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SOME PROPERTIES OF DERIVED HOPF ALGEBRAS OF λ-MODIFIED DIFFERENTIAL HOPF ALGEBRAS

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In [1] we defined a λ-modified differential Hopf algebra A (or simply a (d, λ)-Hopf algebra) and introduced the derived Hopf algebra $\Phi_\lambda(A)=\Psi_\lambda(A)$, maps ξ_λ, η_λ, etc., in order to characterize coprimitivity and primitivity of A. In this note we study some properties of the derived Hopf algebra. Definitions and notations are referred to [1] in the present work.

1. Throughout the present work we understand that K is a field of characteristic $p\neq 0$, $\lambda\in K$ and all modules are G_2-modules over K unless otherwise stated.

Let M be a differential G_2-module. Suppose p is odd.

For each (j, k), $1\leqq j, k\leqq p$, consider the map

$$1+\lambda d_jd_k : M^{\otimes p} \to M^{\otimes p}$$

where 1 is the identity map of $M^{\otimes p}=M\otimes\cdots\otimes M$ (p times) and d_i is the i-th partial differential of $M^{\otimes p}$ for $1\leqq i\leqq n$, [1], (2.2). Since the partial differential are anti-commutative we see immediately

(1.1) i) $(1+\lambda d_jd_k)(1+\lambda d_id_h)=(1+\lambda d_id_h)(1+\lambda d_jd_k)$,

ii) $(1+\lambda d_jd_k)(1+\lambda d_kd_j)=1$,

iii) $1+\lambda d_jd_k$ *is an automorphism of a differential* G_2*-module*,

iv) $1+\lambda d_jd_k$ *is natural*, i.e.,

$$(1+\lambda d_jd_k)f^{\otimes p}=f^{\otimes p}(1+\lambda d_jd_k)$$

for any map $f: M\to N$ *of differential* G_2*-modules.*

We define a natural automorphism

$$B_{p,\lambda} : M^{\otimes p} \to M^{\otimes p}$$

by

(1.2) $B_{p,\lambda}=\prod_{1\leqq j,k\leqq p, k-j\geqq(p+1)/2}(1+\lambda d_jd_k)$

as the composition of maps $1+\lambda d_j d_k$.
By (1.1) ii) $B_{p,-\lambda}$ is the inverse morphism of $B_{p,\lambda}$.

Let $C_p : M^{\otimes p} \to M^{\otimes p}$ be the cyclic permutation and $C_{p,\lambda} : M^{\otimes p} \to M^{\otimes p}$ be the λ-modified cyclic permutation, [1], **3.3.**. As is easily seen we have

(1.3) $d_1 C_p = C_p d_p$, $d_{i+1} C_p = C_p d_i$ *for* $1 \leqq i \leqq p-1$ *and* $C_{p,\lambda} = (1+\lambda d_1 d) C_p$.

Then we obtain

(1.4) **Lemma.** *The following relation*

$$C_{p,\lambda} B_{p,\lambda} = B_{p,\lambda} C_p$$

holds. In particular, $B_{p,\lambda}(x^{\otimes p})$ *is* $C_{p,\lambda}$*-fixed for any* $x \in A$.

Proof. Making use of (1.1), (1.2) and (1.3) we get

$$\begin{aligned} C_{p,\lambda} B_{p,\lambda} &= (1+\lambda d_1 d) C_p \prod_{1 \leqq j<k \leqq p, k-j \geqq (p+1)/2} (1+\lambda d_j d_k) \\ &= \prod_{2 \leqq j \leqq (p+1)/2} (1+\lambda d_1 d_j) \prod_{(p+3)/2 \leqq k \leqq p} (1+\lambda d_1 d_k) \\ &\quad \prod_{2 \leqq j<k \leqq p, k-j \geqq (p+1)/2} (1+\lambda d_j d_k) \prod_{2 \leqq j \leqq (p+1)/2} (1+\lambda d_j d_1) C_p \\ &= \prod_{1 \leqq j<k \leqq p, k-j \geqq (p+1)/2} (1+\lambda d_j d_k) C_p = B_{p,\lambda} C_p . \end{aligned}$$

q.e.d.

