraction

$$A^n_{PL} \times \Delta[1] \xrightarrow{1 \times \bar{c}} A^n_{PL} \times A^0_{PL} \xrightarrow{\mu} A^n_{LP}$$

where $c \in (A^0_{PL})_1 = Q[t]$, $c(t) = t$, μ is the multiplication of the algebra A^*_{PL}.

Let $A^n \times_\pi W\pi \to \bar{W}\pi$ be the π-bundle with fibre A^n obtained by ϕ and the universal π-bundle $W\pi \to \bar{W}\pi$ (see [6]). If $\tau(\pi)$ is the canonical twisting function $\tau(\pi)[g_1, g_2, \cdots, g_q] = g_1$, then $A^n \times_\pi W\pi \to \bar{W}\pi$ is identified with the TCP $A^n \times_{\tau(\pi)} \bar{W}\pi \to \bar{W}\pi$. This Kan fibration $A^n \times_{\tau(\pi)} \bar{W}\pi \to \bar{W}\pi$ has the 0-section $s\colon \bar{W}\pi \to A^n \times_{\tau(\pi)} \bar{W}\pi$ $s[g_1, \cdots, g_q] = (0, [g_1, \cdots, g_q])$ whose image $\{0\} \times_{\tau(\pi)} \bar{W}\pi$ is also denoted by $\bar{W}\pi$. Let $\theta(\tau)\colon X \to \bar{W}\pi$ be the map $\theta(\tau)(x) = [\tau(x), \tau(\partial_0 x), \cdots, \tau(\partial_0^{q-1}x)]$, $x \in X_q$. Let $A_\Phi^n(X, Y; \tau) = \boldsymbol{S}((X, Y), (A^n \underset{\tau(\pi)}{\times} \bar{W}\pi, \bar{W}\pi))_{\bar{W}\pi}$ denote the set of liftings (or maps over $\bar{W}\pi$)

$$\begin{tikzcd} & (A^n \underset{\tau(\pi)}{\times} \bar{W}\pi, \bar{W}\pi) \arrow[d, "p"] \\ (X, Y) \arrow[ur] \arrow[r, "\theta(\tau)"'] & (\bar{W}\pi, \bar{W}\pi) \end{tikzcd},$$

then we can define the group structure on $A_\Phi^n(X, Y; \tau)$ by the fibrewise addition, the fibrewise inversion and the 0-section. Further we define the differential $(\delta \underset{\tau(\pi)}{\times} 1)_*$ so that $(A_\Phi^*(X, Y; \tau), (\delta \underset{\tau(\pi)}{\times} 1)_*)$ is a DG abelian group. We remark that the sequence

$$0 \to A_\Phi^n(X, Y; \tau) \to A_\Phi^n(X; \tau) \to A_\Phi^n(Y; \tau|_Y) \to 0$$

is exact, since $\boldsymbol{S}$ is a closed model category (see [10]) and $A^n \underset{\tau(\pi)}{\times} \bar{W}\pi \to \bar{W}\pi$ is a trivial fibration. By Axiom (a) we find $Z^n A = Ker\{\delta\colon A^n \to A^{n+1}\}$ is an Eilenberg-MacLane complex of type (G, n), especially $Z^n C$ is a $K(G, n)$. Let $[(X, Y), (Z^n A \underset{\tau(\pi)}{\times} \bar{W}\pi, \bar{W}\pi)]_{\bar{W}\pi}$ denote the set of vertical homotopy classes of liftings

$$\begin{tikzcd} & (Z^n A \underset{\tau(\pi)}{\times} \bar{W}\pi, \bar{W}\pi) \arrow[d, "p"] \\ (X, Y) \arrow[ur] \arrow[r, "\theta(\tau)"'] & (\bar{W}\pi, \bar{W}\pi) \end{tikzcd}.$$

The set $[(X, Y), (Z^n A \underset{\tau(\pi)}{\times} \bar{W}\pi, \bar{W}\pi)]_{\bar{W}\pi}$ has also the group structure induced by that of $A_\Phi^n(X, Y; \tau)$. $Z^n C \underset{\pi}{\times} W\pi = Z^n C \underset{\tau(\pi)}{\times} \bar{W}\pi = K(G, n) \underset{\tau(\pi)}{\times} \bar{W}\pi$ is sometimes denoted by $L_\phi(G, n)$ (see [5]).

Theorem 1.1.

$$H^n(A_\Phi^*(X, Y; \tau), (\delta \underset{\tau(\pi)}{\times} 1)_*) = [(X, Y), (Z^n A \underset{\tau(\pi)}{\times} \bar{W}\pi, \bar{W}\pi)]_{\bar{W}\pi}.$$

Theorem 1.2. *There is a natural chain isomorphism*

$$(C_\Phi^*(X, Y; \tau), \delta) \simeq (\boldsymbol{S}((X, Y), (L_\phi(G, *), \bar{W}\pi))_{\bar{W}\pi}, (\delta \underset{\tau(\pi)}{\times} 1)_*).$$

Corollary 1.3.

$$H_\Phi^n(X, Y; \tau) \simeq [(X, Y), (L_\phi(G, n), \bar{W}\pi)]_{\bar{W}\pi}.$$

Therefore as a corollary we have the vertical homotopy classification of the cohomology groups of (X, Y) with local coefficients in L.

