Ikeda, A.
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FORMALLY SELF ADJOINTNESS FOR THE DIRAC OPERATOR ON HOMOGENEOUS SPACES

AKIRA IKEDA

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Introduction. In [5], Wolf proved that the Dirac operator is essentially self adojoint over a Riemannian spin manifold M and he used it to give explicit realization of unitary representations of Lie groups.

Let K be a Lie group and α a Lie group homomorphism of K into $SO(n)$ which factors through Spin (n). He defined the Dirac operator on spinors with values in a certain vector bundle under the assumption that the Riemannian connection on the oriented orthonormal frame bundle P over M can be reduced to some principal K-bundle over M by the homomorphism α.

The purpose of this paper is to give the Dirac operator on a homogeneous space in a more general situation using an invariant connection, and to determine connections that define the formally self adjoint Dirac operator.

Let G be a unimodular Lie group and K a compact subgroup of G. We assume G/K has an invariant spin structure. First, we define the Dirac operator D on spinors using an invariant connection on the oriented orthonormal frame bundle P over G/K. Next, we introduce an invariant connection $\nabla^{\mathcal{V}}$ to a homogeneous vector bundle $\mathcal{V}$ associated to a unitary representation of K, then we define the Dirac operator $D \underset{\nabla^{\mathcal{V}}}{\hat{\otimes}} 1$ on spinors with values in $\mathcal{V}$ according to [4]. As for a metric on spinors, we use a Lemma given by Parthasarathy in [3]. Using this metric and an invariant measure on G/K, we define a hermitian inner product on the space of spinors with values in $\mathcal{V}$. Then we determine connections that define the formally self adjoint Dirac operator with respect to this inner product. In some cases (cf. Remarks in **4**), $D \underset{\nabla^{\mathcal{V}}}{\hat{\otimes}} 1$ is always formally self adjoint if an invariant connection on $\mathcal{V}$ is a metic connection. Moreover, in the same way as Wolf [3], we see that if $D \underset{\nabla^{\mathcal{V}}}{\hat{\otimes}} 1$ is formally self adjoint, then $D \underset{\nabla^{\mathcal{V}}}{\hat{\otimes}} 1$ and $(D \underset{\nabla^{\mathcal{V}}}{\hat{\otimes}} 1)^2$ are essentially self adjoint.

1. Spin construction

Let $\mathfrak{m}$ be an n-dimensional oriented real vector space with an inner product

$\langle\,,\rangle$. We define the Clifford algebra Cliff$(\mathfrak{m})$ over $\mathfrak{m}$ by $T(\mathfrak{m})/I$, where $T(\mathfrak{m})$ is the tensor algebra over $\mathfrak{m}$ and I is the ideal generated by all elements $v\otimes v+\langle v, v\rangle 1$, $v\in\mathfrak{m}$. The multiplication of Cliff$(\mathfrak{m})$ will be denoted by $x\cdot y$. Let p: $T(\mathfrak{m})\rightarrow$ Cliff$(\mathfrak{m})$ denote the canonical projection. Then Cliff$(\mathfrak{m})$ is decomposed into the direct sum Cliff$^+(\mathfrak{m})\oplus$Cliff$^-(\mathfrak{m})$ of the p-images of elements of even and odd degree of $T(\mathfrak{m})$, and $\mathfrak{m}$ is identified with a subspace of Cliff$(\mathfrak{m})$ through the projection p. Let $\{e_1, e_2, \cdots, e_n\}$ be an oriented orthonormal base of $\mathfrak{m}$. The map, $e_{i_1}\cdot e_{i_2}\cdot\cdots\cdot e_{i_p}\mapsto(-1)^p e_{i_p}\cdot\cdots e_{i_2}\cdot e_{i_1}$ defines a linear map of Cliff$(\mathfrak{m})$ and the image of $x\in$Cliff$(\mathfrak{m})$ by this linear map is denoted by $\bar{x}$. The Spin group is defined by

$$\mathrm{Spin}(\mathfrak{m}) = \{x\in \mathrm{Cliff}^+(\mathfrak{m})\colon x \text{ is invertible, } x\cdot\mathfrak{m}\cdot x^{-1}\subset\mathfrak{m} \text{ and } x\cdot\bar{x}=1\}$$

Spin$(\mathfrak{m})$ is a two fold covering group of $SO(\mathfrak{m})$ through the following map π: Spin$(\mathfrak{m})\rightarrow SO(\mathfrak{m})$ defined by $\pi(x)v=x\cdot v\cdot x^{-1}$ for $x\in$Spin$(\mathfrak{m})$ and $v\in\mathfrak{m}$. When $n\geqq 3$, Spin$(\mathfrak{m})$ is the universal covering group of $SO(\mathfrak{m})$. Moreover, the subspace $\mathfrak{spin}(\mathfrak{m})$ of Cliff$(\mathfrak{m})$ spanned by $\{e_i\cdot e_j\}_{i<j}$ becomes a Lie algebra by the bracket operation $[x, y]=x\cdot y-y\cdot x$. This is identified with the Lie algebra of Spin$(\mathfrak{m})$ in such a way that exp: $\mathfrak{spin}(\mathfrak{m})\rightarrow$Spin$(\mathfrak{m})$ is nothing but the restriction of the exponential map of the algebra Cliff$(\mathfrak{m})$ into Cliff$(\mathfrak{m})$. The differential $\dot{\pi}$ of π is given by

$$\dot{\pi}(x)v = x\cdot v - v\cdot x \quad \text{for } x\in\mathfrak{spin}(\mathfrak{m}) \text{ and } v\in\mathfrak{m}\,. \tag{1.1}$$

