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IMMERSION AND EMBEDDING PROBLEMS FOR COMPLEX FLAG MANIFOLDS

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Introduction

For a partition $n=n_1+n_2+\cdots+n_r$ of an integer n, let

$$W = W(n_1, \cdots, n_r) = U(n)/U(n_1)\times\cdots\times U(n_r)$$

be the complex (generalized) flag manifold. For example $W(k, n-k)=G_{k,n-k}$ is the complex Grassmann manifold and $W(1, 1, \cdots, 1)=F(n)$ is the (usual) flag manifold $U(n)/T^n$ where T^n is a maximal torus in $U(n)$. Then we have the natural bundle projection π: $F(n)\to W$ and the induced map

$$\pi^*\colon K(W) \to K(F(n)) = Z[\gamma_1, \gamma_2, \cdots, \gamma_n]/I^+$$

is a monomorphism (see §2). We write $M\subset R^n$ the existence of an embedding and $M\subseteq R^n$ the existence of an immersion of the differentiable manifold M in the Euclidean space R^n.

The purpose of this paper is to prove the following non-immersion and non-embedding theorem for the complex flag manifolds.

Theorem 4.1. *Let $2m=\dim W=n^2-(n_1^2+\cdots+n_r^2)$. For a positive inetger k, if the element*

$$2^m\prod_{(i,j)\in A}\{1+(\gamma_i-\gamma_j)\sum_{l=1}^{\infty}\left(-\frac{1}{2}\right)^l(\gamma_i+\gamma_j)^{l-1}\}$$

of $K(F(n))$ is not divisible by 2^{k+1}, then

$$\text{(i)}\quad W\not\subset R^{4m-2k}, \qquad \text{(ii)}\quad W\not\subseteq R^{4m-2k-1}\,.$$

For the definition of the set A, see (3.1).

As an application of Theorem 4.1, we also prove the following non-existence theorem of immersions and embeddings for some complex Grassmann manifolds $G_{2,n-2}$ for odd integers n.

Theorem 6.1.* *For each integer* $u \geqq 0$, *we put* $\beta(u) = 2\alpha(u) - \nu_2(u+1) + 1$. (For the definition of $\alpha(u)$ and $\nu_2(u+1)$, see p. 128) *Then we have*

$$\text{(i)}\quad G_{2,2n+1} \not\subset R^{8(2u+1)-2\beta(u)}, \qquad \text{(ii)}\quad G_{2,2u+1} \not\subseteq R^{8(2u+1)-2\beta(u)-1}.$$

We give the first few examples of non-embeddabilities:

$$G_{2,1} \not\subset R^{8-2}, \quad G_{2,3} \not\subset R^{24-4}, \quad G_{2,5} \not\subset R^{40-6}, \quad G_{2,7} \not\subset R^{56-6},$$
$$G_{2,9} \not\subset R^{72-6}, \quad G_{2,11} \not\subset R^{88-8}, \quad G_{2,13} \not\subset R^{104-6}, \quad G_{2,15} \not\subset R^{120-8}.$$

Problems of immersions and embeddings for flag manifolds have been investigated by many topologists. Hoggar [10] showed that $G_{2,n-2} \not\subset R^{3m}$ and that $G_{2,n-2} \not\subseteq R^{3m-1}$ where $2m = dim_R\, G_{2,n-2} = 4(n-2)$. He made use of the geometrical dimensions introduced by Atiyah [1]. Our results claim stronger facts that $G_{2,n-2} \not\subset R^{4m-2\beta}$ and that $G_{2,n-2} \not\subseteq R^{4m-2\beta-1}$ because $\beta/m \to 0$ as $n \to \infty$. Our method relies on a theorem of Nakaoka [13] which seems much close to the Atiyah-Hirzebruch's integrality theorem [3]. Tornehave [15] investigated the existence of immersion of flag manifolds $W(n_1, \cdots, n_r \subseteq R^{n^2-r})$ using the theory of Lie algebras and Hirsch's theorem [7]. Kee Yuen Lam [12] also proved the same result making use of his new functor μ^2. Connell [6] discussed on the existence and the non-existence of immersions of some low dimensional flag manifolds. Among his results, there are

$$\text{(i)}\quad G_{2,2} \subseteq R^{14}, \qquad \text{(ii)}\quad G_{2,2} \not\subseteq R^{12},$$
$$\text{(iii)}\quad G_{2,3} \subseteq R^{23}, \qquad \text{(iv)}\quad G_{2,3} \not\subseteq R^{19}.$$

The last statement (iv) agrees with a consequence of our result.

This paper is arranged as follows. In §1, we recall the immersion and embedding theorem of Nakaoka [13]. The structure of K-rings and tangent bundles of W and $F(n)$ are discussed in §§2–3. §4 is devoted to the proof of the main theorem (Theorem 4.1). Here we make use of Atiyah's γ-operations and the fact that the tangent bundle $\tau(W)$ has its splitting on $F(n)$. §5 is on some preliminaries for §6, where we discuss non-immersion and non-embedding of some complex Grassmann manifolds $G_{2,n-2}$. Calculations used here are quite elementry although a little bit complicated.

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1. Immersion and embedding of almost complex manifolds

For a complex vector bundle ξ over a finite CW-complex X, let $\gamma^i(\xi) \in K(X)$

* More complete results are obtained in [18].

denote the Atiyah class of ξ [2]. The map γ_t: $\mathrm{Vect}_C(X)\to 1+K(X)[t]^+$ defined by $\gamma_t(\xi)=\sum_{i\geqq 0}\gamma^i(\xi)t^i$ is multiplicative: $\gamma_t(\xi\oplus\eta)=\gamma_t(\xi)\gamma_t(\eta)$. We define the dual Atiyah class $\bar{\gamma}^i(\xi)\in K(X)$ by $\bar{\gamma}^0(\xi)=1$ and $\sum_{i+j=k}\gamma^i(\xi)\bar{\gamma}^j(\xi)=0$ for $k>0$. Then $\bar{\gamma}_t(\xi)=\sum_{i\geqq 0}\bar{\gamma}^i(\xi)\,t^i$ is the inverse element of $\gamma_t(\xi)$ in the multiplicative group $1+K(X)[t]^+$.

If M is an almost complex manifold of $2m$-dimension, that is, its tangent bundle $\tau(M)$ has a structure of m-dimensional complex vector bundle, then we write $\gamma^i(M)$ (resp. $\bar{\gamma}^i(M)$) for $\gamma^i(\tau(M)-m)$ (resp. $\bar{\gamma}^i(\tau(M)-m)$). We see that $\bar{\gamma}^i(M)=0$ if $i>m$. The following theorem due to Nakaoka [13, Theorem 8] is the starting point of our investigations.

Theorem 1.1. *Let M be a closed almost complex manifold of real dimension $2m$ such that $K(M)$ has no elements of finite order. Then if M can be embedded (resp. immersed) in R^{4m-2k}, the element $\sum_{i=0}^{m}2^{m-i}\bar{\gamma}^i(M)\in K(M)$ is divisible by 2^{k+1} (resp. 2^k).*

Note that the element in Theorem 1.1 is rewritten as

$$\sum_{i=0}^{m}2^{m-i}\bar{\gamma}^i(M)=2^m\sum_{i=0}^{m}\bar{\gamma}^i(M)\Big(\frac{1}{2}\Big)^i=2^m\bar{\gamma}_{1/2}(M)$$

where $\bar{\gamma}_{1/2}(M)$ is regarded as the element of $K(M)\otimes Z[\frac{1}{2}]$. If N is another almost complex manifold of dimension $2n$, it holds that

$$2^{m+n}\bar{\gamma}_{1/2}(M\times N)=2^m\bar{\gamma}_{1/2}(M)\otimes 2^n\bar{\gamma}_{1/2}(N)$$

The following theorem is a generalization of Theorem 9 of Nakaoka [13] and the proof relies on Sanderson-Schwarzenberger [14, Theorem 1].