REMARK. In [1], (5.10) we proved that there exists an element $b_{p,\lambda}(x)$ such that $x^{\otimes p} + b_{p,\lambda}(x)$ is $C_{p,\lambda}$-fixed. Putting $B_{p,\lambda}(x^{\otimes p}) = x^{\otimes p} + b_{p,\lambda}(x)$, the above lemma describes $b_{p,\lambda}(x)$ explicitly.

Put

$$\begin{aligned} &\Delta_0 = 1 - C_p,\ \Delta_\lambda = 1 - C_{p,\lambda},\ \tilde{\Delta}_0 = 1 - C_p \otimes C_p , \\ &\Sigma_0 = \sum_{k=0}^{p-1} C_p^k,\ \Sigma_\lambda = \sum_{k=0}^{p-1} C_{p,\lambda}^k \quad \text{and} \quad \tilde{\Sigma}_0 = \sum_{k=0}^{p-1} C_p^k \otimes C_p^k . \end{aligned}$$

For a differential G_2-module M we put

$$\begin{aligned} \Phi_0(M) &= \operatorname{Ker} \Delta_0 / \operatorname{Im} \Sigma_0, & \Phi_\lambda(M) &= \operatorname{Ker} \Delta_\lambda / \operatorname{Im} \Sigma_\lambda , \\ \Psi_0(M) &= \operatorname{Ker} \Sigma_0 / \operatorname{Im} \Delta_0 \quad \text{and} & \Psi_\lambda(M) &= \operatorname{Ker} \Sigma_\lambda / \operatorname{Im} \Delta_\lambda . \end{aligned}$$

By (1.4) the map $B_{p,\lambda}$ induces natural isomorphisms

(1.5) $\Phi(B_{p,\lambda}) : \Phi_0(M) \to \Phi_\lambda(M)$ *and* $\Psi(B_{p,\lambda}) : \Psi_0(M) \to \Psi_\lambda(M)$

of G_2-modules.

A permutation $U_p : (M \otimes M)^{\otimes p} \to M^{\otimes p} \otimes M^{\otimes p}$ and a λ-modified permutation $U_{p,\lambda} : (M \otimes M)^{\otimes p} \to M^{\otimes p} \otimes M^{\otimes p}$ are defined by

$$U_p = T_p (T_{p-1} T_{p+1}) \cdots (T_2 T_4 \cdots T_{2p-2})$$

and

$$U_{p,\lambda} = T_{p,\lambda} (T_{p-1,\lambda} T_{p+1,\lambda}) \cdots (T_{2,\lambda} T_{4,\lambda} \cdots T_{2p-2,\lambda})$$

where T_i is the i-th partial switching map and $T_{i,\lambda}$ is the i-th partial λ-modified switching map for $1\leqq i\leqq 2p-1$, i.e., $T_{i,\lambda}=(1+\lambda d_i d_{i+1})\,T_i$, [1], (2.16).
Since $T_i d_i=d_{i+1}T_i$, $T_i d_{i+1}=d_i T_i$ and $T_i d_j=d_j T_i$ for $j\neq i,\ i+1$ we have the following relation

(1.6) $U_{p,\lambda}=\prod_{1\leqq j<k\leqq p}(1+\lambda d_k d_{p+j})U_p$.

2. Let p be a prime number and S_k be the set of k-tuples of integers defined by

$$S_k=\{(i_1,\cdots,i_k);\ 0\leqq i_1<\cdots<i_k\leqq p-1\},\ 1\leqq k<p\,.$$

Elements $(i_1,\cdots,i_k)$ and $(i'_1,\cdots,i'_k)$ of S_k are said to be related provided

$$(i_2-i_1,\cdots,i_k-i_1)=(i'_{j+1}-i'_j,\cdots,i'_k-i'_j,p+i'_1-i'_j,\cdots,p+i'_{j-1}-i'_j)$$

for some j.