Now, put $a_i=\sqrt{-1}\,e_{2i-1}\cdot e_{2i}$, $1\leqq i\leqq\left[\frac{n}{2}\right]$, then $a_i^2=1$ and $a_i\cdot a_j=a_j\cdot a_i$. We consider the right multiplication by a_i's on Cliff$(\mathfrak{m})\otimes C$. For a multi-index $q=(q_1, q_2, \cdots, q_{[\frac{n}{2}]})$, where $q_i=1$ or -1, we put

$$L_q^{\pm}=\left\{x\in\mathrm{Cliff}^{\pm}(\mathfrak{m})\otimes C\colon x\cdot a_i = q_i x,\ 1\leqq i\leqq\left[\frac{n}{2}\right]\right\}.$$

These spaces give irreducible representations of Spin$(\mathfrak{m})$ by the left multiplication. When n is odd, these representations are equivalent each other. Any one of these representations is called the spin representation. Choosing a multi-index q, we put $L=L_q^+$ and denote by s the representation of Spin$(\mathfrak{m})$ on L. When n is even, just two inequivalent irreducible representations appear, according to the sign of $\pm\prod q_i$. Each of these representations is called the positive or negative spin representation according to the sign of $\pm\prod q_i$. Choosing a multi-index q with $\prod q_i=1$, we put $L^+=L_q^+$, $L^-=L_q^-$, $L=L^++L^-$ and denote by s^+, s^- and s the representations of Spin$(\mathfrak{m})$ on L^+, L^- and L respectively. We identify each element of $\mathfrak{m}$ with an element of Cliff$(\mathfrak{m})\otimes C$ by the natural inclusion. Then if n is even, by the left Clifford multiplication the following symbol maps are induced;

$$\left\{\begin{array}{l}\varepsilon^{\pm}: \mathfrak{m}\otimes L^{\pm} \to L^{\mp} \\ \varepsilon\ : \mathfrak{m}\otimes L \ \to L\,.\end{array}\right. \tag{1.2}$$

If n is odd, by the left multiplication, we have the following map;

$$\varepsilon': \mathfrak{m}\otimes L_q^+ \to L_q^-.$$

Identifying L_q^+ with L_q^- through the spin module isomorphism induced by right multiplication of e_n, we also have the symbol map;

$$\varepsilon: \mathfrak{m}\otimes L \to L\,. \tag{1.2)$'$}$$

The definition yield the properties (i), (ii) in the following lemma.

Lemma 1. *We have*
(i) *The symbol maps* ε *commute with the action of* Spin$(\mathfrak{m})$, *i.e., it holds* $\varepsilon(\pi(x)v\otimes x\cdot l)=x\cdot\varepsilon(v\otimes l)$ *for* $x\in$ Spin$(\mathfrak{m})$, $v\in\mathfrak{m}$, $l\in L$.
(ii) *If* $\varepsilon(v\otimes l)=0$ (resp. $\varepsilon^{\pm}(v\otimes l)=0$) *for some* $v\in\mathfrak{m}$ *and* $l\in L$ (resp. *for some* $v\in\mathfrak{m}$ *and* $l\in L^{\pm}$), *then* $v=0$ *or* $l=0$.
(iii) (*Lemma* 5.1, §5 *in* [3]). *There exist a hermitian inner product* $\langle\ \rangle$, *on* L *satisfying*

$$\langle\varepsilon(v\otimes l), l'\rangle+\langle l, \varepsilon(v\otimes l')\rangle = 0 \qquad \textit{for } v\in\mathfrak{m}$$

and $l, l'\in L$.

REMARK. We give explicitly a base of $L_q^{\pm}$ and an inner product on L satisfying the above condition. When $n=2m$ (resp. $n=2m+1$), let $e_1, e_1', \cdots, e_m, e_m'$ (resp. $e_1, e_1', \cdots, e_m, e_m', e_n$) be an oriented orthonormal base of $\mathfrak{m}$ Put $f_i=\dfrac{e_i-\sqrt{-1}\,e_i'}{2}$, $f_i'=-\dfrac{e_i+\sqrt{-1}\,e_i'}{2}$ and $a_i=-\sqrt{-1}\,e_i\cdot e_i'$, then we have

$$\begin{aligned}
&f_i'\cdot a_i = -f_i' \\
&f_i\cdot a_i = f_i \\
&f_i'\cdot a_j = a_j\cdot f_i' \qquad \text{if } \ i\neq j \\
&f_i\cdot a_j = a_j\cdot f_i \qquad \text{if } \ i\neq j \\
&f_i\cdot f_i = f_i'\cdot f_i' = 0 \\
&f_i\cdot f_i'+f_i'\cdot f_i = 1 \\
&f_i\cdot f_j'+f_j'\cdot f_i = 0 \qquad \text{if } \ i\neq j\,.
\end{aligned}$$

For a multi-index $q=(q_1, q_2, \cdots, q_m)$ we define

$$g_i = \begin{cases} f_i & \text{if } q_i = 1 \\ f'_i & \text{if } q_i = -1, \end{cases}$$

$$h_i = \begin{cases} f_i & \text{if } q_i = -1 \\ f'_i & \text{if } q_i = 1 \end{cases}$$

and put

$$\tau_q = h_1 \cdot h_2 \cdots \cdot h_m .$$

Then

$$\{g_{i_1} \cdot \cdots \cdot g_{i_p} \cdot \tau_q \in \mathrm{Cliff}^{\pm}(\mathfrak{m}) \otimes C : 1 \leqslant i_1 < \cdots < i_p \leqslant m$$

(resp.

$$\{g_{i_1} \cdot \cdots \cdot g_{i_p} \cdot \tau_q, g_{i_1}, \cdot \cdots \cdot g_{i_p} \cdot \tau_q \cdot e_n \in \mathrm{Cliff}^{\pm}(\mathfrak{m}) \otimes C : 1 \leqslant i_1 < \cdots < i_p \leqslant m\})$$

is a base of $L_q^{\pm}$. The inner product which makes the above base into an orthonormal base satisfies the condition of Lemma 1-(iii).

