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ON MULTIPLY TRANSITIVE GROUPS XIV

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1. Introduction

Let G be a 4-fold transitive group on Ω. If the stabilizer of four points i,j,k,l in G has an orbit of length one in $\Omega-\{i,j,k,l\}$, then G is S_5, A_6 or M_{11} by a theorem of H. Nagao [2]. We now consider the case in which the stabilizer of four points in G has an orbit of length two and have the following results.

Theorem 1. *Let G be a 4-fold transitive group on $\Omega=\{1,2,\cdots,n\}$. If the stabilizer of four points in G has an orbit of length two, then G is S_6.*

Corollary. *Let D be a 4-(v,k,λ) design, where $k=5$ or 6 and $\lambda=1$ or 2. If an automorphism group G of D is 4-fold transitive on the set of points of D, then D is a 4-$(11,5,1)$ design or a trivial design: a 4-$(5,5,1)$ design, a 4-$(6,5,2,)$ design or a 4-$(6,6,1)$ design.*

The case $k=5$ and $\lambda=1$ has proved by H. Nagao [2]. Hence in this paper we shall prove the remaining cases.

We shall use the same notations as in [3].

2. Proof of Theorem 1

Let G be a group satisfying the assumption of Theorem 1. If the order of the stabilizer of four points in G is not divisible by three, then $G=S_6$ by Theorem of [7]. Hence from now on we assume that the order of the stabilizer of four points in G is divisible by three and proceed by way of contradiction.

Let $\{i_1,i_2\}$ be a G_{1234}-orbit of length two and P be a Sylow 3-subgroup of G_{1234}. Then $I(P)\supseteq\{1,2,3,4,i_1,i_2\}$. By a theorem of E. Witt [9], $N_G(P)^{I(P)}$ is 4-fold transitive on $I(P)$. By assumption, $3\nmid|(N_G(P)^{I(P)})_{1234}|$ and $(N_G(P)^{I(P)})_{1234}$ has an orbit $\{i_1,i_2\}$. Hence by Theorem of [7], $N_G(P)^{I(P)}=S_6$ and $|I(P)|=6$.

Hence G_{1234} has exactly one orbit $\{i_1,i_2\}$ of length two and lengths of orbits in $\Omega-\{1,2,3,4,i_1,i_2\}$ are all divisible by three. Furthermore any Sylow 3-subgroup of G_{1234} fixes exactly six points $1,2,3,4,i_1,i_2$.

Let Q be a 3-subgroup of G such that the order of Q is maximal among all 3-subgroups of G fixing more than six points. Set $N=N_G(Q)^{I(Q)}$. Then for any four points i,j,k,l of $I(Q)$ any Sylow 3-subgroup of $N_{i\,j\,k\,l}$ fixes exactly six points i,j,k,l,i',j', where $\{i',j'\}$ is the unique $G_{i\,j\,k\,l}$-orbit of length two, and is semiregular on $I(Q)-\{i,j,k,l,i',j'\}$. Hence to complete the proof of Theorem 1, it will suffice to show the following

Lemma. *Let G be a permutation group on $\Omega=\{1,2,\cdots,n\}$, where $n>6$. Then it is impossible that G satisfies the following condition:*

Let i,j,k,l be any four points of Ω. Then there exist two points i_1,i_2 in $\Omega-\{i,j,k,l\}$ such that any Sylow 3-subgroup of $G_{i\,j\,k\,l}$ fixes six points i,j,k,l,i_1,i_2 and is semiregular on $\Omega-\{i,j,k,l,i_1,i_2\}$.

Proof. Suppose by way of contradiction that G is a counter-example to Lemma. For four points i,j,k,l let i_1,i_2 be two points in $\Omega-\{i,j,k,l\}$ uniquely determined by Sylow 3-subgroups of $G_{i\,j\,k\,l}$. Set $\Delta(i,j,k,l)=\{i,j,k,l,i_1,i_2\}$. Then $\{\Delta(i,j,k,l)\,|\,i,j,k,l\in\Omega\}$ forms a 4-$(v,6,1)$ design on Ω and G is an automorphism group of this design.