(2.1) *This relation is an equivalence relation.*

Proof. Denote by $(i_1,\cdots,i_k)\underset{j}{\sim}(i'_1,\cdots,i'_k)$ if $(i_2-i_1,\cdots,i_k-i_1)=(i'_{j+1}-i'_j,\cdots,i'_k-i'_j,p+i'_1-i'_j,\cdots,p+i'_{j-1}-i'_j)$ for some j. Then we see immediately that $(i_1,\cdots,i_k)\underset{1}{\sim}(i_1,\cdots,i_k)$, $(i_1,\cdots,i_k)\underset{j}{\sim}(i'_1,\cdots,i'_k)$ implies $(i'_1,\cdots,i'_k)\underset{\widetilde{k-j+2}}{\sim}(i_1,\cdots,i_k)$ and $(i_1,\cdots,i_k)\underset{j}{\sim}(i'_1,\cdots,i'_k)$ and $(i'_1,\cdots,i'_k)\underset{j'}{\sim}(i''_1,\cdots,i''_k)$ imply $(i_1,\cdots,i_k)\underset{\widetilde{j+j'-1}}{\sim}(i''_1,\cdots,i''_k)$. q.e.d.

Let $\tilde{S}_k$ be the quotient set of S_k defined by the above equivalence relation and $\pi: S_k\to\tilde{S}_k$ be the natural projection.

(2.2) **Lemma.** *If $\pi(i_1,\cdots,i_k)=\pi(0,s_2,\cdots,s_k)$ for $1\leqq k<p$, then there exists a unique integer j, $1\leqq j\leqq k$, such that*

$$(s_2,\cdots,s_k)=(i_{j+1}-i_j,\cdots,i_k-i_j,p+i_1-i_j,\cdots,p+i_{j-1}-i_j)\,.$$

Proof. By definition there exists a required integer j, $1\leqq j\leqq k$. We shall show that such an integer j is unique when $1\leqq k<p$. Suppose that $(s_2,\cdots,s_k)=(i_{j+1}-i_j,\cdots,p+i_{j-1}-i_j)=(i_{j'+1}-i_j,\cdots,p+i_{j'-1}-i_{j'})$. Then we have

$$\textstyle\sum_{t=2}^k s_t=(j-1)p+\sum_{t=1}^k i_t-k\cdot i_j=(j'-1)p+\sum_{t=1}^k i_t-k\cdot i_{j'}\,.$$

Hence $(j-j')p=k(i_j-i_{j'})$ and $j=j'$. q.e.d.

We may choose elements of form $(0,s_2,\cdots,s_k)$ as representatives of the above equivalence classes in S_k because $\pi(i_1,\cdots,i_k)=\pi(0,i_2-i_1,\cdots,i_k-i_1)$. We identify this set of representatives with $\tilde{S}_k$. Using (2.2) we have the correspondence τ_k between S_k and $\tilde{S}_k\times Z_p$, $1\leqq k<p$, defined by

(2.3) $$\tau_k(i_1, \cdots, i_k) = ((0, s_2, \cdots, s_k), i_j)$$

where $(0, s_2, \cdots, s_k)$ is the representative of $\pi(i_1, \cdots, i_k)$ and $(s_2, \cdots, s_k) = (i_{j+1}-i_j, \cdots, i_k-i_j, p+i_1-i_j, \cdots, p+i_{j-1}-i_j)$.