2. Invariant connections on homogeneous spaces

Let G be a Lie group and K a closed subgroup of G. We denote by $\mathfrak{g}$, $\mathfrak{k}$ their Lie algebras. We assume that the pair (G, K) is reductive, i.e., there exists a subspace $\mathfrak{m}$ of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$ (direct sum) and $Ad(K)\mathfrak{m} \subset \mathfrak{m}$. We fix such decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$ and identify $\mathfrak{m}$ with the tangent space at the origin o of G/K. Let

$$\rho : K \to GL(V)$$

be a real or complex representation of K. $1 \in GL(V)$ denotes the identity automorphism of V. The differential of ρ will be denoted by

$$\dot{\rho} : \mathfrak{k} \to \mathfrak{gl}(V) .$$

Now, we consider G-invariant connections on the principal $GL(V)$-bundle $P = G \times_{\rho} GL(V)$ over G/K, which is the quotient space of $G \times GL(V)$ under the equivalence relation $(g, h) \sim (gk, \rho(k)^{-1}h)$ for $g \in G$, $k \in K$ and $h \in GL(V)$. The equivalence class in P containing $(g, h) \in G \times GL(V)$ will be denoted by $\{g, h\}$. G acts on P as bundle automorphisms by the left translation

$$L_x : \{g, h\} \to \{xg, h\} \qquad x \in G .$$

Proposition 1. *There exists a one to one correspondence between the set of G-invariant connections in $P = G \times_{\rho} GL(V)$ and the set of R-linear mappings $M_{\mathfrak{m}} : \mathfrak{m} \to \mathfrak{gl}(V)$ such that*

$$(2.1) \qquad M_{\mathfrak{m}}(\alpha(k)X) = \rho(k) M_{\mathfrak{m}}(X) \rho(k)^{-1} \qquad \text{for } X \in \mathfrak{m} ,$$

and $k\in K$.
The connection form ω of the G-invariant connection in P corresponding to $M_{\mathfrak{m}}$ is given by

$$(2.2)\qquad \begin{cases} M_{\mathfrak{m}}(X)=\omega_{u_0}(\tilde{X}) & \text{for } X\in\mathfrak{m}\,, \\ \dot{\rho}(X)=\omega_{u_0}(\tilde{X}) & \text{for } X\in\mathfrak{k}\,, \end{cases}$$

where u_0 is the origin $\{e, 1\}$ of $G\times_{\rho} GL(V)=P$ and $\tilde{X}$ is a vector field on P generated by $L_{\exp tx}$.

Proof. See [1].

For a linear map $M_{\mathfrak{m}}$ satisfying the condition (2.1), the corresponding G-invariant connection will be called the connection induced by $M_{\mathfrak{m}}$.

Let $\mathcal{V}=G\times_{\rho} V$ be the vector bundle over G/K associated to (ρ, V), which is the quotient space of $G\times V$ under the equivalence relation $(g, v)\sim(gk, \rho^{-1}(k)v)$ for $g\in G$, $k\in K$ and $v\in V$. We denote by $C^{\infty}(\mathcal{V})$ the space of all C^{∞}-sections to the bundle $\mathcal{V}$. Then $C^{\infty}(\mathcal{V})$ is identified as follows with the space $C_0^{\infty}(G, V)$ of all C^{∞}-functions $\tilde{\phi}\colon G\to V$ which satisfy $\tilde{\phi}(gk)=\rho(k)^{-1}\tilde{\phi}(g)$ for all $g\in G$ and $k\in K$; Let p be the natural projection of G onto G/K and q the projection of $G\times V$ onto $\mathcal{V}$, then the identification $C^{\infty}(\mathcal{V})\ni\phi\mapsto\tilde{\phi}\in C_0^{\infty}(G, V)$ is given by

$$q(g, \tilde{\phi}(g))=\tilde{\phi}(p(g)) \qquad \text{for } g\in G\,.$$

The principal bundle $P=G\times_{\rho} GL(V)$ is identified with the bundle of frames of $\mathcal{V}$ in a natural way, and $\mathcal{V}$ is identified with the vector bundle $P\times_{GL(V)} V$ associated to P by the natural action of $GL(V)$ on V. Thus, for a linear map $M_{\mathfrak{m}}$ satisfying (2.1), the connection in P induced by $M_{\mathfrak{m}}$ defines the covariant derivative

$$\nabla^{\mathcal{V}}\colon C^{\infty}(\mathcal{V})\to C^{\infty}(\mathcal{I}^*\otimes\mathcal{V})$$

on $\mathcal{V}$, where $\mathcal{I}^*$ denotes the cotangent bundle of G/K (cf. [1]). We call $\nabla^{\mathcal{V}}$ the covariant derivative on $\mathcal{V}$ induced by $M_{\mathfrak{m}}$.

Now, we calculate explicitly the covariant derivative $\nabla^{\mathcal{V}}$. Note that $\mathcal{I}^*\otimes\mathcal{V}$ is identified with the associated bundle $G\times_{\alpha^*\otimes\rho}(\mathfrak{m}^*\otimes V)$, where

$$\alpha^*\colon K\to GL(\mathfrak{m}^*)$$

is the representation contragradient to the adjoint representation α of K on $\mathfrak{m}$, and hence, for each $\phi\in C^{\infty}(\mathcal{V})$, $\nabla^{\mathcal{V}}\phi$ defines a C^{∞}-function $\widetilde{\nabla^{\mathcal{V}}\phi}$ from G into $\mathfrak{m}^*\otimes V$.