We may assume that $\Delta(1,2,3,4)=\{1,2,3,4,5,6\}$. Let a be an element of order three in $G_{1\,2\,3\,4}$. Then we may assume that

$$a=(1)\,(2)\cdots(6)\,(7\ 8\ 9)\cdots.$$

Since $a\in N_G(G_{1\,7\,8\,9})$, there is an element b of order three in $G_{1\,7\,8\,9}$ such that $ab=ba$. Then we may assume that

$$b=(1)\,(2)\,(3)\,(4\ 5\ 6)\,(7)\,(8)\,(9)\cdots.$$

Similarly $G_{4\,7\,8\,9}$ has an element c of order three such that $ac=ca$. Since $\Delta(1,7,8,9)\neq\Delta(4,7,8,9)$, $\Delta(1,7,8,9)\cap\Delta(4,7,8,9)=\{7,8,9\}$. Hence $\Delta(4,7,8,9)=\{4,5,6,7,8,9\}$ and we may assume that

$$c=(1\ 2\ 3)\,(4)\,(5)\,(6)\,(7)\,(8)\,(9)\cdots.$$

Then a Sylow 3-subgroup of $\langle a,b,c\rangle$ has the same form as $\langle a,b,c\rangle$ on $\{1,2,\cdots,9\}$. By assumption, $3\nmid|\langle a,b,c\rangle_{1,2,\cdots 9}|$ and so the order of a Sylow 3-subgroup of $\langle a,b,c\rangle$ is 3^3. Hence we may assume that $\langle a,b,c\rangle$ is a 3-group. Then since $I(b^cb^{-1})\supseteq\{1,2,\cdots,9\}$, $b^cb^{-1}=1$. Hence $\langle a,b,c\rangle$ is an elementary abelian 3-group.

Let r be the number of $\langle a,b\rangle$-orbits of length three. For a point i of a $\langle a,b\rangle$-orbit of length three, $\langle a,b\rangle_i$ is of order three and so exactly two elements of $\langle a,b\rangle-\{1\}$ fixes $1,2,3$ and three points of the $\langle a,b\rangle$-orbit containing i. Hence $2r\leqq|\langle a,b\rangle-1|=8$ and so $r\leqq4$. Since $\{4,5,6\}$ and $\{7,8,9\}$ are $\langle a,b\rangle$-orbits of length three, $r=2,3$ or 4.

Suppose that $\langle a,b,c\rangle$ has an orbit of length nine. Since $\langle a,b,c\rangle$ is an abelian group of order twenty-seven, $\langle a,b,c\rangle$ has an element of order three fixing nine points of this $\langle a,b,c\rangle$-orbit of length nine, contrary to the assumption. Thus the lengths of $\langle a,b,c\rangle$-orbits are three or twenty-seven and every $\langle a,b\rangle$-orbit of length three is a $\langle a,b,c\rangle$-orbit.

Case I. $r=4$.

Assume that $\langle a,b\rangle$-orbits of length three are $\{4,5,6\}$, $\{7,8,9\}$, $\{10,11,12\}$ and $\{13,14,15\}$. Then we may assume that

$$\begin{aligned} a &= (1)(2)\cdots(6)(7\ 8\ 9)(10\ 11\ 12)(13\ 14\ 15)\cdots, \\ b &= (1)(2)(3)(4\ 5\ 6)(7)(8)(9)(10\ 11\ 12)\cdots, \\ c &= (1\ 2\ 3)(4)(5)(6)(7)(8)(9)(10\ 11\ 12)\cdots. \end{aligned}$$

Since $|I(ab^{-1})|\leqq 6$ and $|I(ac^{-1})|\leqq 6$, $b=c=(13\ 15\ 14)$ on $\{13,14,15\}$. Then $I(bc^{-1})\supseteq\{7,8,\cdots,15\}$ and so $|I(bc^{-1})|\geqq 9$, contrary to the assumption.

Case II. $r=2$ or 3.