Theorem 1.2. *Let M be the same as in Theorem 1.1. For a positive integer k, if the element $\sum_{i=0}^{m}2^{m-i}\bar{\gamma}^i(M)$ is not divisible by 2^{k+1}, then*

$$\text{(i)}\quad M\not\subset R^{4m-2k},\qquad \text{(ii)}\quad M\not\subseteq R^{4m-2k-1}.$$

Before we prove Theorem 1.2, we put a remark on the exponent of 2 in the binomial coefficient $\binom{a}{b}$. Let $\nu_2(n)$ denote the exponent of 2 in n and $\alpha(n)$ the number of 1's in the diadic expansion of n. Since the equality $\nu_2(n!)=n-\alpha(n)$ holds by the elementary number theory, we have the following

Lemma 1.3. $\nu_2(\binom{a}{b})=\alpha(b)+\alpha(a-b)-\alpha(a)$.

Proof of Theorem 1.2. (i) Straightfoward from Theorem 1.1. (ii) Suppose

$M \subseteq R^{4m-2k-1}$. We fix an integer $s=2^t>m$. By James [11] it holds that $CP^s \subset R^{4s-1}$ and therefore by Sandeson-Schwarzenberger [14, Lemma] it holds that $M\times CP^s \subset R^{4m+4s-2k-2}$. Thus by Theorem 1.1 the element

$$2^{m+s}\bar{\gamma}_{1/2}(M\times CP^s) = 2^m\bar{\gamma}_{1/2}(M)\otimes 2^s\bar{\gamma}_{1/2}(CP^s)$$

is divisible by 2^{k+1}. On the other hand, the isomorphism $\tau(CP^s)\oplus 1_C \cong (s+1)\eta$ implies $\gamma_t(CP^s)=(1+tx)^{s+1}$ and $\bar{\gamma}_t(CP^s)=(1+tx)^{-s-1}$ where η is the canonical line bundle over CP^s and $x=\eta-1_C\in K(CP^s)$. Therefore we have

$$2^s\bar{\gamma}_{1/2}(CP^s) = 2^s(1+x/2)^{-s-1}\ (\mathrm{mod}\ x^{s+1}) = \sum_{i=0}^{s}(-1)^i\binom{s+i}{i}2^{s-i}x^i\,.$$

Since $\binom{s+i}{i}2^{s-i}$ $(0\leqq i<s)$ are divisible by 4 and $\binom{2s}{s}$ is divisible by 2 but not by 4 (see Lemma 1.3), $2^s\bar{\gamma}_{1/2}(CP^s)$ is divisible by 2 but not by 4. Hence $2^m\bar{\gamma}_{1/2}(M)$ must be divisible by 2^{k+1}. This leads to a contradiction.

2. *K*-ring of flag manifolds

Let $(n_1, n_2, \cdots, n_r)$ be a partition of an interg n: $n=n_1+n_2+\cdots+n_r$ and let

$$W = W(n_1, n_2, \cdots, n_r) = U(n)/U(n_1)\times U(n_2)\times\cdots\times U(n_r)$$

be a complex flag manifold. For example for $(1, 1, \cdots, 1)$ we have the usual flag manifold $F(n)=U(n)/T^n$ where T^n is a maximal torus of $U(n)$. For $(k, n-k)$ we have the complex Grassamann manifold $G_{k,n-k}$ of all k-planes in C^n and for $(1, n-1)$, W is just the complex projective space CP^{n-1}.

In this paragraph, we determine the ring structure of $K(F(n))$ and $K(W)$ explicitly. Generally for a compact Lie group G and its closed subgroup H, the ring homomorphism $\alpha\colon R(H)\to K(G/H)$ is constructed by Atiyah-Hirzebruch [4] as follows. For an isomorphism class $x=[V]\in R(H)$ of an H-vector space V, $\alpha(x)$ is the isomorphism class of vector bundle $V\to G\times_H V\to G/H$ associated with the natural principal H-bundle over G/H. If V is moreover a G-vector space, that is, x is in the image of $i^*\colon R(G)\to R(H)$, the bundle map $\alpha\colon G\times_H V\to G/H\times V$ difined by $\alpha(g\times_H v)=(gH, gv)$ is an isomorphism and hence $\alpha(x)=(dim\ V)1_C$. Therefore α is factored through the natural projection p:

$$\begin{tikzcd} R(H) \arrow[rr, "\alpha"] \arrow[dr, "p"'] & & K(G/H) \\ & R(H)\otimes_{R(G)} Z \arrow[ur, "\bar{\alpha}"'] & \end{tikzcd}$$

The following theorem is due to Hodgkin [9, Corollary of Lemma 9.2].

Theorem 2.1. *Let G be a compact connected Lie group with $\pi_1(G)$ free and* *let H be a closed connected subgroup of G with maximal rank. Then the ring homomorphism $\bar{\alpha}$: $R(H)\otimes_{R(G)}Z\to K(G/H)$ is an isomorphism.*

We use these facts for $G=U(n)$ and $H=T^n$ or $\prod_j U(n_j)$. First we will investigate the case $F(n)$ and then the general case $W(n_1, n_2, \cdots, n_r)$. As is well known we have

$$R(T^n)=Z[\alpha_1, \alpha_1^{-1}, \alpha_2, \alpha_2^{-1}, \cdots, \alpha_n, \alpha_n^{-1}]$$
$$R(U(n))=Z[\lambda_1, \lambda_2, \cdots, \lambda_n, \lambda_n^{-1}]$$

and λ_i is mapped on the *i-th* elementary symmetric polynomial of $\alpha_1, \alpha_2, \cdots, \alpha_n$ by the monomorphism i^*: $R(U(n))\to R(T^n)$. Let ξ_i be the image of α_i by the ring homomorpism α: $R(T^n)\to K(F(n))$, then $\xi_1\oplus\xi_2\oplus\cdots\oplus\xi_n$ is the vector bundle associated with the principal T^n bundle $T^n\to U(n)\to F(n)$. Let $\sigma^k(x_1, x_2, \cdots, x_n)$ denote the k-th elementary symmetric polynomial in variables $x_1, x_2, \cdots, x_n$. The element $\sigma^k(\alpha_1, \alpha_2, \cdots, \alpha_n)$ has the same dimension as $\binom{n}{k}1_C$ and they coincide with each other in $\bigotimes_{j=1}^{r} R(U(n_j))\otimes_{R(U(n))}Z$. Therefore $\sigma^k(\xi_1, \xi_2, \cdots, \xi_n)=\binom{n}{k}$ holds in $K(F(n))$. In particuler $\xi_1\xi_2\cdots\xi_n=1$ holds and we have $\xi_j^{-1}=\prod_{k\neq j}\xi_k$. Therefore the ring $K(F(n))$ is isomorphic to the quotient ring of $Z[\xi_1, \xi_2, \cdots, \xi_n]$ factored by the ideal generated by

$$\{\sigma^k(\xi_1, \xi_2, \cdots, \xi_n)-\binom{n}{k};\ k>0\}\ .$$

For the convenience of the later use we adopt the generators $\gamma_i=\xi_i-1$. Then we can choose the elements

$$\{\sigma^k(\gamma_1, \gamma_2, \cdots, \gamma_n),\ k>0\}$$

as a new generator system of the ideal. Hence we have the following

Proposition 2.2.

$$K(F(n))=Z[\gamma_1, \gamma_2, \cdots, \gamma_n]/I^+$$

where I^+ is the ideal generated by $\{\sigma^k(\gamma_1, \gamma_2, \cdots, \gamma_n);\ k>0\}$.