Proposition 2. *Let $\{X_i\}_{i=1,\dots,n}$ be a base of $\mathfrak{m}$ and $\{\omega^i\}_{i=1,\dots,n}$ its dual base. For the covariant derivative $\nabla^{\mathcal{V}}$ on $\mathcal{V}$ induced by $M_{\mathfrak{m}}$ and $\phi\in C^{\infty}(\mathcal{V})$, we have*

$$\widetilde{\nabla^{\mathcal{V}}\phi} = \sum_{i=1}^{n} \omega^i \otimes (X_i\tilde{\phi} + M_{\mathfrak{m}}(X_i)\tilde{\phi}), \tag{2.3}$$

where $X_i\tilde{\phi}$ is the Lie derivative of $\tilde{\phi}$ with respect to the vector field X_i on G, and $M_{\mathfrak{m}}(X_i)\tilde{\phi}$ is a C^∞-function on G defined by $(M_{\mathfrak{m}}(X_i)\tilde{\phi})(g) = M_{\mathfrak{m}}(X_i)\tilde{\phi}(g)$ for $g \in G$.

Proof. Through the identification $\mathcal{V} = P \underset{GL(V)}{\times} V$, for $\phi \in C^\infty(\mathcal{V})$. We define $\tilde{\tilde{\phi}}$ to be a C^∞-map from P into V in the same way as $\tilde{\phi}$; precisely, let $q: P \times V \to \mathcal{V}$ and $p: P \to G/K$ be the projections, then $\tilde{\tilde{\phi}}$ is defined by the relation

$$q(u, \tilde{\tilde{\phi}}(u)) = \phi(p(u)) \qquad \text{for } u \in P.$$

$\tilde{\tilde{\phi}}$ satisfies $\tilde{\tilde{\phi}}(uh) = h^{-1}\tilde{\tilde{\phi}}(u)$ for $h \in GL(V)$, and $\tilde{\tilde{\phi}}(\{g, 1\}) = \tilde{\phi}(g)$ for the class $\{g, 1\} \in P$ represented by $(g, 1) \in G \times GL(V)$. We denote by l_g and L_g the left translations by g on G/K and P respectively. For $X \in \mathfrak{m} = T_0(G/K)$, the horizontal lift to P of $(l_g)_*X$ is $(L_g)_*\tilde{X}_{u_0} - \omega(L_{g*}\tilde{X}_{u_0})^*_{L_g u_0}$ at the point $L_g u_0$, where $\omega(L_{g*}\tilde{X}_{u_0})^*$ is the fundamental vector field on P generated by $\omega(L_{g*}\tilde{X}_{u_0})$. ω is a G-invariant connection, so that $(L_g)_*\tilde{X}_{u_0} - \omega(L_{g*}\tilde{X}_{u_0})^*_{L_g u_0} = L_{g*}\tilde{X}_{u_0} - \omega_{u_0}(\tilde{X})^*_{L_g u_0}$. Then we have,

$$\begin{aligned}
\widetilde{\nabla^{\mathcal{V}}_{l_{g*}X}\phi}(g) &= \widetilde{\widetilde{\nabla^{\mathcal{V}}_{l_{g*}X}\phi}}(\{g, 1\}) \\
&= [(L_{g*}\tilde{X}_{u_0})\tilde{\tilde{\phi}} - \omega_{u_0}(\tilde{X})^*_{L_g u_0}\tilde{\tilde{\phi}}](\{g, 1\}) \\
&= \tilde{X}_{u_0}(L_g^*\tilde{\tilde{\phi}}) - \frac{d}{dt}\tilde{\tilde{\phi}}(\{g, 1\} \exp t\omega_{u_0}(\tilde{X}))|_{t=0} \\
&= \frac{d}{dt}((L_g^*\tilde{\tilde{\phi}})(\{\exp tx, 1\}) - \frac{d}{dt}(\exp t\omega_{u_0}(\tilde{X}))^{-1}\tilde{\tilde{\phi}}(\{g, 1\})|_{t=0} \\
&= \frac{d}{dt}\tilde{\tilde{\phi}}(\{g \exp tx, 1\})|_{t=0} + \omega_{u_0}(\tilde{X})\tilde{\phi}(g) \\
&= (X_g\tilde{\phi})(g) + M_{\mathfrak{m}}(X)\tilde{\phi}(g) \qquad \text{for each } g \in G.
\end{aligned}$$

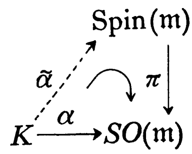
This implies (2.3). q.e.d.

Assume that V has an inner product or a hermitian inner product $\langle\ ,\ \rangle$ according to V is a real or complex vector space, such that ρ is an orthogonal or unitary representation with respect to $\langle\ ,\ \rangle$. Then $\langle\ ,\ \rangle$ defines a metric $\langle\ ,\ \rangle$ on the associated vector bundle $\mathcal{V}$. The connection in P induced by $M_{\mathfrak{m}}$ is called a metric connection if $M_{\mathfrak{m}}(\mathfrak{m})$ is contained in the Lie algebra $\mathfrak{o}(V)$ of the orthogonal group $O(V)$ or in the Lie algebra $\mathfrak{u}(V)$ of the unitary group $U(V)$. This condition is equivalent to that the metric $\langle\ ,\ \rangle$ on $\mathcal{V}$ is parallel with respect to the covariant derivative $\nabla^{\mathcal{V}}$ on $\mathcal{V}$ induced by $M_{\mathfrak{m}}$.

3. Dirac operators on homogeneous spaces

In what follows, we assume that G is a connected unimodular Lie group and K is a compact subgroup of G. Then the pair (G, K) is reductive, and so we retain the notation in the previous section.