Corollary 1.4.

$$H^n(X, Y; L) \cong [(X, Y), (L_{\phi(L)}(G, n), \bar{W}\pi_1 X)]_{\bar{W}\pi_1 X}.$$

2. Proofs of main theorems

As the proof of Theorem 1.1 is essentially the same as that of [1], we only outline it. Bundle-theoretic replacement of the notion of exact sequence leads to

$$\mathrm{Ker}\,\{A^n_\phi(X, Y; \tau) \to A^{n+1}_\phi(X, Y; \tau)\} = \boldsymbol{S}((X, Y), (Z^n A \underset{\tau(\pi)}{\times} \bar{W}\pi, \bar{W}\pi))_{\bar{W}\pi}.$$

The equality

$$\mathrm{Im}\,\{A^{n-1}_\phi(X, Y; \tau) \to A^n_\phi(X, Y; \tau)\} =$$
$$\{f \in \boldsymbol{S}((X, Y), (Z^n A \underset{\tau(\pi)}{\times} \bar{W}\pi, \bar{W}\pi))_{\bar{W}\pi} \mid f \text{ is fibre homotopic to the trivial fibre map}\}$$

is proved by Axiom (b) and the CHEP (Covering Homotopy Extension Property) of $A^{n-1} \underset{\tau(\pi)}{\times} \bar{W}\pi \to Z^n A \underset{\tau(\pi)}{\times} \bar{W}\pi$ which follows from the next

Lemma 2.1. *If π acts simplicially on simplicial abelian groups A and B, and if $\psi: A \to B$ is a π-equivariant simplicial epimorphism, then $A \underset{\tau(\pi)}{\times} \bar{W}\pi \to B \underset{\tau(\pi)}{\times} \bar{W}\pi$ is a Kan fibration.*

Proof. Suppose $x_0, \cdots, x_{k-1}, x_{k+1}, \cdots, x_{q+1} \in (A \underset{\tau(\pi)}{\times} \bar{W}\pi)_q$ satisfy $\partial_i x_j = \partial_{j-1} x_i$, $i<j$, $i, j \neq k$, and $(b, [g_1, \cdots, g_q]) \in (B \underset{\tau(\pi)}{\times} \bar{W}\pi)_{q+1}$ satisfies $\partial_i(b, [g_1, \cdots, g_q]) = (\psi \underset{\tau(\pi)}{\times} 1) x_i$, $i \neq k$. Since $A \underset{\tau(\pi)}{\times} \bar{W}\pi \to \bar{W}\pi$ is a Kan fibration, there exists $(a, [g_1, \cdots, g_q]) \in (A \underset{\tau(\pi)}{\times} \bar{W}\pi)_{q+1}$ such that $\partial_i(a, [g_1, \cdots, g_q]) = x_i$, $i \neq k$. It follows that $\partial_i b = \partial_i \psi(a)$, $i \neq k$. We note that ψ is a principal *Ker* ψ bundle, therefore it is a Kan fibration. It follows that there exists $a' \in A_{q+1}$ such that $\partial_i a = 0$, $i \neq k$, and $\psi a' = b - \psi a$. Then $(a+a', [g_1, \cdots, g_q]) \in (A \underset{\tau(\pi)}{\times} \bar{W}\pi)_{q+1}$ satisfies $\partial_i(a+a', [g_1, \cdots, g_q]) = x_i$, $i \neq k$, and $(\psi \underset{\tau(\pi)}{\times} 1)(a+a', [g_1, \cdots, g_q]) = (b, [g_1, \cdots, g_q])$. This completes the proof.

As for Example 1 we remark that $C^{n-1} \underset{\tau(\pi)}{\times} \bar{W}\pi \to Z^n C \underset{\tau(\pi)}{\times} \bar{W}\pi$ can be written directly as a TCP so that we need not Lemma 2.1.

To prove Theorem 1.2 we must generalize [6; §24] to our situation. Since $C^n_\phi(X, Y; \tau) = \mathit{Ker}\,\{C^n_\phi(X; \tau) \to C^n_\phi(Y; \tau|_Y)\}$, $\boldsymbol{S}((X, Y), (L_\phi(G, n), \bar{W}\pi))_{\bar{W}\pi} = \mathit{Ker}\,\{\boldsymbol{S}(X, L_\phi(G, n))_{\bar{W}\pi} \to \boldsymbol{S}(Y, L_\phi(G, n))_{\bar{W}\pi}\}$, a proof of the absolute version of Theorem 1.2 suffices to prove Theorem 1.2. We define the n-cochain $u(=u_n) \in C^n_\phi(C^n \underset{\tau(\pi)}{\times} \bar{W}\pi; \tau(\pi)p)$ by $u(c, [g_1, \cdots, g_n]) = c(0, 1, \cdots, n)$ of which we call the fundamental n-cochain. If $x \in (C^n \underset{\tau(\pi)}{\times} \bar{W}\pi)_q$, then $\bar{x}^*: C^n_\phi(C^n \underset{\tau(\pi)}{\times} \bar{W}\pi; \tau(\pi)p) \to C^n(\Delta[q]; \tau(\pi)p\bar{x})$ is induced by $\bar{x}: \Delta[q] \to C^n \underset{\tau(\pi)}{\times} \bar{W}\pi$. Let $E: C^k_\phi(\Delta[q]; \tau(\pi)p\bar{x}) \to C^k_q = C^k(\Delta[q]; G)$ be a map defined by

$$Efy = \tau(\pi)p\bar{x}(0, y_0)^{-1}fy,\ y = (y_0, y_1, \cdots, y_k) \in \Delta[q]_k,\ f \in C^k_\Phi(\Delta[q];\ \tau(\pi)p\bar{x}).$$

Lemma 2.2.

