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THE PROJECTIVE DIMENSION OF THE COMPLEX BORDISM OF EILENBERG-MACLANE SPACES

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Let MU_*X be the complex bordism of the space X and let MU_* be the complex bordism coefficient ring [7]. There is a standard conjecture that the projective dimension of $MU_*K(Z/(p),n)$ (hom. dim. ${}_{MU_*}MU_*K(Z/(p),n)$, [4]) should be n. The conjecture was motivated by its truth for $n=0,1$ and by the early establishment of the lower bound hom. dim. ${}_{MU_*}MU_*K(Z/(p),n)\geqslant n$[2, 3, 4]. The purpose of this note is to disprove the conjecture in the strongest possible way. Let p be a prime and let $Z/(p^n)$ denote the integers modulo p^n.

Theorem. *Over MU_*, the projective dimensions of the complex bordism modules $MU_*K(Z, m)$, $m\geqslant 3$, and $MU_*K(Z/(p^n), m)$, $m\geqslant 2$, $n\geqslant 1$, are infinite.*

Richard Kane informs us that he can prove this result directly from Brown-Peterson considerations; we have not seen his work.

We would like to thank Kathleen Sinkinson who, using [6], helped us to make the low dimensional computations of $BP_*K(Z/(p), 2)$ which led us to the first counterexample to the "standard conjecture."

Once the psychological barrier of the conjecture was removed, we realized that we could apply an early lower bound test of Conner and Smith to obtain our theorem. (We follow the convention that all cohomology coefficients are $Z/(p)$.)

Steenrod Operations Test [3]. Suppose $\theta_1, \theta_2, \cdots, \theta_t$ are Steenrod operations in (Q_0), the two-sided ideal generated by the mod p Bockstein. If $\theta_1 \cdots \theta_t$ acts nontrivially on H^*X, then the projective dimension of MU_*X over MU_* is at least t.

Recall the Milnor primitive operations Q_s which satisfy the Milnor relation [5]

$$P^tQ_s = Q_sP^t+Q_{s+1}P^{t-p^s}$$

where Q_0 is the mod p Bockstein and P^t is the t-th reduced power operation.

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When $t=p^s$, this relation defines Q_{s+1} inductively; clearly Q_{s+1} is a member of (Q_0). We shall show that the t-fold composition $Q_{2t}\cdots Q_{t+1}$ acts nontrivially on H^*K where K is one of the Eilenberg-MacLane spaces of the theorem. Since the integer $t>0$ is arbitrary, the Steenrod Operations Test proves the theorem.

Lemma 1. *Let α be a cohomology class of dimension at most 6. Let $Q_0\alpha=0$.*

(i) $Q_{s+1}\alpha=-Q_0P^{p^s}\cdots P^1\alpha,\ s\geqslant 0$;

(ii) $Q_1Q_{s+1}\alpha=Q_0P^{p^s+1}Q_0P^{p^{s-1}}\cdots P^1\alpha,\ s\geqslant 1$.

Proof. If p is odd, the Adem relations imply that $P^{p^s-1}P^{p^{s-1}}=0$. (In the Adem expansion, all the binomial coefficients are of form $\binom{m}{n}$ where $n>m$.) Since $Q_0\alpha=0$, the $s=0$ case of (i) follows from the definition of Q_1. Assuming the $s-1\geqslant 0$ case of (i), we have:

$$
\begin{aligned}
Q_{s+1}\alpha &= P^{p^s}Q_s\alpha-Q_sP^{p^s}\alpha = P^{p^s}(-Q_0P^{p^{s-1}}\cdots P^1\alpha)-0 && \text{(dimension hypothesis)}\\
&= -(Q_0P^{p^s}+Q_1P^{p^s-1})P^{p^{s-1}}\cdots P^1\alpha && \text{(Milnor relations)}\\
&= -Q_0P^{p^s}P^{p^{s-1}}\cdots P^1\alpha-0 && \text{(initial observation)}\\
Q_1Q_{s+1}\alpha &= -(P^1Q_0-Q_0P^1)Q_0P^{p^s}\cdots P^1\alpha &&\\
&= 0+Q_0P^1(P^{p^s}Q_0-Q_1P^{p^s-1})(P^{p^{s-1}}\cdots P^1\alpha) && \text{(Milnor relations)}\\
&= Q_0P^{p^s+1}Q_0P^{p^{s-1}}\cdots P^1\alpha-0 && \text{(Adem relations).}
\end{aligned}
$$

The mod 2 Adem relations imply that $P^{2^s-1}P^{2^{s-1}}=Sq^{2^{s+1}-2}Sq^{2^s}=Sq^{2^{s+1}-1}Sq^{2^s-1}$ and that $Sq^{2^s-1}P^{2^{s-2}}=Sq^{2^s-1}Sq^{2^{s-1}}=0$. So $P^{2^s-1}P^{2^{s-1}}P^{2^{s-2}}=0$ and the odd primary proof is usable when $p=2$, $s>1$. Direct computation establishes the $p=2$, $s=0$ and 1 cases.