We choose a G-invariant Riemannian metric $\langle\ ,\ \rangle$ on G/K. This defines an inner product $\langle\ ,\ \rangle$ on $\mathfrak{m}$. We assume that G/K is orientable and the isotropy representation α of K on $\mathfrak{m}$ has a lifting $\tilde{\alpha}$ to $\mathrm{Spin}(\mathfrak{m})$, i.e., there exists a homomorphism $\tilde{\alpha}$ of K into $\mathrm{Spin}(\mathfrak{m})$ such that the following diagram is commutative;



Take the representation (s, L), $(s^{\pm}, L^{\pm})$ of $\mathrm{Spin}(\mathfrak{m})$ defined in **1** and define the representations (σ, L), $(\sigma^{\pm}, L^{\pm})$ of K by

$$\sigma = s\circ\tilde{\alpha}\,, \quad \sigma^{\pm} = s^{\pm}\circ\tilde{\alpha}\,.$$

The vector bundles over G/K associated to these representations are denoted by $\mathcal{L}$, $\mathcal{L}^{\pm}$ respectively.

Let $\Lambda_{\mathfrak{m}}$ be a linear map of $\mathfrak{m}$ into $\mathfrak{o}(\mathfrak{m})$ satisfying the condition;

$$\Lambda_{\mathfrak{m}}(\alpha(k)X) = \alpha(k)\Lambda_{\mathfrak{m}}(X)\alpha(k)^{-1}$$

for $k\in K$ and $X\in\mathfrak{m}$.
We define a linear map

$$\tilde{\Lambda}_{\mathfrak{m}}\colon \mathfrak{m} \to \mathfrak{spin}(\mathfrak{m})$$

by

$$\tilde{\Lambda}_{\mathfrak{m}} = \dot{\Pi}^{-1}\circ\Lambda_{\mathfrak{m}}\,.$$

Then $\tilde{\Lambda}_{\mathfrak{m}}$ satisfies the condition

$$\tilde{\Lambda}_{\mathfrak{m}}(\alpha(k)X) = \tilde{\alpha}(k)\cdot\tilde{\Lambda}_{\mathfrak{m}}(X)\cdot\tilde{\alpha}(k)^{-1} \qquad \text{for } k\in K,\ X\in\mathfrak{m}.$$

We imbed $\mathfrak{spin}(\mathfrak{m})$ into $\mathfrak{gl}(L)$ (resp. $\mathfrak{gl}(L^{\pm})$) through Clifford left multiplication (the differential of the spin representations). Then the above condition implies

$$\tilde{\Lambda}_{\mathfrak{m}}(\alpha(k)X) = \sigma(k)\tilde{\Lambda}_{\mathfrak{m}}(X)\sigma(k)^{-1}$$

(resp.

$$\tilde{\Lambda}_{\mathfrak{m}}(\alpha(k)X) = \sigma^{\pm}(k)\tilde{\Lambda}_{\mathfrak{m}}(X)\sigma^{\pm}(k)^{-1})$$

for $k\in K$ and $X\in\mathfrak{m}$.

We denote by $\mathcal{T}$, $\mathcal{T}^*$ the tangent and cotangent bundles over G/K. The isomorphism of $\mathcal{T}^*$ onto $\mathcal{T}$ through the Riemannian metric on G/K is denoted by h. And we denote by μ (resp. $\mu^{\pm}$) the map from $C^{\infty}(\mathcal{T}\otimes\mathcal{L})$ (resp. $C^{\infty}(\mathcal{T}\otimes\mathcal{L}^{\pm})$) to

$C^\infty(\mathcal{L})$ (resp. $C^\infty(\mathcal{L}^\pm)$) induced by the bundle map defined by the symbol map (1.2), (1.2)$'$. This can be well defined by Lemma 1-(i). We denote by ∇ (resp. $\nabla^\pm$) the covariant derivative induced by $\tilde{\Lambda}_{\mathfrak{m}}$ on $\mathcal{L}$ (resp. $\mathcal{L}^\pm$). We define the Dirac operators D, $D^\pm$ as follows;

$$D = \mu\circ(h\otimes 1)\circ\nabla\colon C^\infty(\mathcal{L})\xrightarrow{\nabla}C^\infty(\mathcal{T}^*\otimes\mathcal{L})\xrightarrow{h\otimes 1}C^\infty(\mathcal{T}\otimes\mathcal{L})\xrightarrow{\mu}C^\infty(\mathcal{L}),$$

$$D^\pm = \mu^\pm\circ(h\otimes 1)\circ\nabla^\pm\colon C^\infty(\mathcal{L}^\pm)\xrightarrow{\nabla^\pm}C^\infty(\mathcal{T}^*\otimes\mathcal{L}^\pm)\xrightarrow{h\otimes 1}C^\infty(\mathcal{T}\otimes\mathcal{L}^\pm)\xrightarrow{\mu^\pm}C^\infty(\mathcal{L}^\mp).$$

Lemma 2. *Let $\{X_i\}_{i=1,\dots,n}$ be an orthonormal base of $\mathfrak{m}$ with respect to $\langle\ ,\ \rangle$, and $\{\omega^i\}_{i=1\dots n}$ its dual base.*
(i) *For $\phi\in C^\infty(\mathcal{L})$, we have*

$$\widetilde{D\phi} = \sum_{i=1}^{n}\varepsilon\{X_i\otimes(X_i\tilde{\phi}+\tilde{\Lambda}_{\mathfrak{m}}(X_i)\tilde{\phi})\}\,. \tag{3.1}$$

(ii) *The same formulas hold for an element of $C^\infty(\mathcal{L}^\pm)$.*

Proof. Immediate consequence of Proposition 2 and the definition of the Dirac operator D.