We repeat the same procedure for $W=W(n_1, n_2, \cdots, n_r)$. For a partition $(n_1, n_2, \cdots, n_r)$ of n, we define a sequence of integers $(m_0, m_1, \cdots, m_r)$ inductively as follows:

$$m_0=0,\quad m_j=m_{j-1}+n_j\quad (1\leqq i\leqq r)\ .$$

For the representation ring of $\prod_j U(n_j)$ we have

$$R(\prod_j U(n_j)) = \bigotimes_{j=1}^{r} R(U(n_j)) = \bigotimes_{j=1}^{r} Z[\lambda_1^{(j)}, \lambda_2^{(j)}, \cdots, \lambda_{n_j-1}^{(j)}, \lambda_{n_j}^{(j)}, (\lambda_{n_j}^{(j)})^{-1}]$$

and $i^*: R(\prod_j U(n_j)) \to R(T^n)$ maps $\lambda_p^{(j)}$ on the p-th fundamental symmetric polynomial in variables $\{\alpha_i: m_{j-1} < i \leqq m_j\}$. We denote $\sigma_p^{(j)}$ for the image of $\lambda_p^{(j)}$ by the map $\alpha: R(\prod_j U(n_j)) \to K(W)$. Since the element

$$\sum_{i_1+\cdots+i_r=k}^{r} \lambda_{i_1}^{(1)} \lambda_{i_2}^{(2)} \cdots \lambda_{i_r}^{(r)} \in \bigotimes_{j=1}^{r} R(U(n_j))$$

has the same dimension as $\binom{n}{k} 1_C$, they conicide with each other in $\bigotimes_{j=1}^{r} R(U(n_j)) \otimes_{R(U(n))} Z$. Therefore $\sum_{i_1+\cdots+i_r=k} \sigma_{i_1}^{(1)} \sigma_{i_2}^{(2)} \cdots \sigma_{i_r}^{(r)} = \binom{n}{k}$ holds in $K(W)$. In particular $\sigma_{n_1}^{(1)} \sigma_{n_2}^{(2)} \cdots \sigma_{n_r}^{(r)} = 1$ holds and we obtain $(\sigma_{n_j}^{(j)})^{-1} = \prod_{k \neq j} \sigma_{n_k}^{(k)}$. Therefore the ring $K(W)$ is isomorphic to the quotient ring of

$$\bigotimes_{j=1}^{r} Z[\sigma_1^{(j)}, \sigma_2^{(j)}, \cdots, \sigma_{n_j}^{(j)}]$$

factored by the ideal generated by the elements

$$\{\sum_{i_1+\cdots+i_r=k} \sigma_{i_1}^{(1)} \sigma_{i_2}^{(2)} \cdots \sigma_{i_r}^{(r)} - \binom{n}{k};\ k > 0\} .$$

Again we change the generators as follows. The homomorphism $\pi^*: K(W) \to K(F(n))$ induced by the projection of the fibre bundle $\prod_j F(n_j) \to F(n) \to W$ is a monomorphism. In fact, since the odd dimensional parts of the cohomology groups $H^{2i+1}(F(n), Z)$ and $H^{2i+1}(W, Z)$ vanish (Bott [16; Theorem A]), the induced homomorphism $\pi^*: H^*(W, Z) \to H^*(F(n), Z)$ is monic because the Serre spectral sequence of the above fibre bundle collapses (Serre [17]). Moreover, the Atiyah-Hirzebruch spectral sequence of W also collapses and hence the Chern character $ch: K(W) \to H^*(W, Q)$ is monic [4]. Therefore, the commutative diagram

$$\begin{tikzcd} K(W) \arrow[r, "\pi^*"] \arrow[d, "ch"'] & K(F(n)) \arrow[d, "ch"] \\ H^*(W, Q) \arrow[r, "\pi^*"] & H^*(F(n), Q) \end{tikzcd}$$

leads that the homomorphism $\pi^*: K(W) \to K(F(n))$ is monic. We define the element $c_p^{(i)}$ such that $\pi^*(c_p^{(j)})$ is the p-th elementary symmetric polynomial in $\{\gamma_i;\ m_{j-1} < i \leqq m_j\}$. Then $\sigma_p^{(j)}$ and $c_p^{(j)}$ differ in $Z[\gamma_1, \gamma_2, \cdots, \gamma_n]$ only by an element of the submodule generated by $\{c_k^{(j)};\ k < p\}$ or, the same, by $\{\sigma_k^{(j)};\ k < p\}$. Hence we can adopt $c_p^{(j)}$ as ring generators of $K(W)$.

Proposition 2.3.

$$K(W)=\bigotimes_{j=1}^{r} Z[c_1^{(j)}, c_2^{(j)}, \cdots, c_{n_j}^{(j)}]/J^+$$

where J^+ *is the ideal generated by*

$$\{\sum_{i_1+\cdots+i_r=k} c_{i_1}^{(1)}c_{i_2}^{(2)}\cdots c_{i_r}^{(r)};\ k>0\}\ .$$

3. Tangent bundles of $F(n)$ and W

The tangent bundles of $F(n)$ and $W=W(n_1, n_2, \cdots, n_r)$ are investigated by such authors as Hirzebruch [8, §13] and Kee Yuen Lam [12] as follows:

Proposition 3.1.

(1) *Let* $\xi_1\oplus\cdots\oplus\xi_n$ *be the vector bundle associated with the principal bundle* $T^n\to U(n)\to F(n)$, *then we have*

$$\tau(F(n))\cong\sum_{i>j}\xi_i\otimes\xi_j^*\ .$$

(2) *Let* $\zeta_1\oplus\cdots\oplus\zeta_r$ *be the vector bundle associated with the principal bundle* $U(n_1)\times\cdots\times U(n_r)\to U(n)\to W$, *then we have*

$$\tau(W)\cong\sum_{\alpha>\beta}\zeta_\alpha\otimes\zeta_\beta^*\ .$$

With a partition $(n_1, n_2, \cdots, n_n)$ of an integer n, we associate an increasing sequence $(m_0, m_1, \cdots, m_r)$ defined as follows:

$$m_0=0,\quad m_i=m_{i-1}+n_i\ (0<i\leqq r)\ .$$

Let π: $F(n)\to W$ be the natural projection. Since π^*: $K(W)\to K(F(n))$ is a monomorphism and it holds that $\pi^*(\zeta_\alpha)=\sum_{m_{\alpha-1}<i\leqq m_\alpha}\xi_i$, we have the splitting

$$\pi^*(\tau(W))=\sum_{(i,j)\in A}\xi_i\otimes\xi_j^* \tag{3.1}$$

where $B=\bigcup_{\alpha=1}^{r}\{(i,j);\ m_{\alpha-1}<j<i\leqq m_\alpha\}$ and $A=\{(i,j);\ 1\leqq j<i\leqq n\}-B$.

4. Immersion and embedding of flag manifolds

As we saw in §1, for the probrem of immersion and embedding of flag manifolds, we have to know $\bar{\gamma}_{1/2}(W)$. Note that the following three procedures are commutative with each other.

(a) To get γ_t of a vector bundle from γ_t of its splitting line bundles.