(2.4) **Lemma.** *τ_k is a one to one correspondence.*

Proof. Suppose that $\tau_k(i_1, \cdots, i_k)=\tau_k(i'_1, \cdots, i'_k)$, i.e., $\pi(i_1, \cdots, i_k)=\pi(i'_1, \cdots, i'_k)$ and $i_j=i'_{j'}$. Then

$$(i_{j+1}, \cdots, i_k, p+i_1, \cdots, p+i_{j-1})=(i'_{j'+1}, \cdots, i'_k, p+i'_1, \cdots, p+i'_{j'-1}),$$

hence $(i_1, \cdots, i_k)=(i'_1, \cdots, i'_k)$. And also

$$\tau_k(s_{j+1}+i-p, \cdots, s_k+i-p, i, s_2+i, \cdots, s_j+i)=((0, s_2, \cdots, s_k), i)$$

for $p-s_{j+1}\leqq i<p-s_j$. Therefore τ_k is one to one. q.e.d.

(2.4) means that a equivalence class in S_k, $1\leqq k<p$, is a subset which consists of just p elements.

Let M be a differential G_2-module over K, char $K=p$ and $t: M\to M$ be a map of period p, i.e., $t^p=1$. Put $\Delta=1-t$ and $\Sigma=\sum_{i=0}^{p-1} t^i$. We consider maps $x_i: M\to M$, $1\leqq i\leqq p$, such that

(2.5) $x_1t=tx_p$, $x_{i+1}t=tx_i$ *for* $1\leqq i\leqq p-1$ *and* $x_ix_j=x_jx_i$ *for* $1\leqq i, j\leqq p$.

If $\tau_k(i_1, \cdots, i_k)=((0, s_2, \cdots, s_k), i_j)$ it follows immediately from (2.5) that

$$t^{i_j}x_1x_{s_2+1}\cdots x_{s_k+1}t^{p-i_j}=x_{i_1+1}\cdots x_{i_k+1}\,.$$

Denote by σ_k the k-th elementary symmetric polynomial of p variables. Since τ_k is one to one by (2.4) we can express $\sigma_k(x_1, \cdots, x_p)$ as

$$\sigma_k(x_1, \cdots, x_p)=\sum_{(0,s_2,\cdots,s_k)\in\tilde{S}_k}\sum_{i=0}^{p-1} t^ix_1x_{s_2+1}\cdots x_{s_k+1}t^{p-i}$$

for $1\leqq k<p$. As is easily seen we have

$$\sum_{i=0}^{p-1} t^ix_1x_{s_2+1}\cdots x_{s_k+1}t^{p-i}(\mathrm{Ker}\,\Delta)\subset \mathrm{Im}\,\Sigma$$

and

$$\sum_{i=0}^{p-1} t^ix_1x_{s_2+1}\cdots x_{s_k+1}t^{p-i}(\mathrm{Ker}\,\Sigma)\subset \mathrm{Im}\,\Delta\,.$$

Hence

$$\sigma_k(x_1, \cdots, x_p)(\mathrm{Ker}\,\Delta)\subset \mathrm{Im}\,\Sigma \quad \text{and} \quad \sigma_k(x_1, \cdots, x_p)(\mathrm{Ker}\,\Sigma)\subset \mathrm{Im}\,\Delta$$

for $1\leqq k<p$. Thus we obtain

(2.6) **Lemma.**

$$(1+x_1)\cdots(1+x_p)\,|\,\mathrm{Ker}\,\Delta \equiv 1+x_1\cdots x_p \quad \mathrm{mod}\ \mathrm{Im}\,\Sigma$$

and

$$(1+x_1)\cdots(1+x_p)\,|\,\mathrm{Ker}\,\Sigma \equiv 1+x_1\cdots x_p \quad \mathrm{mod}\ \mathrm{Im}\,\Delta\,.$$

