Goto, S.
Osaka J. Math.
15 (1978), 47–50

INVARIANT SUBRINGS UNDER THE ACTION BY A FINITE GROUP GENERATED BY PSEUDO-REFLECTIONS

SHIRO GOTO

(Received October 20, 1976)

1. Introduction

In this note, let R be a regular local ring with maximal ideal $\mathfrak{m}$ and let k be the residue field of R. Assume that R contains k as a subfield. Let G be a finite subgroup of $\mathrm{Aut}_k R$. Assume that $(|G|, \mathrm{ch}\, k)=1$ if k has positive characteristic. Further we assume that G is generated by pseudo-reflections relative to the induced action on the Zariski tangent space $\mathfrak{m}/\mathfrak{m}^2$ of R. (Hence R^G is again a regular local ring and R is a finitely generated R^G-module (c.f. [3]).) For an arbitrary Macaulay local ring B with maximal ideal $\mathfrak{n}$, we put $\mathrm{r}(B)= \dim_{B/\mathfrak{n}} \mathrm{Ext}^d_B(B/\mathfrak{n}, B)$ ($d=\dim B$) and call it the type of B. Recall that B is a Gorenstein local ring if and only if $\mathrm{r}(B)=1$. The aim of this paper is to prove the following

Theorem. *Let $\mathfrak{a}$ be an ideal of R and assume that $\mathfrak{a}$ is stable under the G-action on R. We denote $R/\mathfrak{a}$ by A. Then we have*
(1) *If A is a Macaulay local ring, then the ring A^G of invariants is again a Macaulay local ring and the inequality $\mathrm{r}(A^G)\leqq \mathrm{r}(A)$ holds.*
(2) *If A is a complete intersection, then the ring A^G of invariants is again a complete intersection.*

It is known that A^G is a Macaulay local ring if A is a Macaulay local ring (c.f. Proposition 13, [2]).

As a consequence of this theorem we have

Corollary (c.f. Watanabe, [4]). *If A is a Gorenstein local ring, then the ring A^G of invariants is again a Gorenstein local ring.*

2. Proof of the theorem

An R-module M is called an (R, G)-module if the group G acts on the additive group of the module M so that the identity $s(ax)=s(a)s(x)$ holds for every $s\in G$, $a\in R$, and $x\in M$. An R-homomorphism of (R, G)-modules is called a homomorphism of (R, G)-modules if it is compatible with G-action. For an (R, G)-module M, M^G is an R^G-module and it is contained in the R^G-module M as a direct summand. The projection ρ_M: $M \to M^G$ is given by $\rho_M(x) = 1/g \cdot \sum_{s \in G} s(x)$ which is called the Reynolds operator for M, where $g = |G|$. Note that $[\quad]^G$ is an exact functor.

Let N be a finitely generated R-module and let $i \geqq 0$ be an integer. We put $\beta_i^R(N) = \dim_k \operatorname{Tor}_i^R(k, N)$ and call it the *i-th* Betti number of N. Recall that, if the sequence $\cdots \to F_i \to \cdots \to F_1 \to F_0 \to N \to 0$ is a minimal free resolution of N, then the number $\beta_i^R(N)$ is equal to the rank of the R-module F_i.

First we give the following lemma.

Lemma 1. *Let M be an (R, G)-module and assume that M is finitely generated as an R-module. Then there exists an exact sequence*

$$\cdots \to F_i \to \cdots \to F_1 \to F_0 \to M \to 0$$

of (R, G)-modules such that each (R, G)-module F_i is a finitely generated free R-module with $\operatorname{rank}_R F_i = \beta_i^R(M)$.

Proof. We put $r = \beta_0^R(M)$ $(= \dim_k M/\mathfrak{m}M)$. Notice that the sequence $0 \to \mathfrak{m}M \to M \xrightarrow{\varepsilon} M/\mathfrak{m}M \to 0$ of (R, G)-modules can be regarded as an exact sequence of G-spaces over the field k. Then, by virtue of Maschke's theorem, we can find an r-dimensional G-subspace V of M so that $\varepsilon(V) = M/\mathfrak{m}M$. (Recall that $(|G|, \operatorname{ch} k) = 1$ if k has positive characteristic. This is one of our standard assumptions.) Let $\{e_i\}_{1 \leqq i \leqq r}$ be a k-basis of V and let $[a_{ij}(s)]$ denote the matrix representation of an element s of G relative to this basis. Let F be a finitely generated free R-module of rank r and let $\{X_i\}_{1 \leqq i \leqq r}$ be an R-free basis of F. We put $s(X_j) = \sum_{i=1}^{r} a_{ij}(s) X_i$ for every $s \in G$ and for every $1 \leqq j \leqq r$, and we define a G-action on F by

$$s\left(\sum_{j=1}^{r} a_j X_j\right) = \sum_{j=1}^{r} s(a_j) s(X_j)$$

where $s \in G$ and $a_j \in R$. Then the R-module F becomes an (R, G)-module under this action. Moreover, if we define an R-linear map f: $F \to M$ by $f(X_j) = e_j$ for every $1 \leqq j \leqq r$, then f is a surjective homomorphism of (R, G)-modules. (Note that $M = \sum_{i=1}^{r} Re_i$ by Nakayama's lemma.) Inductively we can construct an exact sequence of (R, G)-modules mentioned above.