(b) To get $\bar{\gamma}_t(\xi)$ from $\gamma_t(\xi)$

(c) Substituting $t=\dfrac{1}{2}$.

Therefore we have the following "commutative diagram" of three procedures:

$$\begin{tikzcd}
 & \gamma_t(E) \arrow[rrr, "(b)"] \arrow[ddr] & & & \bar{\gamma}_t(E) \arrow[ddr, "(c)"] & \\
\gamma_t(L) \arrow[ur, "(a)"] \arrow[rrr, dash] \arrow[ddr, "(c')"'] & & & \bar{\gamma}_t(L) \arrow[ur] \arrow[ddr] & & \\
 & & \gamma_{1/2}(E) \arrow[rrr] & & & \bar{\gamma}_{1/2}(E) \\
 & \gamma_{1/2}(L) \arrow[ur] \arrow[rrr, "(b')"'] & & & \bar{\gamma}_{1/2}(L) \arrow[ur, "(a')"'] &
\end{tikzcd}$$

So let us take the path $(a)^{-1}$ (c') (b') (a') instead of the path (b) (c). Recall that the projection π: $F(n) \rightarrow W$ induces the monomorphism π^*: $K(W) \rightarrow K(F(n))$ (see §2) and $\pi^*\tau(W) = \sum_{(i,j)\in\Delta} \xi_i \otimes \xi_j^*$ (see §3). Hence

$$\begin{aligned}
\pi^*\gamma_t(W) &= \pi^*(\gamma_t((W)-m)) = \gamma_t\Big(\sum_{(i,j)\in\Delta} (\xi_i \otimes \xi_j^* - 1)\Big) \\
&= \prod_{(i,j)\in\Delta} \gamma_t(\xi_i \otimes \xi_j^* - 1).
\end{aligned}$$

Recall that for a line bundle η, we have $\gamma_t(\eta-1) = 1+(\eta-1)t$ [2]. As we have put $\gamma_i = \xi_i - 1$, the equality $\xi_i \otimes \xi_i^* = 1$ implies $\xi_i^* = 1/(1+\gamma_i)$. Therefore

$$\begin{aligned}
\gamma_t(\xi_i \otimes \xi_j^* - 1) &= 1+(\xi_i \otimes \xi_j^* - 1)t \\
&= 1+\left(\frac{1+\gamma_i}{1+\gamma_j} - 1\right)t = 1+\left(\frac{\gamma_i - \gamma_j}{1+\gamma_j}\right)t.
\end{aligned}$$

Substituting $t = \frac{1}{2}$ and taking its inverse element:

$$\bar{\gamma}_{1/2}(\xi_i \otimes \gamma_j^* - 1) = \left\{1+\frac{\gamma_i - \gamma_j}{1+\gamma_j}\left(\frac{1}{2}\right)\right\}^{-1} = \frac{1+\gamma_j}{1+\frac{1}{2}(\gamma_i+\gamma_j)}$$

$$= 1 - \frac{\frac{1}{2}(\gamma_i - \gamma_j)}{1+\frac{1}{2}(\gamma_i+\gamma_j)} = 1+(\gamma_i - \gamma_j)\sum_{l=1}^{\infty}\left(-\frac{1}{2}\right)^l (\gamma_i+\gamma_j)^{l-1}.$$

Therefore we have

$$\pi^*(\bar{\gamma}_{1/2}(W)) = \prod_{(i,j)\in\Delta} \{1+(\gamma_i - \gamma_j)\sum_{l=1}^{\infty}\left(-\frac{1}{2}\right)^l (\gamma_i+\gamma_j)^{l-1}\}.$$

Combining this result with Theorem 1.2 we obtain the following

Theorem 4.1. *Let $2m = \dim W = n^2 - (n_1^2 + \cdots + n_r^2)$.*
For a positive integer k, if the element

$$2^m \prod_{(i,j)\in A} \{1+(\gamma_i-\gamma_j)\sum_{l=1}^{\infty}\left(-\frac{1}{2}\right)^l(\gamma_i+\gamma_j)^{l-1}\}$$

of $K(F(n))$ is not divisible by 2^{k+1}, then we have

$$\text{(i)}\quad W \not\subset R^{4m-2k}, \qquad \text{(ii)}\quad W \not\subseteq R^{4m-2k-1}.$$

It does not seem easy to find from this theorem the dimension of Euclidean space in which $W(n_1, n_2, \cdots, n_r)$ cannot be embedded or immersed. In the following paragraph, we will discuss non-immersion and non-embedding for only the case $W(2, n-2)=G_{2,n-2}$ for odd integer n.

5. Preliminaries

In §2, we have determined the ring structure of K-ring of $F(n)$ and $W=W(n_1, n_2, \cdots, n_r)$ as follows:

$$K(F(n)) = Z[\gamma_1, \gamma_2, \cdots, \gamma_n]/I^+$$

$$K(W) = \bigotimes_{j=1}^{r} Z[c_1^{(j)}, c_2^{(j)}, \cdots, c_{n_j}^{(j)}]/J^+ .$$

For the next paragraph, we observe some algebraic properties of these rings. Although K-ring has no geometrical grading, giving $\deg \gamma_i=1$ and $\deg c_i^{(j)}=i$, we regard $K(F(n))$ and $K(W)$ as graded algebras. It is possible because the ideals I^+ and J^+ are generated by homogeneous elements. (see §2).

First in $K(F(n))$, it holds that

$$\gamma_i^n = 0 \quad (i = 1, 2, \cdots, n) . \tag{5.1}$$

In fact let $\pi_i\colon F(n)\to CP^{n-1}$ be such natural projection that the induced homomorphism $\pi_i^*\colon K(CP^{n-1})=Z[x]/(x^n)\to K(F(n))$ satisfies $\pi_i^*(x)=\gamma_i$. Then $x^n=0$ implies $\gamma_i^n=0$.

Next, as far as the applications discussed in §6 are concerned, it is sufficient to observe the case $W=G_{k,n-k}$. In this case, we have

$$K(G_{k,n-k}) = Z[c_1, c_2, \cdots, c_k, c_1', c_2', \cdots, c_{n-k}']/J^+$$

and J^+ is generated by

$$\{c_i+c_{i-1}c_1'+\cdots+c_1c_{i-1}'+c_i', \quad 1\leqq i\leqq k(n-k)\} . \tag{5.2}$$

Of course we understand that $c_j=0$ if $j>k$ and $c_j'=0$ if $j>n-k$.

Proposition 5.1. *In the ring $K(G_{k,n-k})$, we have*

$$c_l' = \sum_{||I||=l} (-1)^{|I|}\binom{|I|}{i_1, i_2, \cdots, i_k} c_1^{i_1}c_2^{i_2}\cdots c_k^{i_k} \tag{5.3}$$

where $|I|=\sum_{j=1}^{k} i_j$ *and* $\|I\|=\sum_{j=1}^{k} j i_j$ *for* $I=(i_1, i_2, \cdots, i_k)$.