For $0 \leqq s \leqq p$, define maps

$$D^s_{p,\lambda} : M^{\otimes p} \to M^{\otimes p} \quad \text{and} \quad \tilde{D}^s_{p,\lambda} : M^{\otimes p} \otimes M^{\otimes p} \to M^{\otimes p} \otimes M^{\otimes p}$$

by

(2.7) $D^s_{p,\lambda} = \prod_{1 \leqq j \leqq p-s}(1+\lambda d_j d_{s+j}) \prod_{1 \leqq k \leqq s}(1+\lambda d_{p-s+k} d_k)$

and

$$\tilde{D}^s_{p,\lambda} = \prod_{1 \leqq j \leqq p-s}(1+\lambda d_j d_{p+s+j}) \prod_{1 \leqq k \leqq s}(1+\lambda d_{p-s+k} d_{p+k})$$

respectively. Putting

$$x_j = \lambda d_j d_{s+j},\ \tilde{x}_j = \lambda d_j d_{p+s+j} \quad \text{for } 1 \leqq j \leqq p-s$$

and

$$x_{p-s+k} = \lambda d_{p-s+k} d_k,\ \tilde{x}_{p-s+k} = \lambda d_{p-s+k} d_{p+k} \quad \text{for } 1 \leqq k \leqq s,$$

by (1.3) we have

(2.8) $\quad x_1 C_p = C_p x_p,\ \tilde{x}_1(C_p \otimes C_p) = (C_p \otimes C_p)\tilde{x}_p$

$\quad x_{i+1} C_p = C_p x_i,\ \tilde{x}_{i+1}(C_p \otimes C_p) = (C_p \otimes C_p)\tilde{x}_i$ *for* $1 \leqq i \leqq p-1$.

and

$$x_i x_j = x_j x_i,\ \tilde{x}_i \tilde{x}_j = \tilde{x}_j \tilde{x}_i \quad \textit{for } 1 \leqq i, j \leqq p\,.$$

By an easy calculation we see

(2.9) $x_1 \cdots x_p = 0$ *and* $\tilde{x}_1 \cdots \tilde{x}_p = (-1)^{p(p-1)/2} \lambda^p d_1 \cdots d_{2p}$.

Remark that

(2.10) $D^s_{p,\lambda} = (1+x_1)\cdots(1+x_p)$ and $\tilde{D}^s_{p,\lambda} = (1+\tilde{x}_1)\cdots(1+\tilde{x}_p)$.

Then, by (2.6), (2.8) and (2.9) we have

$$D^s_{p,\lambda}\,|\,\mathrm{Ker}\,\Delta_0 \equiv 1 \quad \mathrm{mod}\ \mathrm{Im}\Sigma_0, \quad D^s_{p,\lambda}\,|\,\mathrm{Ker}\,\Sigma_0 \equiv 1 \quad \mathrm{mod}\ \mathrm{Im}\,\Delta_0$$

$$\tilde{D}^s_{p,\lambda}\,|\,\mathrm{Ker}\,\tilde{\Delta}_0 \equiv 1+(-1)^{p(p-1)/2}\lambda^p d_1 \cdots d_{2p} \quad \mathrm{mod}\ \mathrm{Im}\,\tilde{\Sigma}_0$$

and

$$\tilde{D}^s_{p,\lambda}\,|\,\mathrm{Ker}\,\tilde{\Sigma}_0 \equiv 1+(-1)^{p(p-1)/2}\lambda^p d_1 \cdots d_{2p} \quad \mathrm{mod}\ \mathrm{Im}\,\tilde{\Delta}_0\,.$$

More generally we obtain by an induction on n that

$$(2.11)\quad \prod_{1\leq k\leq n} D^{s_k}_{p,\lambda}|\operatorname{Ker}\Delta_0\equiv 1 \qquad \operatorname{mod}\operatorname{Im}\Sigma_0\,,$$
$$\prod_{1\leq k\leq n} D^{s_k}_{p,\lambda}|\operatorname{Ker}\Sigma_0\equiv 1 \qquad \operatorname{mod}\operatorname{Im}\Delta_0$$