When $r=2$, $\langle a,b\rangle$-orbits of length three are $\{4,5,6\}$ and $\{7,8,9\}$. If $|\Omega|=9$, then we have a 4-(9,6,1) design. Then the number of blocks is $\dfrac{\binom{9}{4}}{\binom{6}{4}}=\dfrac{9\cdot 8\cdot 7\cdot 6}{6\cdot 5\cdot 4\cdot 3}$, wehich is not an integer. Thus $|\Omega|>9$. Hence we may assume that

$$\begin{aligned} a &= (1)(2)\cdots(6)(7\ 8\ 9)(10\ 11\ 12)(13\ 14\ 15)(16\ 17\ 18)\cdots, \\ b &= (1)(2)(3)(4\ 5\ 6)(7)(8)(9)(10\ 13\ 16)(11\ 14\ 17)(12\ 15\ 18)\cdots. \end{aligned}$$

Then

$$ab = (1)(2)(3)(4\ 5\ 6)(7\ 8\ 9)(10\ 14\ 18)(11\ 15\ 16)(12\ 13\ 17)\cdots,$$

and $I(ab)=\{1,2,3\}$. Since $\langle a,b,c\rangle$ has no orbit of length nine, the lengths of $\langle a,b,c\rangle$-orbits in $\{10,11,\cdots,n\}$ are twenty-seven. Thus $n\equiv 9 \pmod{27}$ and $n>9$.

Next when $r=3$, we may assume that $\langle a,b\rangle$-orbits of length three are $\{4,5,6\}$, $\{7,8,9\}$ and $\{10,11,12\}$. If $|\Omega|=12$, then we have a 4-(12,6,1) design. Then the number of blocks containing a given point is $\dfrac{\binom{11}{3}}{\binom{5}{3}}=\dfrac{11\cdot 10\cdot 9}{5\cdot 4\cdot 3}$, which is not an integer. Thus $|\Omega|>12$. Hence we may assume that

$$\begin{aligned} a &= (1)(2)\cdots(6)(7\ 8\ 9)(10\ 11\ 12)(13\ 14\ 15)(16\ 17\ 18)(19\ 20\ 21)\cdots, \\ b &= (1)(2)(3)(4\ 5\ 6)(7)(8)(9)(10\ 11\ 12)(13\ 16\ 19)(14\ 17\ 20) \end{aligned}$$

$(15\ 18\ 21)\cdots$.

Then

$$ab=(1)(2)(3)(4\ 5\ 6)(7\ 8\ 9)(10\ 12\ 11)(13\ 17\ 21)(14\ 18\ 19)$$
$$(15\ 16\ 20)\cdots,$$

and $I(ab)=\{1,2,3\}$. Then by the same reason as above $n\equiv 12 \pmod{27}$ and $n>12$.

For any two points i,j, of $\{2,3,\cdots,6\}$ set $\Gamma(i,j)=\{(i_1\ i_2\ i_3)\,|\,a=(i_1\ i_2\ i_3)\cdots$ and $i,j\in\Delta(1,i_1,i_2,i_3)\}$. For any 3-cycle $(i_1\,i_2\,i_3)$ of a $\Delta(1,i_1,i_2,i_3)^a=\Delta(1,i_1,i_2,i_3)$. Hence there are two points i,j in $\{2,3,\cdots,6\}$ such that $\Delta(1,i_1,i_2,i_3)=\{1,i,j,i_1,i_2,i_3\}$. Thus $(i_1\ i_2\ i_3)\in\Gamma(i,j)$.

For any 3-cycle $(i'\ j'\ k')$ of ab $\Delta(i',j',k',1)^{ab}=\Delta(i',j',k',1)$. Hence $\Delta(i',j',k',1)=\{i',j',k',1,2,3\}$. Thus for any point i' of $\{4,5,\cdots,n\}$ $\Delta(1,2,3,i')=\{1,2,3,i',j',k'\}$, where $(i'\ j'\ k')$ is a 3-cycle of ab. Hence $\Gamma(2,3)=\{(7\ 8\ 9)\}$ or $\{(7\ 8\ 9),\ (10\ 11\ 12)\}$ for $r=2$ or 3 respectively.

Conversely for any two points i,j of $\{2,3,\cdots,6\}$ if there is a 3-cycle $(i_1\ i_2\ i_3)$ of a such that $(i_1\ i_2\ i_3)\in\Gamma(i,j)$, then $G_{i\,i_1\,i_2\,i_3}$ has an element b' of order three such that $ab'=b'a$. Since $\Delta(i,i_1,i_2,i_3)\cap\Delta(1,i_1,i_2,i_3)\supseteq\{i,i_1,i_2,i_3\}$, $\Delta(i,i_1,i_2,i_3)=\Delta(1,i_1,i_2,i_3)=\{1,i,j,i_1,i_2,i_3\}$. Then ab' is of order three and $I(ab')=\{1,i,j\}$. Hence by the same argument as is used for $\Gamma(2,3)$, $|\Gamma(i,j)|=1$ or 2 for $r=2$ or 3 respectively.