Lemma 2. *Let M be an (R, G)-module and assume that M is finitely generated as an R-module. Then the inequality $\beta_i^R(M) \geq \beta_i^{R^G}(M^G)$ holds for every integer $i \geqq 0$.*

Proof. Let $\cdots \to F_i \to \cdots \to F_1 \to F_0 \to M \to 0$ be an exact sequence of (R, G)-modules obtained for M by Lemma 1. Since R is a finitely generated free R^G-module and since F_i^G is a direct summand of F_i as an R^G-module, we see that F_i^G is a finitely generated free R^G-module for every integer $i \geqq 0$. Therefore, as the sequence $\cdots \to F_i^G \to \cdots \to F_1^G \to F_0^G \to M^G \to 0$ of R^G-modules is exact, to prove this lemma we have only to show that $\mathrm{rank}_R F \geqq \mathrm{rank}_{R^G} F^G$ for every (R, G)-module F which is finitely generated and free as an R-module. We put $r = \mathrm{rank}_{R^G} F^G$ and let $\{x_i\}_{1 \leqq i \leqq r}$ be an R^G-free basis of F^G. We denote by K the quotient field of R and consider $K \otimes_R F$ as a (K, G)-module naturally (*i.e.*, We define $s(c \otimes x) = s(c) \otimes s(x)$ for $s \in G$, $c \in K$, and $x \in F$.).

Now assume that $r > \mathrm{rank}_R F$. Then $\{1 \otimes x_i\}_{1 \leqq i \leqq r}$ is not linearly independent over K and so we may express, without loss of generality, $1 \otimes x_1 = \sum_{i=1}^{r} c_i \otimes x_i$ for some $c_i \in K$. Let s be an element of G. Then, as $s(x_i) = x_i$ for every $1 \leqq i \leqq r$, we have $1 \otimes x_1 = \sum_{i=2}^{r} s(c_i) \otimes x_1$. Thus we see that $1 \otimes x_1 = \sum_{i=2}^{r} (1/g \cdot \sum_{s \in G} s(c_i)) \otimes x_i$ where $g = |G|$. Now we put $b_i = 1/g \cdot \sum_{s \in G} s(c_i)$. Then, since $b_i \in K^G$ and since K^G coincides with the quotient field of R^G, we can find a non-zero element a of R^G so that $ab_i \in R^G$ for every $2 \leqq i \leqq r$. Therefore there is an identity $ax_1 = \sum_{i=2}^{r} a_i x_i$ in F^G where $a_i = ab_i$. But this is impossible, since $\{x_i\}_{1 \leqq i \leqq r}$ is linearly independent over R^G. Thus we conclude that $r \leqq \mathrm{rank}_R F$.

Proof of the theorem.

First consider (1) and suppose that A is a Macaulay local ring. It is known that A^G is a Macaulay local ring (c.f. Proposition 13, [2]). We put $s = \dim R - \dim A^G$. (Note that $s = \dim R^G - \dim A^G$, since $\dim R^G = \dim R$ and $\dim A^G = \dim A$.) Then we have, by Lemma 3.5 of [1], that $\beta_s^R(A) = \mathrm{r}(A)$. Similarly we have $\beta_s^{R^G}(A^G) = \mathrm{r}(A^G)$, since R^G is a regular local ring by the standard assumption. Thus we conclude that $\mathrm{r}(A^G) \leqq \mathrm{r}(A)$ by Lemma 2.

Now consider (2) and suppose that A is a complete intersection. Then $\beta_1^R(A) = s$ and hence $\beta_s^{R^G}(A^G) \leqq s$ by Lemma 2. On the other hand, since R^G is a regular local ring and since $s = \dim R^G - \dim A^G$, we know that $\beta_1^{R^G}(A^G) \geqq s$. Therefore $\beta_1^{R^G}(A^G) = s$ and this implies that A^G is again a complete intersection.

Acknowledgement. The author thanks Dr. S. Mori for his useful suggestions and discussions.

NIHON UNIVERSITY

References

[1] Y. Aoyama and S. Goto: *On the type of graded Cohen-Macaulay rings*, J. Math. Kyoto Univ. **15** (1975), 19–23.

[2] M. Hochster and J.A. Eagon: *Cohen-Macaulay rings, invariant theory, and the generic perfection of determinantal loci*, Amer. J. Math. **93** (1971), 1020–1058.

[3] J.-P. Serre: *Groupes finis d'automorphismes d'anneaux locaux réguliers*, Colloq. d'Alg. E. N. S., (1967).

[4] K. Watanabe: *Invariant subrings of a Gorenstein local ring by a finite group generated by pseudo-reflections*, to appear.