Let (ρ, V) be a finite dimensional unitary representation of K and $\mathcal{V}$ the vector bundle associated to (ρ, V). Then $\mathcal{V}$ carries the invariant metric induced from the hermitian inner product on V. Let $M_{\mathfrak{m}}$ be a linear map of $\mathfrak{m}$ into $\mathfrak{gl}(V)$ satisfying the condition (2.1), and $\nabla^{\mathcal{V}}$ the covariant derivative on $\mathcal{V}$ induced by $M_{\mathfrak{m}}$. In order to define our Dirac operators from $C^\infty(\mathcal{L}\otimes\mathcal{V})$ (resp. $C^\infty(\mathcal{L}^\pm\otimes\mathcal{V})$) to $C^\infty(\mathcal{L}\otimes\mathcal{V})$ (resp. $C^\infty(\mathcal{L}^\pm\otimes\mathcal{V})$) We use the following theorem.

Theorem P (Theorem 3, §9, Chapter IV in [4]). *Let $\mathcal{U}, \mathcal{V}, \mathcal{W}$ be vector bundles over a C^∞-manifold M, $D\colon C^\infty(\mathcal{U})\to C^\infty(\mathcal{W})$ a first order linear differential operator on M and $\nabla^{\mathcal{V}}$ a covariant derivtive for $\mathcal{V}$. Then there is a unique first order differential operator*

$$T\colon C^\infty(\mathcal{U}\otimes\mathcal{V}) \to C^\infty(\mathcal{W}\otimes\mathcal{V})$$

such that

$$T(f\otimes h)(x) = (Df\otimes h)(x) \quad \textit{whenever } (\nabla^{\mathcal{V}}h)(x) = 0\,.$$

We denote this operator by $D \underset{\nabla^{\mathcal{V}}}{\hat{\otimes}} 1$. From (2.3), (3.1) and using the above theorem, we can define a differential operator $D \underset{\nabla^{\mathcal{V}}}{\hat{\otimes}} 1$ (resp. $D^\pm \underset{\nabla^{\mathcal{V}}}{\hat{\otimes}} 1$), which we call the Dirac operators.

Proposition 3. *For $\phi\in C^\infty(\mathcal{L}\otimes\mathcal{V})$, we have*

$$\widetilde{(D \underset{\nabla^{\mathcal{V}}}{\hat{\otimes}} 1)\phi} = \sum_{i=1}^{n}\varepsilon_V[X_i\otimes\{X_i\tilde{\phi}+(\tilde{\Lambda}_{\mathfrak{m}}(X_i)\otimes 1)\tilde{\phi}+(1\otimes M_{\mathfrak{m}}(X_i))\tilde{\phi}\}] \tag{3.2}$$

where $\varepsilon_V = \varepsilon \otimes 1: \mathfrak{m} \otimes L \otimes V \to L \otimes V$ *and* $\{X_i\}_{i=1,\dots,n}$ *is an orthonormal base of* $\mathfrak{m}$.

Proof. It suffices to prove that the right hand side of (3.2) is a first order differential operator from $C^\infty(\mathcal{L}\otimes\mathcal{V})$ to $C^\infty(\mathcal{L}\otimes\mathcal{V})$ and satisfies the condition of theorem P. But it is easy to see these conditions by making use of Proposition 2. q.e.d.

4. Formal self adjointness of Dirac operators

We denote by dx the invariant measure on G/K induced by the G-invariant Riemannian metric. Since G is a unimodular Lie group, there exists a bi-invarint measure dg on G such that for any C^∞-function f with compact support we have

$$\int_G p^* f dg = \int_{G/K} f dx$$

where p is the projection $G \to G/K$. Then we have in virtue of the invariance of dg

$$(4.1) \qquad \int_G Xf dg = 0 \qquad \text{for all } X \in \mathfrak{g}\,.$$

We fix inner products on L and $L^\pm$ satisfying the Lemma 1-(iii). Then $\mathcal{L}$ and $\mathcal{L}^\pm$ carry the metrics induced from the above inner products and also $\mathcal{L}\otimes\mathcal{V}$, $\mathcal{L}^\pm\otimes\mathcal{V}$ carry the metrics induced from the metrics of $\mathcal{L}$, $\mathcal{L}^\pm$ and $\mathcal{V}$. The inner product $(\ ,\ )$ on the space $C_c^\infty(\mathcal{L}\otimes\mathcal{V})$ of all C^∞-sections with compact support is defined by

$$(4.2) \qquad (\phi, \psi) = \int_{G/K} \langle \phi, \psi \rangle dx = \int_G \langle \tilde{\phi}, \tilde{\psi} \rangle_0 dg$$

where $\langle\ ,\ \rangle$ is the metric on the vector bundle, and $\langle\ ,\ \rangle_0$ is the inner product of $L\otimes V$ or $L^\pm \otimes V$.

Proposition 4. *We have the formula for the formal adjoint operator* $(D \underset{\nabla^{\mathcal{V}}}{\hat{\otimes}} 1)^*$ *of* $(D \underset{\nabla^{\mathcal{V}}}{\hat{\otimes}} 1)$ *as follows;*

$$(4.3) \qquad \widetilde{(D \underset{\nabla^{\mathcal{V}}}{\hat{\otimes}} 1)^* \phi} = \sum_{i=1}^{n} [\varepsilon_V \{X_i \otimes (X_i \phi)\} + (\tilde{\Lambda}_{\mathfrak{m}}(X_i)\otimes 1)\varepsilon_V(X_i \otimes \tilde{\phi}) - \varepsilon_V\{X_i \otimes (1\otimes M^*_{\mathfrak{m}}(X_i))\tilde{\phi}\}]$$

where $M^*_{\mathfrak{m}}$ *is the adjoint operator of* $M_{\mathfrak{m}}$ *with respect to the inner product of* V.