Thus for any two points i,j of $\{2,3,\cdots,6\}$ $|\Gamma(i,j)|\leqq r-1$. On the other hand the number of unordered pairs (i,j), where $i,j\in\{2,3,\cdots,6\}$, is $\binom{5}{3}=10$. Hence the number of 3-cycles of a is at most $10(r-1)$.

Suppose that $r=2$. Then $n\leqq 6+3\cdot 10=36$. Since $n\equiv 9 \pmod{27}$ and $n>9$, $n=36$. Thus we have a 4-$(36,6,1)$ design. Then the number of blocks containing a given point is $\dfrac{\binom{35}{3}}{\binom{5}{3}}=\dfrac{35\cdot 34\cdot 33}{5\cdot 4\cdot 3}$, which is not an integer. Thus $r\neq 2$.

Suppose that $r=3$. Then $n\leqq 6+3\cdot 20=66$. Since $n\equiv 9 \pmod{27}$ and $n>12$, $n=39$ or 66. If $n=39$, then we have a 4-$(39,6,1)$ design. Then the number of blocks is $\dfrac{\binom{39}{4}}{\binom{6}{4}}=\dfrac{39\cdot 38\cdot 37\cdot 36}{6\cdot 5\cdot 4\cdot 3}$, which is not an integer. Thus $n\neq 39$.

From now on we sasume that $n=66$. Then for any two points i,j of $\{2,3,\cdots,6\}$ there is a 3-subgroup fixing exactly three points $1,i,j$. Let i_1,i_2,i_3 be any three points of Ω. Then there is an element of order three which fixes exactly six points containing i_1,i_2,i_3. Then by the same argument as is usd for a, there is a 3-subgroup fixing exactly three points i_1,i_2,i_3. Thus by a theorem of D.

Livingstone and A. Wagner [1], G is 3-fold transitive on Ω.

Let Γ_1, Γ_2 and Γ_3 are G_{123}-orbits such that $4 \in \Gamma_1$, $7 \in \Gamma_2$ and $10 \in \Gamma_3$. For any point i of $\Omega - \{1,2,3\}$ $\Delta(1,2,3,i) = \{1,2,3,i,j,k\}$, where $ab = (i\ j\ k)\cdots$. Hence any element of order three and fixing exactly three points 1,2,3 has a 3-cycle $(i\ j\ k)$ or $(i\ k\ j)$. Thus $\langle ab \rangle$ is a unique 3-subgroup of G_{123} which is of order three and semiregular on $\Omega - \{1,2,3\}$. Since G is 3-fold transitive on Ω, for any three points i_1, i_2, i_3 there is a unique 3-subgroup of order three which fixes exactly three points i_1, i_2, i_3 and is semiregular on $\Omega - \{i_1, i_2, i_3\}$.

Hence for a 3-cycle (13 17 21) of ab, there is an element c' of order three such that $I(c') = \{13, 17, 21\}$ and $abc' = c'ab$. Then we may assume that

$$c' = (1\ 2\ 3)(13)(17)(21)\cdots .$$

Furthermore since $\{10, 11, 12\}$ is a $\langle a, b \rangle$-orbit of length three, we may assume that

$$c = (1\ 2\ 3)(4)(5)(6)(7)(8)(9)(10\ 11\ 12)\cdots ,$$

and $\langle a, b, c \rangle$ is semiregular on $\Omega - \{1, 2, \cdots, 12\}$.
Then since $c^{-1}c' \in G_{123}$ and c fixes Γ_1, Γ_2 and Γ_3, c' fixes Γ_1, Γ_2 and Γ_3.

Assume that c' fixes $\{4,5,6\}$. Then $\Delta(4,5,6,13)^{c'} = \Delta(4,5,6,13)$. Hence $\Delta(4,5,6,13) = \{4,5,6,13,17,21\}$. On the other hand ac is an element of order three fixing exactly three points 4,5,6. Hence ac fixes $\Delta(4,5,6,13)$. This a contradiction since ac does not fix $\{13,17,21\}$. Thus c' does not fix $\{4,5,6\}$. Similarly c' fixes neither $\{7,8,9\}$ nor $\{10,11,12\}$. Since c' fixes Γ_1, Γ_2 and Γ_3, $\{4,5,6\} \subsetneqq \Gamma_1$, $\{7,8,9\} \subsetneqq \Gamma_2$ and $\{10,11,12\} \subsetneqq \Gamma_3$.