and

$$(2.12)\quad \prod_{1\leq k\leq n}\tilde{D}^{s_k}_{p,\lambda}|\operatorname{Ker}\tilde{\Delta}_0\equiv 1+(-1)^{p(p-1)/2}n\cdot\lambda^p d_1\cdots d_{2p}\equiv \tilde{D}^0_{p,n\lambda}|\operatorname{Ker}\tilde{\Delta}_0 \quad \operatorname{mod}\operatorname{Im}\tilde{\Sigma}_0\,,$$
$$\prod_{1\leq k\leq n}\tilde{D}^{s_k}_{p,\lambda}|\operatorname{Ker}\tilde{\Sigma}_0\equiv D^0_{p,n\lambda}|\operatorname{Ker}\tilde{\Sigma}_0 \qquad \operatorname{mod}\operatorname{Im}\tilde{\Delta}_0\,.$$

3. Throughout this section we suppose p is odd. For $\lambda\in K$ we define another element $\mu=\mu(\lambda)\in K$ by

$$\mu=\mu(\lambda)=\lambda/2\,.$$

Let A be a differential algebra (or coalgebra). We define another structure of differential algebra (or coalgebra) on A by endowing with multiplication ${}_\mu\varphi=\varphi(1+\mu d\sigma\otimes d)$ (or comultiplication ${}_\mu\psi=(1-\mu d\sigma\otimes d)\psi$) where σ is the canonical involution [1], (1.1). Denote this by μA. Then we have

(3.1) **Lemma.** i) A *is associative or λ-commutative if and only if μA is associative or commutative,*
ii) ${}_\mu\varphi_n^{w_n}=\varphi_n^{w_n}\prod_{1\leq j<k\leq n+1}(1+\mu d_jd_k)$ (or ${}_\mu\psi_n^{w_n}=\prod_{1\leq j<k\leq n+1}(1-\mu d_jd_k)\psi_n^{w_n}$) *for each $w_n\in W_n$, the set* (1.7) *of* [1],
iii) $F^n(\mu A)=F^n(A)$ (*or* $G^n(\mu A)=G^n(A)$) *for all* $n\geq 1$.

Proof. First we prove ii) by an induction on n. In case $n=1$ it is the definition that ${}_\mu\varphi=\varphi(1+\mu d_1d_2)$ (or ${}_\mu\psi=(1-\mu d_1d_2)\psi$). As in [1], (1.18) we can express as $w_n=(1, w_s, s+1+w_{n-s-1})$ for some s, $0\leq s<n$. Then

$$\begin{aligned}
{}_\mu\varphi_n^{w_n} &= {}_\mu\varphi({}_\mu\varphi_s^{w_s}\otimes{}_\mu\varphi_{n-s-1}^{w_{n-s-1}})\\
&=\varphi(1+\mu d_1d_2)(\varphi_s^{w_s}\otimes\varphi_{n-s-1}^{w_{n-s-1}})\prod_{1\leq j<k\leq s+1}(1+\mu d_jd_k)\prod_{1\leq i<h\leq n-s}(1+\mu d_{s+i+1}d_{s+h+1})\\
&=\varphi_n^{w_n}\prod_{1\leq r\leq s+1,\,1\leq t\leq n-s}(1+\mu d_rd_{s+t+1})\prod_{1\leq j<k\leq s+1}(1+\mu d_jd_k)\\
&\qquad\prod_{1\leq i<h\leq n-s}(1+\mu d_{s+i+1}d_{s+h+1})=\varphi_n^{w_n}\prod_{1\leq j<k\leq n+1}(1+\mu d_jd_k)
\end{aligned}$$
$$\left(\text{or}\quad {}_\mu\psi_n^{w_n}=\prod_{1\leq j<k\leq n+1}(1-\mu d_jd_k)\psi_n^{w_n}\right),$$

where we apply induction hypotheses to s and $n-s-1$.