(1) *E is a chain isomorphism.*

(2) $E\delta_i^* = \begin{cases} \tau(\pi)(px)\delta_0^* E & (i=0) \\ \delta_i^* E & (i \neq 0). \end{cases}$

(3) $E\sigma_i^* = \sigma_i^* E.$

Here δ_i: $\Delta[q-1] \to \Delta[q]$, σ_i: $\Delta[q+1] \to \Delta[q]$ *are the standard (co) face and (co) degeneracy operators.*

Proof. (1) If $y = (y_0, \cdots, y_k) \in \Delta[q]_k$ then

$$\begin{aligned} E\delta f(y) &= \tau(\pi)(p\bar{x}(0, y_0))^{-1}\tau(\pi)(p\bar{x}(y))^{-1}f(\partial_0 y) \\ &\quad + \sum_{i=1}^{k}(-1)^i\tau(\pi)(p\bar{x}(0, y_0))^{-1}f(\partial_i y) \\ &= \tau(\pi)(p\bar{x}(0, y_1)^{-1}f(\partial_0 y) + \sum_{i=1}^{k}(-1)^i\tau(\pi)(p\bar{x}(0, y_0))^{-1}f(\partial_i y) \\ &= \sum_{i=0}^{k}(-1)^i Ef(\partial_i y) \\ &= \delta Ef(y). \end{aligned}$$

(2) Consider the next diagram

$$\begin{array}{ccc} C^k_\Phi(\Delta[q];\ \tau(\pi)p\bar{x}) & \xrightarrow{E} & C^k_q \\ \downarrow \delta_i^* & & \downarrow \partial_i = \delta_i^* \\ C^k_\Phi(\Delta[q-1];\ \tau(\pi)p\overline{\partial_i x}) & \rightarrow & C^k_{q-1}. \end{array}$$

If $y = (y_0, \cdots, y_k) \in \Delta[q-1]_k$, $f \in C^k_\Phi(\Delta[q];\ \tau(\pi)p\bar{x})$, then we find

$$\begin{aligned} E\delta_i^* f(y) &= \tau(\pi)(p\bar{x}\delta_i(0, y_0))^{-1}f(\delta_i y) \\ &= \tau(\pi)(p\bar{x}(\delta_i 0, \delta_i y_0))^{-1}f(\delta_i y) \\ &= \tau(\pi)(p\bar{x}(0, \delta_i 0))\tau(\pi)(p\bar{x}(0, \delta_i y_0))^{-1}f(\delta_i y) \\ &= \tau(\pi)(p\bar{x}(0, \delta_i 0))\delta_i^* Ef(y). \end{aligned}$$

(3) The proof is parallel to that of (2).

Lemma 2.3. *Let* $\tilde{x}$: $C^n_\Phi(C^n \underset{\tau(\pi)}{\times} \bar{W}\pi;\ \tau(\pi)p) \to (C^n \underset{\tau(\pi)}{\times} \bar{W}\pi)_q$ *be the map* $\tilde{x}(f) = (E\bar{x}^* f, px)$. *Then we have* $\tilde{x}(u) = x$.

Proof. Put $x = (c, g) \in (C^n \underset{\tau(\pi)}{\times} \bar{W}\pi)_q$. Let $y = (y_0, \cdots, y_n) \in \Delta[q]_n$ be a non-degenerate simplex. At first we find $y = y^*(0, 1, \cdots, q)$, $y^* = \partial_{i_1}\partial_{i_2}\cdots\partial_{i_{q-n}}$, $0 \leq i_1 < i_2 < \cdots < i_{q-n} \leq q$, $\{y_0, \cdots, y_n, i_1, \cdots, i_{q-n}\} = \{0, 1, \cdots, q\}$. Thus we have

$$
\begin{aligned}
E\bar{x}^*u(y) &= \tau(\pi)(p\bar{x}(0, y_0))^{-1}u(\bar{x}y) = \tau(\pi)(p\bar{x}(0, y_0))^{-1}u(y^*x) \\
&= \begin{cases} u(y^*c, y^*g) & (y_0 = 0) \\ \tau(\pi)(p\bar{x}(0, y_0))^{-1}u(\tau(\pi)(\partial_{i_2}\partial_{i_3}\cdots\partial_{i_{q-n}}g)y^*, y^*g) & (y_0 \neq 0) \end{cases}
\end{aligned}
$$

Since $\tau(\pi)(\partial_i b)=\tau(b)$, $i>1$, we find

$$
\begin{aligned}
\tau(\pi)(\partial_{i_2}\cdots\partial_{i_{q-n}}g) &= \tau(\pi)(p\bar{x}(\partial_{i_2}\cdots\partial_{i_{q-n}})(0, 1, \cdots, q))) \\
&= \tau(\pi)(p\bar{x}(0, y_0, y_1, \cdots, y_n)) \\
&= \tau(\pi)(p\bar{x}(0, y_0)),
\end{aligned}
$$

so that

$$
\begin{aligned}
E\bar{x}^*u(y) &= y^*c(0, 1, \cdots, n) \\
&= cy_*(0, 1, \cdots, n) \\
&= c(y).
\end{aligned}
$$

It follows that $\tilde{x}(u)=(c, g)=x$ as desired.