Proof. For $\phi, \psi \in C_c^\infty(\mathcal{L}\otimes\mathcal{V})$,

$$
\begin{aligned}
((D \underset{\nabla^{\mathcal{V}}}{\hat{\otimes}} 1)\phi, \psi) &= \sum_{i=1}^{n} \int_G \langle \varepsilon_V \{X_i \otimes (X_i \tilde{\phi})\}, \tilde{\psi} \rangle_0 dg \\
&\quad + \sum_{i=1}^{n} \int_G \langle \varepsilon_V \{X_i \otimes (\tilde{\Lambda}_{\mathfrak{m}}(X_i) \otimes 1)\tilde{\phi}\}, \tilde{\psi} \rangle_0 dg \\
&\quad + \sum_{i=1}^{n} \int_G \langle \varepsilon_V \{X_i \otimes (1 \otimes M_{\mathfrak{m}}(X_i))\tilde{\phi}\}, \tilde{\psi} \rangle_0 dg \\
&= -\sum_{i=1}^{n} \int_G \langle X_i \tilde{\phi}, \varepsilon_V (X_i \otimes \tilde{\psi}) \rangle_0 dg \\
&\quad - \sum_{i=1}^{n} \int_G \langle (\tilde{\Lambda}_{\mathfrak{m}}(X_i) \otimes 1)\tilde{\phi}, \varepsilon_V (X_i \otimes \tilde{\psi}) \rangle_0 dg \\
&\quad - \sum_{i=1}^{n} \int_G \langle (1 \otimes M_{\mathfrak{m}}(X_i))\tilde{\phi}, \varepsilon_V (X_i \otimes \tilde{\psi}) \rangle_0 dg \\
&= -\sum_{i=1}^{n} \int_G X_i \langle \tilde{\phi}, \varepsilon_V (X_i \otimes \tilde{\psi}) \rangle_0 dg \\
&\quad + \sum_{i=1}^{n} \int_G \langle \tilde{\phi}, X_i \varepsilon_V (X_i \otimes \tilde{\psi}) \rangle_0 dg \\
&\quad + \sum_{i=1}^{n} \int_G \langle \tilde{\phi}, (\tilde{\Lambda}_{\mathfrak{m}}(X_i) \otimes 1) \varepsilon_V (X_i \otimes \tilde{\psi}) \rangle_0 dg \\
&\quad - \sum_{i=1}^{n} \int_G \langle \tilde{\phi}, (1 \otimes M^*_{\mathfrak{m}}(X_i)) \{\varepsilon_V (X_i \otimes \tilde{\psi})\} \rangle_0 dg \\
&= \sum_{i=1}^{n} \int_G \langle \tilde{\phi}, \varepsilon_V (X_i \otimes X_i \tilde{\psi}) \rangle_0 dg \\
&\quad + \sum_{i=1}^{n} \int_G \langle \tilde{\phi}, (\tilde{\Lambda}_{\mathfrak{m}}(X_i) \otimes 1) \varepsilon_V (X_i \otimes \tilde{\psi}) \rangle_0 dg \\
&\quad - \sum_{i=1}^{n} \int_G \langle \tilde{\phi}, \varepsilon_V \{X_i \otimes (1 \otimes M^*_{\mathfrak{m}}(X_i))\tilde{\psi}\} \rangle_0 dg .
\end{aligned}
$$

Thus we have the proposition 4. q.e.d.

Theorem. *Suppose the connection induced by $M_{\mathfrak{m}}$ is a metric connection. Then a necessary and sufficient condition that $D \underset{\nabla^{\mathcal{V}}}{\hat{\otimes}} 1$ is a formal self adjoint operator (resp. $D^{\pm} \underset{\nabla^{\mathcal{V}}}{\hat{\otimes}} 1$ is the formal adjoint operator of $D^{\mp} \underset{\nabla^{\mathcal{V}}}{\hat{\otimes}} 1$) if and only if the following condition;*

$$
\sum_{i=1}^{n} \Lambda_{\mathfrak{m}}(X_i) X_i = 0 \tag{4.4}
$$

holds, where $\{X_i\}_{i=1\dots n}$ is an orthonormal base of $\mathfrak{m}$.

Proof. From our assumption, $M_{\mathfrak{m}}(X) = -M^*_{\mathfrak{m}}(X)$ for each $X \in \mathfrak{m}$. For $\phi \in C^\infty_c(\mathcal{L} \otimes \mathcal{V})$, from Proposition 4 we have

$$
\widetilde{(D \underset{\nabla^{\mathcal{V}}}{\hat{\otimes}} 1)\phi} - \widetilde{(D \underset{\nabla^{\mathcal{V}}}{\hat{\otimes}} 1)^*\phi} = \sum_{i=1}^{n} [(\tilde{\Lambda}(X_i) \otimes 1)\varepsilon_V (X_i \otimes \tilde{\phi}) - \varepsilon_V \{X_i \otimes (\tilde{\Lambda}_{\mathfrak{m}}(X_i) \otimes 1)\tilde{\phi}\}] .
$$

Thus, we see that the condition,

$$(4.5) \qquad \sum_{i=1}^{n} \{\tilde{\Lambda}_{\mathfrak{m}}(X_i)\varepsilon(X_i \otimes l) - \varepsilon(X_i \otimes \tilde{\Lambda}_{\mathfrak{m}}(X_i)l)\} = 0 \qquad \text{for } l \in L,$$

is necessary and sufficient in order that $D \underset{\nabla^{CV}}{\hat{\otimes}} 1$ becomes the formal self adjoint operator. From Lemma 1-(ii) and (1.1), the condition (4.5) is equivalent to

$$\sum_{i=1}^{n} \Lambda_{\mathfrak{m}}(X_i)X_i = 0 . \qquad \text{q.e.d.}$$

Corollary. *In the following two cases, the condition of theorem is satisfied;*
(i) *α has no fixed point except* 0.
(ii) *The connection induced by $\Lambda_{\mathfrak{m}}$ is Riemannian, i.e., it coincides with the Riemannian connection defined by a G-invariant Riemannian metric g on G/K.*