It follows immediately from ii) and [1], (1.8) (or (1.8*)) that $F^n(\mu A)\subset F^n(A)$ (or $G^n(A)\subset G^n(\mu A)$). On the other hand,

$$\varphi_n^{w_n}={}_\mu\varphi_n^{w_n}\prod_{1\leq j<k\leq n+1}(1-\mu d_jd_k)\ \ (\text{or } \psi_n^{w_n}=\prod_{1\leq j<k\leq n+1}(1+\mu d_jd_k){}_\mu\psi_n^{w_n})\,,$$

hence $F^n(A)\subset F^n(\mu A)$ (or $G^n(\mu A)\subset G^n(A)$). Thus

$$F^n(A)=F^n(\mu A) \text{ (or } G^n(A)=G^n(\mu A)) .$$

i) is obvious by ii) and [1], (3.3). q.e.d.

$\Phi_\lambda(A)$ and $\Phi_0(\mu A)$ (or $\Psi_\lambda(A)$ and $\Psi_0(\mu A)$) become differential algebras (or coalgebras) which have multiplications $\Phi_\lambda(\varphi)$ and $\Phi_0({}_\mu\varphi)$ (or comultiplications $\Psi_\lambda(\psi)$ and $\Psi_0({}_\mu\psi)$) induced by $\varphi_\lambda=\varphi^{\otimes p}U_{p,\lambda}^{-1}$ and ${}_\mu\varphi^{\otimes p}U_p^{-1}$ (or $\psi_\lambda=U_{p,\lambda}\psi^{\otimes p}$ and $U_{p\mu}\psi^{\otimes p}$) respectively. Remark that differentials of them are trivial, [1], (5.12).

Here we obtain the following relationship between $\Phi_\lambda(A)$ and $\Phi_0(\mu A)$ (or $\Psi_\lambda(A)$ and $\Psi_0(\mu A)$).

(3.2) **Proposition.** *The map $B_{p,\lambda}$ induces a natural isomorphism*

$$\Phi(B_{p,\lambda}):\Phi_0(\mu A)\rightarrow\Phi_\lambda(A) \quad (\text{or } \Psi(B_{p,\lambda}):\Psi_0(\mu A)\rightarrow\Psi_\lambda(A))$$

of differential algebras (or coalgebras).

The above proposition follows immediately from the following

(3.3) **Lemma.**

$$\varphi^{\otimes p}U_{p,\lambda}^{-1}(B_{p,\lambda}\otimes B_{p,\lambda})\,|\,\mathrm{Ker}\,\Delta_0\otimes\mathrm{Ker}\,\Delta_0\equiv B_{p,\lambda}\,{}_\mu\varphi^{\otimes p}U_p^{-1}\,|\,\mathrm{Ker}\,\Delta_0\otimes\mathrm{Ker}\,\Delta_0 \mod \mathrm{Im}\,\Sigma_\lambda$$

(or

$$(B_{p,\lambda}\otimes B_{p,\lambda})U_{p\mu}\psi^{\otimes p}\,|\,\mathrm{Ker}\,\Sigma_0\equiv U_{p,\lambda}\psi^{\otimes p}B_{p,\lambda}\,|\,\mathrm{Ker}\,\Sigma_0 \mod (A^{\otimes p})_\lambda\otimes\mathrm{Im}\,\Delta_\lambda+\mathrm{Im}\,\Delta_\lambda\otimes(A^{\otimes p})_\lambda) .$$

Proof. The case of algebras: Using (1.1), (1.2) and (1.6) we compute

$$\begin{aligned}
&\varphi^{\otimes p}U_{p,\lambda}^{-1}(B_{p,\lambda}\otimes B_{p,\lambda})\\
&=\varphi^{\otimes p}U_p^{-1}(\textstyle\prod_{k-j\leq(p-1)/2}(1-\lambda d_k d_{p+j})\\
&\qquad\textstyle\prod_{k-j\geq(p+1)/2}(1-\lambda d_k d_{p+j})(1+\lambda d_j d_k)(1+\lambda d_{p+j}d_{p+k}))\\
&=\varphi^{\otimes p}U_p^{-1}\textstyle\prod_{k-j\leq(p-1)/2}(1-\lambda d_k d_{p+j})\\
&\qquad\textstyle\prod_{k-j\geq(p+1)/2}(1+\lambda(d_j+d_{p+j})(d_k+d_{p+k}))(1-\lambda d_j d_{p+k}))\\
&=B_{p,\lambda}\varphi^{\otimes p}U_p^{-1}\textstyle\prod_{k-j\leq(p-1)/2}(1-\lambda d_k d_{p+j})\prod_{k-j\geq(p+1)/2}(1-\lambda d_j d_{p+k})
\end{aligned}$$