Corollary 2. *Let $\iota_m\in H^mK(Z, m)$ be the fundamental class and let $s>1$. The following are polynomial generators of $H^*(K(Z, m)$ except in the cases when $p=2$, $m=3$ or 4; in these two cases, they are squares of polynomial generators.*

(i) $Q_{s-1}\iota_m$, *for* $m=2k-1\geqslant 3$

(ii) $Q_sQ_1\iota_m$, *for* $m=2k\geqslant 4$.

Proof. By using cohomology suspension, it suffices to prove the result for the lowest cases: $m=3$, 4 for p odd and $m=3$, 4, 5, 6 for $p=2$. When p is odd, Lemma 1 shows that $Q_s\iota_3$ and $Q_sQ_1\iota_4=-Q_1Q_s\iota_4$ are admissible monomials of even degree. By Cartan's computation of the cohomology of $K(Z, m)$ [1], they are polynomial generators.

When $p=2$, $m\leqslant 6$, Lemma 1 shows that $Q_s\iota_m=Sq^1Sq^{2^s}\cdots Sq^2\iota_m=Sq^{2^s+1}Sq^{2^{s-1}}\cdots Sq^2\iota_m$ and that $Q_1Q_s\iota_m=Sq^1Sq^{2^s+2}Sq^1Sq^{2^{s-1}}\cdots Sq^2\iota_m=Sq^{2^s+3}Sq^{2^{s-1}+1}Sq^{2^{s-2}}\cdots Sq^2\iota_m$. So $Q_s\iota_5$ and $Q_1Q_s\iota_6$ are admissible monomials while $Q_s\iota_3$ and $Q_1Q_s\iota_4$ are squares of the admissible monomials $Sq^{2^{s-1}}\cdots Sq^2\iota_3$ and $Sq^{2^{s-1}+1}Sq^{2^{s-2}}\cdots Sq^2\iota_4$, respectively.

Lemma 3. *Let α be an m-dimensional cohomology class and let $s \geqslant 1$.*
(i) $Q_s(P^{k-1}\alpha)=(Q_{s-1}\alpha)^p$, *for* $m=2k-1$;
(ii) $Q_s(P^{k+p-1}Q_1\alpha)=(Q_{s-1}Q_1\alpha)^p$, *for* $m=2k$.

Proof. By the Milnor relations, $Q_sP^{k-1}\alpha=P^{k-1+p^{s-1}}Q_{s-1}\alpha-Q_{s-1}P^{k-1+p^{s-1}}\alpha$. Part (i) follows from the observation that the dimension of $Q_{s-1}\alpha$ is $2(p^{s-1}+k-1)$. Part (ii) follows from an application of (i) to $Q_1\alpha$.

Corollary 4. *For $m \geqslant 3$ and $t \geqslant 1$, define cohomology classes $\gamma(t) \in H^*K(Z,m)$ by*
(i) $\gamma(t)=P^{p^{t-1}k-p^{t-1}}\cdots P^{p^k-p}P^{k-1}\iota_m$, *for* $m=2k-1$;
(ii) $\gamma(t)=P^{kp^{t-1}+p^t-p^{t-1}}\cdots P^{k+p-1}Q_1\iota_m$, *for* $m=2k$.
For $s>t$, we have:
(i) $Q_s\gamma(t)=(Q_{s-t}\iota_m)^{p^t}$, *for* $m=2k-1$;
(ii) $Q_s\gamma(t)=(Q_{s-t}Q_1\iota_m)^{p^t}$, *for* $m=2k$.

Proof. By Lemma 3, $Q_s\gamma(t)=(Q_{s-1}\gamma(t-1))^p$. Iterate this.

Proposition 5. *For $m \geqslant 3$, let $\alpha \in H^*K(Z, m)$ be defined by*:
(i) $\alpha=\iota_m$, *for* $m=2k-1$;
(ii) $\alpha=Q_1\iota_m$, *for* $m=2k$.
Let I be the ideal of $H^*K(Z, m)$ given by $I=((Q_1\alpha)^p, \cdots, (Q_{t-1}\alpha)^p)$. Then $Q_{2t}\cdots Q_{t+1}(\gamma(1)\cdots\gamma(t)) \equiv (Q_t\alpha)^{p+\cdots+p^t}$ modulo I.

Proof. Recall that $Q_s(xy)=(Q_sx)y+(-1)^n x(Q_sy)$ where n is the dimension of x. Iterate Corollary 4.

Proof of the Theorem. Let f: $K(Z/(p^n),m) \to K(Z,m+1)$ be a map realizing the n-th order Bockstein β_n in that $f^*\iota_{m+1}=\beta_n\iota_m$. The induced map f^* is an injection. By Corollary 2, powers of $Q_t\alpha$ are nonzero modulo I if $t>1$. So Proposition 5 shows that the t-fold compositions $Q_{2t}\cdots Q_{t+1}$ are nonzero on the classes $\gamma(1)\cdots\gamma(t) \in H^*K(Z, m+1)$ and $f^*(\gamma(1)\cdots\gamma(t)) \in H^*K(Z/(p^n), m)$, $m \geqq 2$. Our theorem then follows from the Steenrod Operations Test.

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