Theorem 1.2 (absolute version).

Define α: $\boldsymbol{S}(X, L_\phi(G, n))_{\bar{W}\pi} \to C^n_\phi(X; \tau)$ *and* β: $C^n_\phi(X; \tau) \to \boldsymbol{S}(X, L_\phi(G, n))_{\bar{W}\pi}$ *by* $\alpha(f)=f^*u$, $\beta(\gamma)(x)=\tilde{x}(\gamma)$ $(=(E\bar{x}^*\gamma, \theta(\tau)(x)))$. *Then we have*

(1) α *is a homomorphism of groups,*

(2) β *is well defined,*

(3) $\alpha\beta=id$, $\beta\alpha=id$ *and*

(4) $\delta\alpha=\alpha(\delta \underset{\tau(\pi)}{\times} 1)_*$.

Proof. (1) It is easy.

(2) We must see that $\beta(\gamma)$ is simplicial. If $x\in X_{n+1}$ then we find by Lemma 2.2

$$
\begin{aligned}
\beta(\gamma)(\partial_i x) &= (E\delta_i^*\bar{x}^*, \partial_i px) \\
&= \begin{cases} (\tau(\pi)(px)\partial_0 E\bar{x}^*\gamma, \partial_0 px) & (i=0) \\ (\partial_i E\bar{x}^*, \partial_i px) & (i\neq 0) \end{cases} \\
&= \partial_i\beta(\gamma)(x).
\end{aligned}
$$

Similarly we have $\beta(\gamma)(s_i x)=s_i\beta(\gamma)(x)$.

(3) If $x\in X_n$ is non-degenerate we find

$$
\alpha\beta(\gamma)(x) = \beta(\gamma)^*(u)(x) = u(E\bar{x}^*\gamma, px) = E\bar{x}^*\gamma(0, 1, \cdots, n) = \gamma(x).
$$

And if $f\in\boldsymbol{S}(X, L_\phi(G, n))_{\bar{W}\pi}$, $x\in X_q$, then we have

$$
\beta\alpha(f)(x) = \tilde{x}(f^*u) = (E\bar{x}^*f^*u, px) = \widetilde{fx}(u) = fx.
$$

(4) Let $f\in\boldsymbol{S}(X, L_\phi(G, n))_{\bar{W}\pi}$, $x\in X_{n+1}$. Put $fx=(c, px)\in L_\phi(G, n)_{n+1}$, then we find

$$\begin{aligned}
\alpha(\delta \underset{\tau(\pi)}{\times} 1)_* fx &= u_{n+1}(\delta c, px) = \delta c(0, 1, \cdots, n+1) \\
&= \sum_{i=0}^{n+1} (-1)^i c(\delta_i(0, 1, \cdots, n+1)) \\
&= \sum_{i=0}^{n+1} (-1)^i \partial_i c(0, 1, \cdots, n)\,, \\
\delta\alpha f(x) &= \tau(\pi)(px)^{-1} f^* u_n(\partial_0 x) + \sum_{i=1}^{n+1} (-1)^i f^* u_n(\partial_i x) \\
&= \tau(\pi)(px)^{-1} u_n(\tau(\pi)(px)\partial_0 c,\ \partial_0 px) + \sum_{i=1}^{n+1} (-1)^i u_n(\partial_i c,\ \partial_i px) \\
&= \sum_{i=0}^{n+1} (-1)^i \partial_i c(0, 1, \cdots, n)\,,
\end{aligned}$$

so that we find $\alpha(\delta \underset{\tau(\pi)}{\times} 1)_* = \delta\alpha$. This completes the proof.

3. Topological version

Let T be a path connected topological space, and let L be a local system on T. Since we can identify the fundamental groupoid πT of T with that of the singular simplicial set ST, we regard L to be a local system on ST. The fundamental group $\pi_1 T$ is defined to be $\pi_1 ST$ for some fixed $t_0 \in ST_0 (=T)$ as a set, but the multiplication is defined as usual, so that the inversion $I: \pi_1 T \to \pi_1 ST$, $I(g)=g^{-1}$, is an isomorphism. We understand $\phi: \pi_1 T \to Aut\, L(t_0)$ to be $\phi(L)I$, where $\phi(L)$ is the group homomorphism given by L and t_0 as before. The cohomology groups of T with coefficients in L, $H^n(T; L, t_0, (u_t))$, are defined as $H^n_{\phi(L)}(ST; F(t_0, (u_t))$ for some fixed paths $u_t \in \pi T(t_0, t)$, $t \in T$ (see [2]).