Assume that G_{123} has an orbit of length six. Then we may assume that $\{4,5,6,7,8,9\} = \Gamma_1 = \Gamma_2$. Since c' commutes with ab and fixes Γ_1, c' fixes $\{4,5,6\}$, which is a contradiction. Thus G_{123} has no orbit of length six.

Assume that Γ_1, Γ_2 and Γ_3 are three distinct G_{123}-orbits. Since c fixes Γ_1, Γ_2 and Γ_3, $|\Gamma_i| \geqq 3+27=30$ $(i=1,2,3)$. Hence $|\Gamma_1 \cup \Gamma_2 \cup \Gamma_3| \geqq 90$, which is a contradiction.

Assume that $\Gamma_1 = \Gamma_2 \neq \Gamma_3$. Since c fixes Γ_1 and Γ_3, $|\Gamma_1| = 6+27=33$ and $|\Gamma_3| = 3+27=30$. Let R be a Sylow 11-subgroup of G_{123}. Then since $|\Gamma_1| = 33$, $I(R) \cap \Gamma_1 = \phi$. Hence $|I(R)| = 33$, 22 or 11. Since $\langle a, b \rangle$ is the unique 3-subgroup of G_{123} which is of order three and is semiregular on $\Omega - \{1,2,3\}$, $R \leq C_G(\langle a, b \rangle)$. Hence ab fixes $I(R)$. Thus $|I(R)| \equiv 0 \pmod 3$ and so $|I(R)| = 33$. Since R is an 11-group $(\neq 1)$, for any four points i,j,k,l of $I(R)$ $\Delta(i,j,k,l) \subset I(R)$. Thus we have a 4-(33, 6, 1) design on $I(R)$. Then the number of blocks containing given two points is $\dfrac{\binom{31}{2}}{\binom{4}{2}} = \dfrac{31 \cdot 30}{4 \cdot 3}$, which is not an integer. Thus we have a contradiction.

Hence $\Gamma_1=\Gamma_2=\Gamma_3\supseteqq\{4,5,\cdots,12\}$. Then since $\langle a,b\rangle<G_{1\,2\,3}$, the lengths of $G_{1\,2\,3}$-orbits in $\{4,5,\cdots,66\}$ are all divisible by nine. On the other hand $|\{4,5,\cdots,66\}|$ is not divisible by twenty-seven. Hence any Sylow 3-subgroup of $G_{1\,2\,3}$ has an orbit of length nine. Let Q be a Sylow 3-subgroup of $G_{1\,2\,3}$ and i a point of a Q-orbit of length nine. If $Q_i\neq 1$, then $|I(Q_i)|=6$ and Q_i is semiregular on $\Omega-I(Q)$ by the assumption. Then since $|\Omega-I(Q_i)|=60$, $|Q_i|=3$. Thus $|Q|=9$ or $9\cdot 3$.

Assume that $\Gamma_1=\{4,5,\cdots,12\}$. Since c' fixes Γ_1 but does not fix $\{4,5,6\}$, we may assume that

$$c'=(1\ 2\ 3)(4\ 7\ 10)(5\ 8\ 12)(6\ 9\ 11)\cdots.$$

Then

$$c'c^{-1}=(1)(2)(3)(4\ 7\ 12\ 5\ 8\ 11\ 6\ 9\ 10)\cdots.$$

Since a Sylow 3-subgroup of $\langle a,b,c'c^{-1}\rangle$ has the same form as $\langle a,b,c'c^{-1}\rangle$ on $\{1,2,\cdots,12\}$, we may assume that $\langle a,b,c'c^{-1}\rangle$ is a 3-subgroup. Then since $|\langle a,b,c'c^{-1}\rangle|=9\cdot 3$, $\langle a,b,c'c^{-1}\rangle$ is a Sylow 3-subgroup of $G_{1\,2\,3}$.