Proof. (i) for any $k \in K$ we have,

$$\alpha(k)(\sum_{i=1}^{n} \Lambda_{\mathfrak{m}}(X_i)X_i) = \sum_{i=1}^{n} \alpha(k)\Lambda_{\mathfrak{m}}(X_i)\alpha(k)^{-1}\alpha(k)X_i$$
$$= \sum_{i=1}^{n} \Lambda_{\mathfrak{m}}(\alpha(k)X_i)\alpha(k)X_i$$
$$= \sum_{i=1}^{n} \Lambda_{\mathfrak{m}}(X_i)X_i ,$$

where the last equality follows from the fact that $\alpha(k)$ is an orthogonal transformation on $\mathfrak{m}$. Hence from our assumption, we have

$$\sum_{i=1}^{n} \Lambda_{\mathfrak{m}}(X_i)X_i = 0 .$$

(ii) We denote by B the inner product on $\mathfrak{m}$ induced by g. Then the Riemannian connection defined by g is given by

$$\Lambda_{\mathfrak{m}}(X)Y = \frac{1}{2}[X, Y]_{\mathfrak{m}} + U(X, Y)$$

where $[X, Y]$ is the $\mathfrak{m}$-component of $[X, Y]$ and $U(X,Y)$ is the symmetric bilinear mapping of $\mathfrak{m} \times \mathfrak{m}$ into $\mathfrak{m}$ defined by

$$2B(U(X, Y), Z) = B(X, [Z, Y]_{\mathfrak{m}}) + B([Z, X]_{\mathfrak{m}}, Y)$$

for all $X, Y, Z \in \mathfrak{m}$. (cf. Theorem 3.3 Chapter X in [1]-(b)). From the above formula, we have

$$\sum_{i=1}^{n} \Lambda_{\mathfrak{m}}(X_i)X_i = \sum_{i=1}^{n} U(X_i, X_i)$$
$$= \sum_{i,i}^{n} B(U(X_i, X_i), X_j)X_j$$

$$= \sum_{i,j}^{n} B(X_i, [X_j, X_i])X_j ,$$

here we extend B to $\mathfrak{g}$ such that (1) B is an $Ad(k)$-invariant metric on $\mathfrak{g}$, and (2) $\mathfrak{m}$ and $\mathfrak{k}$ are mutually orthogonal with respect to B. We choose an orthonormal base $\{Y_1, \cdots, Y_p\}$ of $\mathfrak{k}$. Then using $[\mathfrak{k}, \mathfrak{m}] \subset \mathfrak{m}$, we have

$$\begin{aligned} \sum_{i=1}^{n} B(X_i, [X_j, X_i]_{\mathfrak{m}}) &= \sum_{i=1}^{n} \{B(X_i, [X_j, X_i]) + B(Y_i, [X_j, X_i])\} \\ &= Tr\, ad_{\mathfrak{g}} X_j \\ &= 0 , \end{aligned}$$

where the last equality holds since G is unimodular. Thus we have

$$\sum_{i=1}^{n} \Lambda_{\mathfrak{m}}(X_i)X_i = 0 . \qquad \text{q.e.d.}$$

Remarks (i) Suppose G is compact and rank G=rank K, then the condition (i) of Corollary is always satisfied. In fcat, let T be a maximal torus of G contained in K. Adjoint representation of T on $\mathfrak{g}$ is decomposed as follows;

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}_1 \oplus \cdots \oplus \mathfrak{p}_{n/2} ,$$

where each $\mathfrak{p}_i$ is two dimensional subspace of $\mathfrak{g}$. On each $\mathfrak{p}_i$, T act as nontrivial rotational elements. Thus the isotropy representation has no fixed point except 0.
(ii) Suppose G is semi-simple and G/K is a symmetric space. Let $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ be a Cartan decomposition. Let H be a non-zero element of $\mathfrak{p}$ and $\mathfrak{a}$ a maximal abelian subspace of $\mathfrak{p}$ containing H. Then there exist an element w of the Weyl group of G/K and an element k of K such that $Ad(k)H=wH \neq H$. Thus the condition of Corollary (i) is satisfied.
(iii) If $D \hat{\otimes}_{\nabla^{CV}} 1$ is formally self adjoint, then in the same way as Wolf [5], We see that $D \hat{\otimes}_{\nabla^{CV}} 1$ and $(D \hat{\otimes}_{\nabla^{CV}} 1)^2$ are essentially self adjoint operators. Because, his proof (Theorem 5.1 and Theorem 6.1 in [5]) has no use that the connection is Riemannian.

Osaka University

References

[1] S. Kobayashi and K. Nomizu:
(a) Foundations of Differential Geometry, vol. I, Interscience, New York, 1963.
(b) Foundations of Differential Geometry, vol. II, Interscience, New York, 1969.
[2] J.L. Koszul: Lectures on Fibre Bundles and Differential Geometry, Tata Institute, 1960.

[3] R. Parthasarathy: *Dirac operator and the discrete series*, Ann. of Math. **96** (1972), 1–30.

[4] R. Palais: Seminar on the Atiyah-Singer Index Theorem, Annals of Mathematics Studies. No. 57, Princeton 1965.

[5] J.A. Wolf: *Essential self adjointness for the Dirac operator and its square*, Indiana. Univ. Math. J. **22** (1973), 611–640.