We have to describe the (vertical) homotopy classification of $H^n(T;L)$ in the category of topological spaces. Roughly speaking, the geometric realization functor induces the isomorphism from the simplicial vertical homotopy classes to the topological ones in our case.

Let $K(G, n) = |\bar{W}^n G| \simeq |Z^n C|$ be the Eilenberg-MacLane complex on which $\pi_1 T$ acts from the right (i.e. $|\phi(L)|(g, a) = a \cdot g$, $a \in K(G, n)$, $g \in \pi_1 T$).

Lemma 3.1. *The geometric realization of the Kan fibration* $Z^n C \underset{\tau(\pi_1 ST)}{\times} \bar{W}\pi_1 ST \to \bar{W}\pi_1 ST$ *is homeomorphic to the fibre bundle* $|Z^n C| \underset{\pi_1 T}{\times} E\pi_1 T \to B\pi_1 T$, *where* $E\pi_1 T \to B\pi_1 T$ *is the universal* $\pi_1 T$ *bundle in the sense of Milgram* (*see* [7]).

Proof. Define the bisimplicial set K by $K_{p,q} = Z^n C_p \times \bar{W}_q \pi_1 ST$, $\partial_i' = \partial_i \times 1$, $s_i' = s_i \times 1$, $\partial_i''(a, b) = \begin{cases} (\tau(\pi_1 ST)(b)a,\ \partial_0 b) & (i=0) \\ (a,\ \partial_i b) & (i \neq 0) \end{cases}$, and $s_i'' = 1 \times s_i$. The geometric realization of the diagonal simplicial set $Z^n C \underset{\tau(\pi_1 ST)}{\times} \bar{W}\pi_1 ST$ of K is naturally homeomorphic to the successive geometric realization of K, which is homeomrophic to the geometric bar construction $B(|Z^n C|, \pi_1 T, *)$ (see [7]) by making use of the simplicial homeomorphism $f=(f_q)$, f_q: $|Z^nC|\times \bar{W}_q\pi_1ST\to |Z^nC|\times(\pi_1T)^q$, $f_q(a, [g_1, \cdots, g_q])=a[g_1, \cdots, g_q]$. Further $B(|Z^nC|, \pi_1T, *)$ is homeomorphic to $|Z^nC|\underset{\pi_1T}{\times}B(\pi_1T, \pi_1T, *)$ by Corollary 8.4 of [7]. This completes the proof.

Proposition 3.2. *Let p: $E\to B$ be a Kan fibration, and let θ: $X\to B$ be a simplicial set over B. Then the geometric realization functor gives the bijection*

$$[X, E]_B \to [|X|, |E|]_{|B|}\,.$$

Corollary 3.3. *We have the topological (vertical) homotopy classification*

$$H^n(T, L)\cong[|ST|, K(G, n)\underset{\pi_1T}{\times} E\pi_1T]_{B\pi_1T}\,.$$

If $T=|X|$ the geometric realization of some connected simplicial set X, for example if T is a path connected regular CW complex, we can regard X as a sub simplicial set of ST by the adjoint i of 1_T: $|X|\to T$. We suppose that $t_0\in X_0$ and $u_t\in\pi X(t_0, t)$ if $t\in X_0$. Then we have the following

Proposition 3.4. *i: $X\to ST$ induces the isomorphism*

$$i^*: H^n(T; L)\to H^n(X; L_{|X})\,.$$

Corollary 3.5. *We have the classification*

$$H^n(T; L)\cong[T, K(G, n)\underset{\pi_1T}{\times} E\pi_1T]_{B\pi_1T}\,.$$

Proof of Proposition 3.2. The adjointness of the geometric realization functor to the singular complex functor defines the natural bijection $\boldsymbol{S}(X, S|E|)_{S|B|}\to \boldsymbol{T}(|X|, |E|)_{|B|}$, where $\boldsymbol{T}(|X|, |E|)_{|B|}$ is the set of topological liftings

$$\begin{tikzcd} & {|E|} \arrow[d, "|p|"] \\ {|X|} \arrow[ur] \arrow[r, "\theta"'] & {|B|} \end{tikzcd}\ .$$

We find easily that the above natural bijection and its inverse map preserve vertical homotopies, so that we have the bijection $[X, S|E|]_{S|B|}\to[|X|, |E|]_{|B|}$. By adjointing the identity maps 1: $|E|\to|E|$, 1: $|B|\to|B|$ we have the commutative diagram

$$\begin{tikzcd} E \arrow[r, "\dot{i}_E"] \arrow[d, "p"'] & S|E| \arrow[d, "S|p|"] \\ B \arrow[r, "\dot{i}_B"] & S|B| \end{tikzcd}\ ,$$

which induces the map $(\dot{i}_E, \dot{i}_B)_*$: $[X, E]_B\to[X, S|E|]_{S|B|}$. The formal properties of adjointness lead to the commutative diagram

$$\begin{tikzcd}[X, E]_B \arrow[rr, "| \ |"] \arrow[dr, "(i_E, i_B)_*"'] & & {[|X|, |E|]_{|B|}} \\ & {[X, |E|]_{S|B|}} \arrow[ur, "\cong"'] & \end{tikzcd}$$

The following Lemma 3.6 completes the proof.