Suppose that $\langle a,b,c'c^{-1}\rangle$ has an orbit of length nine in $\{13,14,\cdots,66\}$. Since $|\{13,14,\cdots,66\}|=54$, The number of $\langle a,b,c'c^{-1}\rangle$-orbits of length nine in $\{3,4,\cdots,66\}$ is at least four. Let $(i_1\,i_2\,i_3)$ be a 3-cycle of a contained in a $\langle a,b,c'c^{-1}\rangle$-orbit of length nine. Then $|\langle a,b,c'c^{-1}\rangle_{1\,2\,3\,i_1\,i_2\,i_3}|=3$ and so two non-identity elements of $\langle a,b,c'c^{-1}\rangle_{1\,2\,3\,i_1\,i_2\,i_3}$ fixes exactly six points $1,2,3,i_1,i_2,i_3$. Thus $\langle a,b,c'c^{-1}\rangle$ has at least $4\cdot 3\cdot 2=24$ elements which fix exactly six points. On the other hand since $(c'c^{-1})^3=ab$, $\langle c'c^{-1}\rangle$ is semiregular on $\Omega-\{1,2,3\}$. Thus $|\langle a,b,c'c^{-1}\rangle|\geqq 24+9=33$, which is a contradiction. Thus Γ_1 is the only $\langle a,b,c'c^{-1}\rangle$-orbit of length nine.

Let i be any point in $\{4,5,\cdots,66\}$. Then $G_{1\,2\,3}$ has an element x of order three and fixing i. Then by the same argument as is used for $\langle a,b,c'c^{-1}\rangle$, there is a Sylow 3-subgroup of $G_{1\,2\,3}$ which has exactly one orbit of length nine containing i. Since this Sylow 3-subgroup is conjugate to $\langle a,b,c'c^{-1}\rangle$ in $G_{1\,2\,3}$, $G_{1\,2\,3}$ is transitive on $\{4,5,\cdots,66\}$, which is a contradiction. Thus Γ_1 is not a $G_{1\,2\,3}$-orbit. This implies that $G_{1\,2\,3}$ has no orbit of length nine.

Suppose that $G_{1\,2\,3}$ is intransitive on $\{4,5,\cdots,66\}$. Since c fixes Γ_1, $|\Gamma_1|=9+27=36$. Thus $G_{1\,2\,3}$-orbits on $\{4,5,\cdots,66\}$ are Γ_1 and one orbit of length twenty-seven. Then $G_{1\,2\,3}$ has a non-identity 3-element fixing a point of the $G_{1\,2\,3}$-orbit of length twenty-seven. Thus a Sylow 3-subgroup of $G_{1\,2\,3}$ is of order more than twenty-seven, which is a contradiction.

Thus $G_{1\,2\,3}$ is transitive on $\{4,5,\cdots,66\}$. Hence G is 4-fold transitive on Ω. Let P be a Sylow 2-subgroup of $G_{1\,2\,3\,4}$. Then by Corollary of [4], $|I(P)|=4$. Since $\Delta(1,2,3,4)=\{1,2,\cdots,6\}$, $\{5,6\}$ is a $G_{1\,2\,3\,4}$-orbit. By Corollary of [5], $I(P_5)=\{1,2,\cdots,6\}=I(a)$ and $|P:P_5|=2$.

Suppose that P_5 is semiregular on $\{7,8,\cdots,66\}$. Then since $|P:P_5|=2$, lengths of P-orbits on $\{7,8,\cdots,66\}$ are $|P|$ or $|P|/2$. Hence for any point i of $\{7,8,\cdots,66\}$, $|P_i|=1$ or 2. Then by Corollary 1 of [6], $|\Omega|\neq 66$, which is a contradiction. Thus P_5 is not semiregular on $\{7,8,\cdots,66\}$.

For any three points i,j,k of $\{1,2,\cdots,6\}$ there is an element x of order three fixing exactly three points i,j,k. Then since $\langle x\rangle$ is the unique 3-subgroup of G_{ijk} which is of order three and semiregular on $\Omega-\{i,j,k\}$, $[a,x]=1$ and so x fixes $I(a)$. Hence $x\in C_G(G_{I(a)})$. Let H be a subgroup generated by all elements of order three fixing exactly three points $i,j,k,$ where i,j,k run over all three points of $\{1,2,\cdots,6\}$. Then $H\leq C_G(G_{I(a)})$ and $H^{I(a)}=A_6$.