Proof. By (5.2) it is sufficient to check

$$\sum_{t+j=s}\{c_t \sum_{\|I\|=j}(-1)^{|I|}\binom{|I|}{i_1, i_2, \cdots, i_k} c^I\} = 0$$

The left hand side is rewritten as

$$\sum_{t=0}^{s}\sum_{\|I\|=s-t}(-1)^{|I|}\binom{|I|}{i_1, i_2, \cdots, i_k} c^I c_t .$$

Put $J_t=(i_1, \cdots, (i_t+1), \cdots, i_k)$ for $1\leqq t\leqq k$ then we have

$$=\sum_{\|I\|=s}(-1)^{|I|}\binom{|I|}{i_1, i_2, \cdots, i_k} c^I+\sum_{t=1}^{s}\sum_{\|I\|=s-t}(-1)^{|J_t|-1}\binom{|J_t|-1}{i_1, i_2, \cdots, i_k} c^{J_t}$$
$$=\sum_{\|J\|=s}(-1)^{|J|}\left\{\binom{|J|}{j_1, j_2, \cdots, j_k}-\sum_{t=1}^{s}\binom{|J|-1}{j_1, \cdots, (j_t-1), \cdots, j_k}\right\} c^J = 0$$

by the formula for the multinomial coefficients and thus Proposition 5.1 is proved.

By Proposition 5.1, we see that all monomials in $K(G_{k,n-k})$ is written only by $c_1, c_2, \cdots$. Moreover, it seems that $K(G_{k,n-k})$ is the free module over Z with a base consisting of the monomials $\{c_{j_1}c_{j_2}\cdots c_{j_r} : j_1+\cdots+j_r\leqq n-k\}$ but the author has succeeded only to prove Proposition 5.3. Before that, we prove the following

Lemma 5.2. *Let n and k be two integers with $0\leqq k\leqq n$, then we have*

$$\sum_{i\geqq 0}(-1)^i\binom{n-i}{i}\binom{n-2i}{k-i} = 1 .$$

Proof. Putting $\left\{\begin{matrix} n \\ k \end{matrix}\right\}=\sum_{i\geqq 0}(-1)^i\binom{n-i}{i}\binom{n-2i}{k-i}$, we show that $\left\{\begin{matrix} n \\ k \end{matrix}\right\}=1$ by induction on n and k. Evidently we have $\left\{\begin{matrix} n \\ 0 \end{matrix}\right\}=\binom{n}{0}\binom{n}{0}=1$ and $\left\{\begin{matrix} n \\ n \end{matrix}\right\}=\binom{n}{0}\binom{n}{n}=1$. Next it is easy to see that

$$\left\{\begin{matrix} n \\ k \end{matrix}\right\}=\left\{\begin{matrix} n-1 \\ k \end{matrix}\right\}+\left\{\begin{matrix} n-1 \\ k-1 \end{matrix}\right\}-\left\{\begin{matrix} n-2 \\ k-1 \end{matrix}\right\} .$$

holds and by the hypothesis of induction, $\left\{\begin{matrix} n \\ k \end{matrix}\right\}=1+1-1=1$. q.e.d.

In what follows, we consider the case $k=2$ and we put $r=n-2$.

Proposition 5.3. *In $K(G_{2,r})=Z[c_1, c_2, c_1', c_2', \cdots, c_r']/J^+$ it holds that the $2r$-dimensional part is generated by c_2^r and other monomials of $2r$-dimension is written as*

$$c_1^{2j}c_2^{r-j}=\left\{\binom{2j}{j}-\binom{2j}{j-1}\right\} c_2^r .$$

Proof. In Proposition 5.1, the convention $c_l'=0$ $(r<l\leqq 2r)$ leads to the relations

$$(5.4)\qquad \sum_{i_1+2i_2=l}(-1)^{i_1+i_2}\binom{i_1+i_2}{i_2}c_1^{i_1}c_2^{i_2}=0 \quad (r<l\leqq 2r)\,.$$

Multiplying c_1^{2k-l} and rewriting $i_2=r-j$ and $i_1+i_2=l-r+j$, we have the relations in homogeneous $2r$-dimensions:

$$\sum_j(-1)^j\binom{l-r+j}{r-j}c_1^{2j}c_2^{r-j}=0 \quad (r<l\leqq 2r)$$

Therefore it is sufficient to solve the following homogeneous linear equations in $r+1$ variables $x_0, x_1, \cdots, x_r$.

$$(5.5)\qquad \begin{cases} A_{r,r+1}=\sum_j(-1)^j\binom{1+j}{r-j}x_j=0 \\ A_{r,r+2}=\sum_j(-1)^j\binom{2+j}{r-j}x_j=0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ A_{r,2r}=\sum_j(-1)^j\binom{r+j}{r-j}x_j=0 \end{cases}$$

We fix integers j, k and r with $r\geqq k+j$. Comparing the coefficients of y^{r-j} in the expansion of the equality

$$(1+y)^{r+j}(1-(1+y)^{-1})^k=y^k(1+y)^{r-k+j}\,,$$

(I owe this equality to K. Shibata) we obtain the relation

$$\sum_{s=0}^{k}(-1)^s\binom{k}{s}\binom{r-s+j}{r-j}=\binom{r-k+j}{r-k-j}\,.$$

Hence we have

$$\sum_{s=0}^{k}(-1)^s\binom{k}{s}A_{r,2r-s}=A_{r-k,2(r-k)} \quad 1\leqq k<r\,.$$

This means that (5.5) is equivalent to the following homogeneous equations

$$(5.6)\qquad \begin{cases} A_{1,2}=\sum_{j=0}^{1}(-1)^j\binom{1+j}{1-j}x_j=0 \\ A_{2,4}=\sum_{j=0}^{2}(-1)^j\binom{2+j}{2-j}x_j=0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ A_{r,2r}=\sum_{j=0}^{r}(-1)^j\binom{r+j}{r-j}x_j=0 \end{cases}$$

This is rewritten as

$$
(5.7) \qquad \begin{cases} \sum_{j>0} (-1)^j \binom{1+j}{1-j} x_j = -x_0 \\ \sum_{j>0} (-1)^j \binom{2+j}{2-j} x_j = -x_0 \\ \cdots\cdots\cdots\cdots\cdots \\ \sum_{j>0} (-1)^j \binom{r+j}{r-j} x_j = -x_0 \end{cases}
$$

and the matrix is a triangular one with the diagonal consisting of 1 and -1 alternatively. Hence the matrix is unimodular and the solution is unique. It is therefore sufficient to show that

$$
(5.8) \qquad x_j = \left\{ \binom{2j}{j} - \binom{2j}{j-1} \right\} x_0
$$

is the solution. In Lemma 5.2 putting $n-i=l+j$ and $i=l-j$, we have $n-2i=2j$. Moreover putting (i) $k-i=j$ and (ii) $k-i=j-1$, we have

$$
\text{(i)} \quad \sum_j (-1)^j \binom{l+j}{l-j} \binom{2j}{j} = (-1)^l \qquad 1 \leqq l \leqq r,
$$

$$
\text{(ii)} \quad \sum_j (-1)^j \binom{l+j}{l-j} \binom{2j}{j-1} = (-1)^l \quad 1 \leqq l \leqq r,
$$

and hence $\sum_j (-1)^j \binom{l+j}{l-j} \left\{ \binom{2j}{j} - \binom{2j}{j-1} \right\} = 0$, $1 \leqq l \leqq r$. This means that (5.8) is just the solution of (5.6) and hence of (5.5).

6. Non-immersion and non-embedding of Grassmann manifolds

For an application of Theorem 4.1, we investigate the dimension of Euclidean spaces in which Grassmann manifolds $G_{k,n-k}$ cannot be immersed or embedded. Only the case $k=2$ and n is odd was succeeded. First we show the results. $\alpha(n)$ denotes the number of l's in the diadic expansion of an integer n and $\nu_p(n)$ denotes the exponent of a prime p in n.