Lemma 3.6. *$(i_E, i_B)_*$ is a bijection.*

Proof. Let $\begin{tikzcd} E' \arrow[r, "i'"] \arrow[d, "p'"] & S|E| \arrow[d, "S|p|"] \\ B \arrow[r, "i_B"] & S|P| \end{tikzcd}$ be the pullback diagram. It is easy to see that $(i', i_B)_*$: $[X, E']_B \to [X, S|E|]_{S|B|}$ is a bijection. We have to find that j_*: $[X, E]_B \to [X, E']_B$ is bijective for j: $E \to E'$ the canonical injection.

Let $p|_M$: $M \to B$ be a minimal fibration of p: $E \to B$ and r: $E \to M$ be a retraction. Since j: $E \to E'$ is a trivial cofibration (i.e. anodyne extension, or map which is both cofibration and weak equivalence) of the closed model category $\boldsymbol{S}$, for the diagram

$$\begin{tikzcd} E \arrow[r, "r"] \arrow[d, "j"'] & M \arrow[d, "p_{|M}"] \\ E' \arrow[r, "p'"'] \arrow[ur, dashed, "r'"] & B \end{tikzcd}$$

the filler r' exists. Let h: $E \times \Delta[1] \to E$ be a homotopy from 1_E to r over B. Since $E \times \Delta[1] \cup E' \times \dot{\Delta}[1] \subset E' \times \Delta[1]$ is a trvial cofibration (see [4; IV, 2.2]), for the diagram

$$\begin{tikzcd} E \times \Delta[1] \cup E' \times \dot{\Delta}[1] \arrow[r, "jh \cup 1_{E'} \cup r'"] \arrow[d, phantom, "\cap"] & E' \arrow[d, "p'"] \\ E' \times \Delta[1] \arrow[r, "p' p_r"'] \arrow[ur, dashed, "H"] & B \end{tikzcd}$$

the filler H exists and makes $p_{|M}$: $M \to B$ the strong deformation retract of p': $E' \to B$. This proves Lemma 3.6.

Proof of Proposition 3.4. By Corollary 1.3 and naturality we have the commutative diagram

$$\begin{tikzcd} H^n(T; L) \arrow[r, phantom, "\cong"] \arrow[d, "i^*"] & {[ST, L_{\phi(L)}(G, n)]_{\overline{W}\pi_1 ST}} \arrow[d, "i^*"] \\ H^n(X, L_{|X}) \arrow[r, phantom, "\cong"] & {[X, L_{\phi(L)}(G, n)]_{\overline{W}\pi_1 ST}} \end{tikzcd}$$

The next Lemma 3.7 completes Proposition 3.4.

Lemma 3.7. *Let θ: $Z\to B$ and p: $E\to B$ be a simplicial set over B and a Kan fibration respectively. If i: $A\to Z$ is a trivial cofibration, then i^*: $[Z, E]_B \to [A, E]_B$ is a bijection.*

Proof. Since i: $A\to Z$ is a trivial cofibration, for every commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & E \arrow[d, "p"] \\ Z \arrow[r] \arrow[ur, dashed, "\theta"] & B \end{tikzcd}$$

the filler exists and this follows the surjectivity of i^*. To prove the injectivity of i^* we have to see that the inclusion $A\times\Delta[1]\cup Z\times\dot{\Delta}[1]\subset Z\times\Delta[1]$ is a trivial cofibration. It follows from [4;IV, 2.2]. This proves Lemma 3.7.

We remark that the relative version of Corollary 3.5 is available. We only give some hints.

(1) *If a Kan fibration p: $E\to B$ has a section s: $B\to E$, then we can choose a minimal fibration $p_{|M}$: $M\to B$ of p such that $M\supset sB$.*

This implies the relative version of Lemma 3.6 and therefore that of Proposition 3.2.

(2) *Let X be a connected simplicial set and let $Y\subset X$ be a connected sub simplicial set. Since $X\cap S|Y|=Y$ and $i_{|Y}$ is a anodyne extension in the commutative diagram*

$$\begin{tikzcd} Y \arrow[r] \arrow[d, "i_{|Y}"] & X \arrow[d, "i_{|X}"] \\ S|Y| \arrow[r] & S|X|, \end{tikzcd}$$

$X\subset X\cup S|Y|=X\coprod_Y S|Y|$ is a anodyne extension, and therefore $X\cup S|Y|\subset S|X|$ is also a trivial cofibration. We easily find that the map $i_X{}^*$: $[(S|X|, S|Y|), (E, sB)]_B\to[(X, Y), (E, sB)]_B$ ia a bijection for a Kan fibration p: $E\to B$ with a section s.

This implies the relative version of Proposition 3.4 and therefore that of Corollary 3.5.

4. Appendix

In this section we generalize some results in [6; §25]. For simplicity we assume that all simplicial sets are one vertexed Kan complexes.