Since $H\leq C_G(a)$, H induces a permutation group on the set of 3-cycles of a. Let K be a subgroup of H fixing $\{7,8,9\}$. Since $\Delta(7,8,9,1)=\{7,8,9,1,2,3\}$ and $\Delta(7,8,9,4)=\{7,8,9,4,5,6\}$, any element of K fixes or interchanges $\{1,2,3\}$ and $\{4,5,6\}$. Let $\bar{K}$ be a subgroup of H which fixes or interchanges $\{1,2,3\}$ and $\{4,5,6\}$. Then $K\leq\bar{K}$ and $|\bar{K}^{I(a)}|=6\cdot 3\cdot 2$. If $\{1,2,3,i,j,k\}$ is a block and $(i\ j\ k)$ is a 3-cycle of a, then $(i\ j\ k)=(7\ 8\ 9)$ or $(10\ 11\ 12)$. Similarly if $\{4,5,6,i,j,k\}$ is a block and $(i\ j\ k)$ is a 3-cycle of a, then $(i\ j\ k)=(7\ 8\ 9)$ or $(10\ 11\ 12)$. Hence any element of $\bar{K}$ fixes or interchanges $\{7,8,9\}$ and $\{10,11,12\}$. Hence $|\bar{K}:K|=1$ or 2.

Since $|H:\bar{K}|=|A_6|/6\cdot 3\cdot 2=10$, $|H:K|=10$ or 20. Since a has twenty 3-cycles, H is transitive or has two orbits of length ten on the set consisting of 3-cycles of a. Furthermore for any 3-cycle $(i\ j\ k)$ of a H has a subgroup which is transitive on $\{i,j,k\}$. Hence H is transitive or has two orbits of length thirty on $\{7,8,\cdots,66\}$.

Since P_5 is not semiregular on $\{7,8,\cdots,66\}$, there is a non-identity element y in P_5 which has a fixed point i in $\{7,8,\cdots,66\}$. Then since $H\leq C_G(y)$, $I(y)\supseteqq i^H$. Hence H has two orbits on $\{7,8,\cdots,66\}$ and so $|I(y)|=6+30=36$. Then since G is a 4-fold transitive group of degree sixty-six, we have a contradiction by a theore of W. A. Manning (See [8], Theorem 15.1). Thus we complet the proof of Theorem 1.

3. Proof of Corollary

(i) Let D be a 4-$(v,5,2)$ design. Let $\{1,2,3,4,i_1\}$ and $\{1,2,3,4,i_2\}$ be two blocks containing $\{1,2,3,4\}$. Then G_{1234} fixes $\{i_1,i_2\}$. If $\{i_1\}$ and $\{i_2\}$ are G_{1234}-orbits, then $G=A_6$ by a theorem of H. Nagao [2]. Hence D is a 4-$(6,5,2)$ design. If $\{i_1,i_2\}$ is a G_{1234}-orbit, then $G=S_6$ by Theorem 1. Hence D is also a 4-$(6,5,2)$ design.

(ii) Let D be a 4-$(v,6,1)$ design. Then by the same reason as (i), D is a 4-$(6,6,1)$ design.

(iii) Let D be a 4-$(v,6,2)$ design. Let $\{1,2,3,4,i_1,j_1\}$ and $\{1,2,3,4,i_2,j_2\}$ be two blocks containing $\{1,2,3,4\}$. Then G_{1234} fixes $\{i_1,j_1,i_2,j_2\}$ and $|\{i_1,j_1,i_2,j_2\}|=3$ or 4. If G_{1234} has an element which has a 3-cycle on $\{i_1,j_1,i_2,j_2\}$, then there are at least three blocks containing $\{1,2,3,4\}$, which is a contradiction.

Let P be a Sylow 3-subgroup of G_{1234}. Then $I(P)\supseteqq\{i,2,3,4,i_1,j_1,i_2,j_2\}$. Hence by Theorem of [7], $|I(P)|=11$ or 12 and $N_G(P)^{I(P)}=M_{11}$ or M_{12}. If $N_G(P)^{I(P)}=M_{12}$, then $(N_G(P)^{I(P)})_{1234}$ is transitive on $I(P)-\{1,2,3,4\}$. This is a contradiction since the number of blocks containing $\{1,2,3,4\}$ is two. If $N_G(P)^{I(P)}=M_{11}$, then a subgroup of $N_G(P)^{I(P)}$ fixing $\{1,2,3,4\}$ as a set has two orbits of length one and six. Hence by the same reason as above, we have a contradiction. Thus we complete the proof of Corollary.

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