Theorem 6.1. *For each integer $u \geqq 0$ we put $\beta(u) = 2\alpha(u) - \nu_2(u+1) + 1$. Then we have*

$$
\text{(i)} \quad G_{2,2u+1} \not\subset R^{8(2u+1)-2\beta(u)}, \qquad \text{(ii)} \quad G_{2,2u+1} \not\subseteq P^{8(2u+1)-2\beta(u)-1}.
$$

REMARK 1. It might be interesting to compare these results with the Atiyah-Hirzebruch's results [3] that (i) $CP^m \not\subset R^{4m-2\alpha(m)}$ and (ii) $CP^m \not\subseteq R^{4m-2\alpha(m)-1}$.

REMARK 2. Connell [6] also proved that $G_{2,3} \not\subseteq R^{19}$.

Proof. By the results in §3 we have

(6.1) $$K(G_{2,n-2})=Z[c_1, c_2, c_1', c_2', \cdots, c_{n-2}']/J^+$$

(6.2) $$K(F(n))=Z[\gamma_1, \gamma_2, \cdots, \gamma_n]/I^+$$

Let π: $F(n)\to G_{2,n-2}$ be the projection of the fibre bundle with the fibre $F(2)\times F(n-2)$, then π^*: $K(G_{2,n-2})\to K(F(n))$ is a monomorphism and $\pi^*(c_i)$ (resp. $\pi^*(c_i'))$ is the i-th symmetric polynomial in γ_1, γ_2 (resp. $\gamma_3, \gamma_4, \cdots, \gamma_n$). In Proposition 5.1 we have shown that c_2^{n-2} generates the $2(n-2)$-dimensional part of the graded module $K(G_{2,n-2})$ and we will show in Lemma 6.4 that the coefficient a of c_2^{n-2} in $\sum_{i=0}^{m} 2^{m-i}\bar{\gamma}^i(G_{2,n-2})$ is

(6.3) $$a=\begin{cases} 0 & n\text{: even} \\ -\dfrac{2(2u+3)}{(2u-1)(u+1)}\dbinom{2u}{u}^2 & n=2u+3 \end{cases}$$

Therefore unfortunately we get no informations if n is even. When n is odd, note that $\nu_2(\binom{2u}{u})=\alpha(u)$ holds by Lemma 1.3. Then we have

(6.4) $$\nu_2(a)=\beta(u)=2\alpha(u)-\nu_2(u+1)+1\,.$$

Since $\sum_{i=0}^{m} 2^{m-i}\bar{\gamma}^i(G_{2,n-2})$ cannot be devided by $2^{\nu_2(a)+1}$, Theorem 6.1 follows from Theorem 1.2. q.e.d.

It is left to get the coefficient a of c_2^{n-2} in

(6.5) $$2^m \prod_{\substack{3\leq i\leq n \\ 1\leq j\leq 2}} \{1+(\gamma_i-\gamma_j)\sum_{l=1}^{\infty}\left(-\frac{1}{2}\right)^l(\gamma_i+\gamma_j)^{l-1}\}$$

which will be done in Lemmas 6.2, 6.3 and 6.4. In Lemma 6.2 we work in the case $G_{k,n-k}$ for arbitraly k, but in Lemmas 6.3 and 6.4 we restrict ourselves to the case $k=2$.

Lemma 6.2.

(a) *For fixed* j, *we can put*

$$\prod_{i=k+1}^{n}\{1+(\gamma_i-\gamma_j)\sum_{l=1}^{\infty}\left(-\frac{1}{2}\right)^l(\gamma_i+\gamma_j)^{l-1}\}$$
$$=\sum_{l=0}^{\infty}\left(-\frac{1}{2}\right)^l\sum_{p=0}^{l}(-1)^p e_{n,l-p,p}\pi^*(c_p)\gamma_j^{l-p}$$

(b) $$e_{n,l-p,p}=\sum_{r=0}^{l-p}(-1)^r\binom{n-k}{r}\binom{l-1}{l-p-r}.$$

Proof. $\bar{\gamma}_{1/2}(\xi_i\otimes\xi_j^*-1) = 1+\sum_{l=1}^{\infty}\left(-\frac{1}{2}\right)^l(\gamma_i-\gamma_j)(\gamma_i+\gamma_j)^{l-1}$

$$= 1+\sum_{l=1}^{\infty}\left(-\frac{1}{2}\right)^l\sum_{p=0}^{l-1}\binom{l-1}{l-p-1}(\gamma_i-\gamma_j)\gamma_i^p\gamma_j^{l-p-1}$$

$$= 1+\sum_{l=1}^{\infty}\left(-\frac{1}{2}\right)^l\left\{\sum_{p=1}^{l}\binom{l-1}{l-p}\gamma_i^p\gamma_j^{l-p}-\sum_{p=0}^{l-1}\binom{l-1}{l-p-1}\gamma_i^p\gamma_j^{l-p}\right\}.$$

In order to introduce a new function, we recall some properties of binomial coefficient $\binom{a}{b}$. Putting $\binom{0}{0}=1$ and $\binom{0}{b}=0$ if $b\neq 0$, $\binom{a}{b}$ is defined by $\binom{a}{b}=\binom{a-1}{b}+\binom{a-1}{b-1}$ for each pair (a, b) of integers. Then $\binom{a}{b}=0$ if $b<0$ or if $0\leqq a<b$. $\binom{a}{0}=1$ for each a and $\binom{a}{a}=1$ if $a\geqq 0$. We define a new function $\begin{bmatrix} a \\ b \end{bmatrix}$ for each pair (a, b) of intergers by

$$\begin{bmatrix} a \\ b \end{bmatrix}=\binom{a}{b}-\binom{a}{b-1} \tag{6.6}$$

Then, we have $\begin{bmatrix} a \\ b \end{bmatrix}=0$ if $b<0$ or if $0\leqq a+1<b$, $\begin{bmatrix} a \\ 0 \end{bmatrix}=1$ for each a and $\begin{bmatrix} a \\ a+1 \end{bmatrix}=-1$ if $a\geqq 0$. Using these the above equations are contineued as follows:

$$=\sum_{l=0}^{\infty}\left(-\frac{1}{2}\right)^l\sum_{p=0}^{l}\begin{bmatrix} l-1 \\ l-p \end{bmatrix}\gamma_i^p\gamma_j^{l-p}.$$

Therefore

$$\prod_{i=k+1}^{n}\bar{\gamma}_{1/2}(\xi_i\otimes\xi_j^*-1)$$

$$=\prod_{i=k+1}^{n}\sum_{l_i=0}^{\infty}\left\{\left(-\frac{1}{2}\right)^{l_i}\sum_{p_i=0}^{l_i}\begin{bmatrix} l_i-1 \\ l_i-p_i \end{bmatrix}\gamma_i^{p_i}\gamma_j^{l_i-p_i}\right\}$$

$$=\sum_{l=0}^{\infty}\left(-\frac{1}{2}\right)^l\sum_{l_{k+1}+\cdots+l_n=l}\sum_{p=0}^{l}\left\{\sum_{p_{k+1}+\cdots+p_n=p}\prod_{i=k+1}^{n}\begin{bmatrix} l_i-1 \\ l_i-p_i \end{bmatrix}\prod_{i=k+1}^{n}\gamma_i^{p_i}\right\}\gamma_j^{l-p}$$

$$=\sum_{l=0}^{\infty}\left(-\frac{1}{2}\right)^l\sum_{p=0}^{l}\left\{\sum_{p_{k+1}+\cdots+p_n=p}\sum_{\substack{l_{k+}+\cdots+l_n=l \\ p_i\leqq l_i}}\prod_{i=k+1}^{n}\begin{bmatrix} l_i-1 \\ l_i-p_i \end{bmatrix}\prod_{i=k+1}^{n}\gamma_i^{p_i}\right\}\gamma_j^{l-p}$$