Let K be an Eilenberg-MacLane complex $K(G, n)$, $n\geq 2$. It is shown in [6; §25] that the group complex $A(K)$ of invertible elements in K^K is isomorphic to $Aut\,G\times K$ with the group structure

$$\begin{pmatrix} f \\ x \end{pmatrix}\begin{pmatrix} g \\ y \end{pmatrix}=\begin{pmatrix} fg \\ fy+x \end{pmatrix}, \quad f, g\in Aut\,G\,, \quad x, y\in K_q\,.$$

The action $A(K)\times K\to K$ is identified with $(Aut\,G\times K)\times K\to K$, $\begin{pmatrix} f \\ x \end{pmatrix}y=fy+x$, $f\in Aut\,G$, $x, y\in K_q$.

Let $p\colon K\underset{\tau}{\times} B\to B$ be a Kan fibration. If we put $\tau(b)=\begin{pmatrix} f(b) \\ x(b) \end{pmatrix}$, then f, x satisfy the following formulae

$$\begin{cases} f(b) =f(\partial_0 b)^{-1}f(\partial_1 b) \\ f(b) =f(\partial_{i+1}b) \quad (i>0) \\ f(b) =f(s_{i+1}b) \quad (i\geq 0) \\ 1_G \;=f(s_0 b) \end{cases}, \qquad \begin{cases} \partial_0 x(b) =f(\partial_0 b)^{-1}(-x(\partial_0 b)+x(\partial_1 b)) \\ \partial_i x(b) = x(\partial_{i+1}b) \quad (i>0) \\ s_i x(b) = x(s_{i+1}b) \quad (i\geq 0) \\ e_q \;= x(s_0 b) \end{cases}.$$

Since $f(b)=f(\partial_2\partial_3\cdots\partial_q b)$, $b\in B_q$ and $f(b)=f(b')$ if b, b' are homotopic 1-simplexes, f factors as $f=\phi F$, where $F\colon B\to\pi_1 B$ is the twisting function defined by $F(b)=[\partial_2\partial_3\cdots\partial_q b]$, $b\in B_q$. It is easily seen that $\phi\colon \pi_1 B\to Aut\,G$, $\phi([b])=f(b)$ is a group homomorphism.

Lemma 4.1. *For a Kan fibration $p\colon K\underset{\tau}{\times} B\to B$ $\pi_1 B$-action on $\pi_n K=G$ is ϕ.*

Proof. Consider the following commutative diagram

$$\begin{tikzcd} K\times(0) \arrow[r] \arrow[d, phantom, "\cap"] & K\underset{\tau}{\times} B \arrow[d, "p"] \\ K\times\Delta[1] \arrow[r, "\bar{b}p_r"'] \arrow[ur, dashed, "h"] & B, \quad b\in B_1\,. \end{tikzcd}$$

The filler h defines a homotopy equivalence $h(\;,(1))\colon K\to K$, and $\pi_n(h(\;,(1)))$ $\in Aut\,G$ is the automorphism given by $[b]\in\pi_1 B$. Since $K(=K(G, n), n\geq 2)$ is one vertexed, $x(b)=0$ for $b\in B_1$ and therefore $\tau(\bar{b}(y))=f(\bar{b}(y))$. Put a priori $h(z, y)=(f(\bar{b}(0, y))(z), \bar{b}(y))$, $z\in K_q$, $y\in\Delta[1]$, then h is simplicial. We find $h(z, (1))=(f(b)z, *)$. This completes the proof.

If we give $\pi_1 B\times K$ a similar multiplication as $Aut\,G\times K$, then we have the twisting function $\tau'\colon B\to\pi_1 B\times K$, $\tau'(b)=\begin{pmatrix} F(b) \\ x(b) \end{pmatrix}$ and the group homomorphism $\phi\times 1\colon \pi_1 B\times K\to Aut\,G\times K$.

Lemma 4.2. *The group of the bundle $p\colon K\underset{\tau}{\times} B\to B$ can always be reduced to $\pi_1 B\times K$.*

Lemma 4.3. $K\underset{\tau(\pi_1 B\times K)}{\times}\bar{W}(\pi_1 B\times K)\to\bar{W}(\pi_1 B\times K)$ *is isomorphic to* $WK\underset{\pi_1 B}{\times} W\pi_1 B\to\bar{W}K\underset{\pi_1 B}{\times} W\pi_1 B$ $(=L_\phi(G, n))$.

Proof. Consider the commutative diagram

$$\begin{array}{ccccc} K \underset{\tau(\pi_1 B\times K)}{\times} \bar{W}(\pi_1 B\times K) & \xrightarrow{H} & WK \underset{\tau(\pi_1 B)}{\times} \bar{W}\pi_1 B & \cong & WK \underset{\pi_1 B}{\times} W\pi_1 B \\ \downarrow & & \downarrow & & \downarrow \\ \bar{W}(\pi_1 B\times K) & \xrightarrow{h} & \bar{W}K \underset{\tau(\pi_1 B)}{\times} \bar{W}\pi_1 B & \simeq & \bar{W}K \underset{\pi_1 B}{\times} W\pi_1 B\,, \end{array}$$