We first show that $\sum_{l_{k+1}+\cdots+l_n=l}\prod_{i=k+1}^{n}\begin{bmatrix} l_i-1 \\ l_i-p_i \end{bmatrix}$ depends only on p but does not depend on the partition $(p_{k+1}, \cdots, p_n)$ of p and moreover it is equal to

$$e_{n,l-p,p}=\sum_{r=0}^{l-p}(-1)^r\binom{n-k}{r}\binom{l-1}{l-p-r}. \tag{6.7}$$

For that we set up a relation of the function $\begin{bmatrix} a \\ b \end{bmatrix}$. Comparing the coefficient of x^t in the expansion of the equality

$$\prod_{i=1}^{q}(1+x)^{-s_i}=(1+x)^{-s}, \quad (s=s_1+\cdots+s_q),$$

we have $\sum_{t_1+\cdots+t_q=t}\prod_{i=1}^{q}\binom{s_i+t_i-1}{t_i}=\binom{s+t-1}{t}$.

From this we easily see that

$$\sum_{t_1+\cdots+t_q=t}\prod_{i=1}^{q}\begin{bmatrix}s_i+t_i-1\\ t_i\end{bmatrix}=\sum_{r=0}^{q}(-1)^r\binom{q}{r}\binom{s+t-1}{t-r}. \tag{6.9}$$

In fact

$$\begin{aligned}&\sum_{t_1+\cdots+t_q=t}\prod_{i=1}\left\{\binom{s_i+t_i-1}{t_i}-\binom{s_i+t_i-1}{t_i-1}\right\}\\ &=\sum_{t_1+\cdots+t_q=t}\sum_{I\supset J}(-1)^r\prod_{i=1}^{q}\binom{s_i'+t_i'-1}{t_i'}\end{aligned}$$

where J runs through all of the subsets of $I=\{1,2,\cdots,q\}$ and r is the number of elements in J. Moreover

$$\begin{aligned}&s_i'=s_i+1 \quad\text{and}\quad t_i'=t_i-1 \quad\text{if}\quad i\in J\\ &s_i'=s_i \quad\text{and}\quad t_i'=t_i \quad\text{if}\quad i\notin J.\end{aligned}$$

Hence the above equation is continued as

$$\begin{aligned}&=\sum_{I\supset J}(-1)^r\sum_{t_1'+\cdots+t_q'=t-r}\prod_{i=1}\binom{s_i'+t_i'-1}{t_i'}\\ &=\sum_{I\supset J}(-1)^r\binom{s+t-1}{t-r}=\sum_{r=0}^{q}(-1)^r\binom{q}{r}\binom{s+t-1}{t-r}.\end{aligned}$$

Replace l_i for s_i+t_i and l_i-p_i for t_i in (6.9). Since $p_{k+1}+\cdots+p_n=p$ is constant, the condition $t_1+\cdots+t_q=t$ is replaced by $l_{k+1}+\cdots+l_n=l$ and hence we have

$$\sum_{l_{k+1}+\cdots+l_n=l}\prod_{i=k+1}^{q}\begin{bmatrix}l_i-1\\ l_i-p_i\end{bmatrix}=\sum_{r=0}^{n-k}(-1)^r\binom{n-k}{r}\binom{l-1}{l-p-r} \tag{6.10}$$

as required.

Next we show that in $K(F(n))$ it holds that

$$\pi^*c_p=(-1)^p\sum_{p_{k+1}+\cdots+p_n=p}\prod_{i=k+1}^{n}\gamma_i^{p_i}. \tag{6.11}$$

In fact,

$$\prod_{1\leqq j\leqq k}(1+\gamma_j)\prod_{k+1\leqq i\leqq n}(1+\gamma_i)=\prod_{1\leqq i\leqq n}(1+\gamma_i)=1$$

implies

$$\pi^*(\sum_p c_p)=\prod_{1\leqq j\leqq k}(1+\gamma_j)=\prod_{k+1\leqq i\leqq n}(1+\gamma_i)^{-1}$$

$$=\prod_{k+1\leqq i\leqq n}\sum_{p_i=0}^{\infty}(-\gamma_i)^{p_i}=\sum_{p=0}^{\infty}(-1)^p\sum_{p_{k+1}+\cdots+p_n=p}\prod_{i=k+1}^{n}\gamma_i^{p_i}$$

Hence we have (6.11) and Lemma 6.2 is proved.

For the calculations in Lemma 6.4, we restrict ourselves to the case $k=2$ and determine the values of some $e_{n,l-p,p}$'s more explicitly. We put

$$e_{ij}=(-1)^j e_{n,n-i,j}\,. \tag{6.12}$$

Lemma 6.3.

(1) *When n is even, putting $n-2=2u$, we have*

$$e_{ij}=\sum_{2r+s=2u+2-i}(-1)^{r+j}\binom{2u}{r}\binom{j+1-i}{s} \qquad \text{if } j+1\geqq i\,.$$

$$e_{ij}=\sum_{2r+s=2u+2-i}(-1)^{r+s+j}\binom{2u-i+j+1}{r}\binom{i-j-1}{s} \qquad \text{if } j+1\leqq i\,.$$

(2) *When n is odd, putting $n-2=2u+1$, we have*

$$e_{ij}=\sum_{2r+s=2u+3-i}(-1)^{r+j}\binom{2u+1}{r}\binom{j+1-i}{s} \qquad \text{if } j+1\geqq i\,.$$

$$e_{ij}=\sum_{2r+s=2u+3-i}(-1)^{r+s+j}\binom{2u-i+j+2}{r}\binom{i-j-1}{s} \qquad \text{if } j+1\leqq i\,.$$

Proof. Comparing the coefficients of x^m in the expansion of

$$(1-x)^k(1+x)^l=\begin{cases}(1-x^2)^k(1+x)^{l-k} & \text{if } l\geqq k\\ (1-x^2)^l(1-x)^{k-l} & \text{if } l\leqq k\end{cases}$$

we have

$$\sum_{r=0}^{m}(-1)^r\binom{k}{r}\binom{l}{m-r}=\begin{cases}\sum_{2r+s=m}(-1)^r\binom{k}{r}\binom{l-k}{s} & \text{if } l\geqq k\\ \sum_{2r+s=m}(-1)^{s+r}\binom{l}{r}\binom{k-l}{s} & \text{if } l\leqq k\end{cases}$$

Applying this to Lemma 6.2 (b) with $k=2$, we have Lemma 6.3. q.e.d.

We give the list of some e_{ij} ($1\leqq i\leqq 5$, $0\leqq i\leqq 2$) which we will use in Lemma 6.4.