where

$$H\left(x, \begin{pmatrix} \boldsymbol{F} \\ \boldsymbol{x} \end{pmatrix}\right) = ((x, F_1^{-1}x_1, F_1^{-1}F_2^{-1}x_2, \cdots, F_1^{-1}F_2^{-1}\cdots F_q^{-1}x_q), \boldsymbol{F}) \quad x\in K_q\,,$$

$$\begin{pmatrix} \boldsymbol{F} \\ \boldsymbol{x} \end{pmatrix} = \begin{pmatrix} F_1, F_2, \cdots, F_q \\ x_1, x_2, \cdots, x_q \end{pmatrix} \in \bar{W}_q(\pi_1 B\times K)\,,$$

$$h\left(\begin{pmatrix} \boldsymbol{F} \\ \boldsymbol{x} \end{pmatrix}\right) = ((F_1^{-1}x_1, \cdots, F_1^{-1}F_2^{-1}\cdots F_q^{-1}x_q), \boldsymbol{F})\,.$$

These H, h are simplicial isomorphisms. This completes the proof.

By making use of the above Lemmas we have the next

Theorem 4.4. *p: $K\underset{\tau}{\times} B\to B$ is classified by the element $o(p)\in H_\Phi^{n+1}(B; F)$ corresponding to $h\theta(\tau')$: $B\to L_\Phi(G, n+1)$.*

Theorem 4.5. *If p_1: $K\underset{\tau_1}{\times} B\to B$, p_2: $K\underset{\tau_2}{\times} B\to B$ are fibre homotopy equivalent, then there is a $\pi_1 B$-equivariant automorphism g, i.e. $\phi_2(1\times g)=g\phi_1$, such that $g_* o(p_1)=o(p_2)$.*

Proof. Put $\tau_1(b)=\begin{pmatrix} f_1(b) \\ x_1(b) \end{pmatrix}$, $\tau_2(b)=\begin{pmatrix} f_2(b) \\ x_2(b) \end{pmatrix}$. Since K is a minimal Kan complex, p_1 and p_2 are minimal fibrations. Therefore they are strongly $A(K)$-equivalent. By Lemma 20.2 of [6] there is a map Θ: $K\underset{\tau_1}{\times} B\to K\underset{\tau_2}{\times} B$, $\Theta(y, b)= \left(\begin{pmatrix} g(b) \\ z(b) \end{pmatrix} y, b\right)$, g: $B\to Aut\, G$, z: $B\to K$, such that

$$\begin{cases} f_2(b)g(b) = g(\partial_0 b)f_1(b)\,, \\ x_2(b)+f_2(b)\partial_0 z(b) = z(\partial_0 b)+g(\partial_0 b)x_1(b)\,, \\ g(b) = g(\partial_i b) \qquad (i>0)\,, \\ \partial_i z(b) = z(\partial_i b) \qquad (i>0)\,, \\ g(b) = g(s_i b)\,, \\ s_i z(b) = z(s_i b)\,. \end{cases}$$

Since $g(b)=g(\partial_q b)=\cdots=g(\partial_1\partial_2\cdots\partial_q b)=g(*)$, $b\in B_q$, g is contstant.

If we put $\tau_3(b)=\begin{pmatrix} f_2(b) \\ gx_1(b) \end{pmatrix}=\begin{pmatrix} gf_1(b)g^{-1} \\ gx_1(b) \end{pmatrix}$, then we have the TCP p_3: $K\underset{\tau_3}{\times} B\to B$. It is easily seen that p_2 and p_3 are strongly $\pi_1 B\times K$-equivalent by the map of TCP's $\Theta'\colon K \underset{\tau_2}{\times} B \to K \underset{\tau_3}{\times} B$, $\Theta'(y, b) = \left(\begin{pmatrix} 1 & \\ -z(b) & \end{pmatrix} y, b\right)$. Therefore we find $o(p_2) = o(p_3)$. We have the map of TCP's $\Theta''\colon K \underset{\tau_1}{\times} B \to K \underset{\tau_3}{\times} B$, $\Theta''(y, b) = (gy, b)$ which is induced by the commutative diagram

$$\begin{tikzcd}
WK \underset{\phi_1\tau(\pi_1 B)}{\times} \bar{W} \arrow[r, "Wg\times 1"] \arrow[d] & WK \underset{\phi_2\tau(\pi_1 B)}{\times} \bar{W} \arrow[d] \\
\bar{W}K \underset{\phi_1\tau(\pi_1 B)}{\times} \bar{W} \arrow[r, "\bar{W}g\times 1"] & \bar{W}K \underset{\phi_2\tau(\pi_1 B)}{\times} \bar{W}.
\end{tikzcd}$$

This completes Theorem 4.5.

Let (X, x_0) be a connected minimal Kan complex, and let $(x_0, X^{(1)}, X^{(2)}, \cdots, X^{(n)}, \cdots)$ denote the natural Postnikov system of X. Then $X^{(n)} \to X^{(n-1)}$ is isomorphic to $p_n\colon K(\pi_n X, n) \underset{\tau}{\times} X^{(n-1)} \to X^{(n-1)}$. A sequence of cocycles of $o(p_n)$'s is called a set of k-invariants of X. As Corollaries of our results we have Theorems 25.7 and 25.8 of [6] without any restriction on π_1-action.

OSAKA CITY UNIVERSITY

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