(1) When n is even, putting $n-2=2u$, we have

$$e_{10}=0 \qquad e_{11}=(-1)^{u+1}\binom{2u}{u} \qquad e_{12}=(-1)^u 2\binom{2u}{u}$$

$$e_{20}=(-1)^u\binom{2u-1}{u} \qquad e_{21}=(-1)^{u+1}\binom{2u}{u} \qquad e_{22}=(-1)^u\binom{2u}{u}$$

$$e_{30}=(-1)^u 2\binom{2u-2}{u-1} \qquad e_{31}=(-1)^{u+1}\binom{2u-1}{u-1} \qquad e_{32}=0$$

$$e_{40}=(-1)^{u-1}\binom{2u-3}{u-1}+(-1)^u 3\binom{2u-3}{u-2}$$

$$e_{41}=(-1)^u\binom{2u-2}{u-1}+(-1)^{u+1}\binom{2u-2}{u-2} \qquad e_{42}=(-1)^{u-1}\binom{2u-1}{u-1}$$

$$e_{50}=(-1)^{u-1}4\binom{2u-4}{u-2}+(-1)^u4\binom{2u-4}{u-3}$$

$$e_{51}=(-1)^u3\binom{2u-3}{u-2}+(-1)^{u+1}\binom{2u-3}{u-3} \qquad e_{52}=(-1)^{u-1}2\binom{2u-2}{u-2}$$

(2) When n is odd, putting $n-2=2u+1$, we have

$$e_{10}=(-1)^{u+1}\binom{2u+1}{u} \qquad e_{11}=(-1)^u\binom{2u-1}{u} \qquad e_{12}=0$$

$$e_{20}=(-1)^{u+1}\binom{2u}{u} \qquad e_{21}=0 \qquad e_{22}=(-1)^u\binom{2u+1}{u}$$

$$e_{30}=0 \qquad e_{31}=(-1)^{u+1}\binom{2u}{u} \qquad e_{32}=(-1)^u\binom{2u+1}{u}$$

$$e_{40}=(-1)^u3\binom{2u-2}{u-1}+(-1)^{u+1}\binom{2u-2}{u-2}$$

$$e_{41}=(-1)^{u-1}2\binom{2u-1}{u-1} \qquad e_{42}=(-1)^u\binom{2u}{u-1}$$

$$e_{50}=(-1)^{u-1}\left\{\binom{2u-3}{u-1}-6\binom{2u-3}{u-2}+\binom{2u-3}{u-3}\right\}$$

$$e_{51}=(-1)^u\binom{2u-2}{u-1}+(-1)^{u+1}3\binom{2u-2}{u-2}$$

$$e_{52}=(-1)^{u-1}\binom{2u-1}{u-1}+(-1)^u\binom{2u-1}{u-2}$$

Lemma 6.4. *In $K(G_{2,n-2})$, the coefficient* a of c_2^{n-2} *in* $2^m\bar{\gamma}_{1/2}(G_{2,n-2})$ *is*

$$a=\begin{cases}0 & n\text{: even}\\ -\dfrac{2(2u+3)}{(2u-1)(u+1)}\dbinom{2u}{u}^2 & n=2u+3\end{cases} \tag{6.13}$$

Proof. Combining (6.5), (a) of Lemma 6.2 and (6.12), we have

$$2^m\bar{\gamma}_{1/2}(G_{2,n-2})=2^m\left\{\sum_{i_1=1}^{n}\sum_{j_1=0}^{2}\left(-\frac{1}{2}\right)^{n+j_1-i_1}e_{i_1j_1}c_{j_1}\gamma_1^{n-i_1}\right\}$$
$$\times\left\{\sum_{i_2=1}^{n}\sum_{j_2=0}^{2}\left(-\frac{1}{2}\right)^{n+j_2-i_2}e_{i_2j_2}c_{j_2}\gamma_2^{n-i_2}\right\}$$

The term of degree $m=2(n-2)$ in this equation is

$$\sum_{i_1+i_2=j_1+j_2+4}e_{i_1j_1}e_{i_2j_2}c_{j_1}c_{j_2}\gamma_1^{n-i_1}\gamma_2^{n-i_2} \tag{6.14}$$

and as $j_1, j_2\leqq 2$, it must hold that $4\leqq i_1+i_2\leqq 8$. So we can list up all terms which appear in (6.14) as follows:

$e_{i_1j_1}e_{i_2j_2}$	$c_{j_1}c_{j_2}\gamma_1^{n-i_1}\gamma_2^{n-i_2}$
$e_{20}e_{20}$	$\gamma_1^{n-2}\gamma_2^{n-2}=c_2^{n-2}$
$e_{10}e_{30}$	$(\gamma_1^2+\gamma_2^2)\gamma_1^{n-3}\gamma_2^{n-3}=-c_2^{n-2}$
$e_{20}e_{31}+e_{21}e_{30}$	$c_1(\gamma_1+\gamma_2)\gamma_1^{n-3}\gamma_2^{n-3}=c_2^{n-2}$
$e_{30}e_{32}+e_{32}e_{30}$	$c_2\gamma_1^{n-3}\gamma_2^{n-3}=c_2^{n-2}$
$e_{31}e_{31}$	$c_1^2\gamma_1^{n-3}\gamma_2^{n-3}=c_2^{n-2}$
$e_{10}e_{41}+e_{11}e_{40}$	$c_1(\gamma_1^3+\gamma_2^3)\gamma_1^{n-4}\gamma_2^{n-4}=-c_2^{n-2}$
$e_{20}e_{42}+e_{22}e_{40}$	$c_2(\gamma_1^2+\gamma_2^2)\gamma_1^{n-4}\gamma_2^{n-4}=-c_2^{n-2}$
$e_{21}e_{41}$	$c_1^2(\gamma_1^2+\gamma_2^2)\gamma_1^{n-4}\gamma_2^{n-4}=0$
$e_{31}e_{42}+e_{32}e_{41}$	$c_1c_2(\gamma_1+\gamma_2)\gamma_1^{n-4}\gamma_2^{n-4}=c_2^{n-2}$
$e_{42}e_{42}$	$c_2^2\gamma_1^{n-4}\gamma_2^{n-4}=c_2^{n-2}$
$e_{10}e_{52}+e_{12}e_{50}$	$c_2(\gamma_1^4+\gamma_2^4)\gamma_1^{n-5}\gamma_2^{n-5}=0$
$e_{11}e_{51}$	$c_1^2(\gamma_1^4+\gamma_2^4)\gamma_1^{n-5}\gamma_2^{n-5}=-c_2^{n-2}$
$e_{21}e_{52}+e_{22}e_{51}$	$c_1c_2(\gamma_1^3+\gamma_2^3)\gamma_1^{n-5}\gamma_2^{n-5}=-c_2^{n-2}$
$e_{32}e_{52}$	$c_2^2(\gamma_1^2+\gamma_2^2)\gamma_1^{n-5}\gamma_2^{n-5}=-c_2^{n-2}$
$e_{11}e_{62}+\ _{12}e_{61}$	$c_1c_2(\gamma_1^5+\gamma_2^5)\gamma_1^{n-6}\gamma_2^{n-6}=0$
$e_{22}e_{62}$	$c_2^2(\gamma_1^4+\gamma_2^4)\gamma_1^{n-6}\gamma_2^{n-6}=0$
$e_{12}e_{72}$	$c_2^2(\gamma_1^6+\gamma_2^6)\gamma_1^{n-7}\gamma_2^{n-7}=0$

Note that the relations on the right hand side is obtained from Proposition 5.3. Therefore the coefficient a of c_2^{n-2} in (6.2) is obtained as follows:

$$\begin{aligned}a=&\ e_{20}e_{20}-e_{10}e_{30}+e_{20}e_{31}+e_{21}e_{30}+e_{30}e_{32}+e_{32}e_{30}\\&+e_{31}e_{31}-e_{10}e_{41}-e_{11}e_{40}-e_{20}e_{42}-e_{22}e_{40}+e_{31}e_{42}\\&+e_{41}e_{32}+e_{42}e_{42}-e_{11}e_{51}-e_{21}e_{52}-e_{22}e_{51}-e_{32}e_{52}\end{aligned}$$

Applying the list given bellow Lemma 6.3 to this equation, we have (6.13).

q.e.d.

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