where $\prod$ runs over $1\leq j<k\leq \mathrm{p}$. By (2.7) we note that

$$\begin{aligned}
&\textstyle\prod_{1\leq j<k\leq p}(\prod_{k-j\geq(p+1)/2}(1-\lambda d_j d_{p+k})\prod_{k-j\leq(p-1)/2}(1-\lambda d_k d_{p+j}))\\
&=\textstyle\prod_{1\leq s\leq(p-1)/2}(\prod_{1\leq j\leq s}(1-\lambda d_j d_{2p-s+j})\prod_{1\leq j\leq p-s}(1-\lambda d_{s+j}d_{p+j}))\\
&=\textstyle\prod_{1\leq s\leq(p-1)/2}\tilde{D}_{p,-\lambda}^{p-s} .
\end{aligned}$$

Consequently we obtain

$$\varphi^{\otimes p}U_{p,\lambda}^{-1}(B_{p,\lambda}\otimes B_{p,\lambda}) = B_{p,\lambda}\varphi^{\otimes p}U_p^{-1}\prod_{1\leq s\leq (p-1)/2}\tilde{D}_{p,-\lambda}^{p-s}\,.$$

On the other hand we obtain

$$_{\mu}\varphi^{\otimes p}U_p^{-1} = \varphi^{\otimes p}U_p^{-1}\prod_{1\leq j\leq p}(1+\mu d_j d_{p+j})=\varphi^{\otimes p}U_p^{-1}\tilde{D}_{p,\mu}^{0}\,.$$

Hence, by making use of (1.4) and (2.12) we have

$$\begin{aligned}&\varphi^{\otimes p}U_{p,\lambda}^{-1}(B_{p,\lambda}\otimes B_{p,\lambda})\,|\,\mathrm{Ker}\,\Delta_0\otimes\mathrm{Ker}\,\Delta_0\\ &=B_{p,\lambda}\varphi^{\otimes p}U_p^{-1}\prod_{1\leq s\leq (p-1)/2}\tilde{D}_{p,-\lambda}^{p-s}\,|\,\mathrm{Ker}\,\Delta_0\otimes\mathrm{Ker}\,\Delta_0\\ &\equiv B_{p,\lambda}\varphi^{\otimes p}U_p^{-1}\tilde{D}_{p,-((p-1)/2)\lambda}^{0}\,|\,\mathrm{Ker}\,\Delta_0\otimes\mathrm{Ker}\,\Delta_0\\ &\equiv B_{p,\lambda\mu}\varphi^{\otimes p}U_p^{-1}\,|\,\mathrm{Ker}\,\Delta_0\otimes\mathrm{Ker}\,\Delta_0 \qquad \mathrm{mod}\ \mathrm{Im}\,\Sigma_\lambda\end{aligned}$$

because $\mathrm{Ker}\,\Delta_0\otimes\mathrm{Ker}\,\Delta_0\subset\mathrm{Ker}\,\tilde{\Delta}_0$.

In case of coalgebras, by the same argument as above we obtain

$$U_{p,\lambda}\psi^{\otimes p}B_{p,\lambda}=(B_{p,\lambda}\otimes B_{p,\lambda})\prod_{1\leq s\leq (p-1)/2}\tilde{D}_{p,\lambda}^{p-s}U_p\psi^{\otimes p}$$

and

$$U_{p\mu}\psi^{\otimes p} = \tilde{D}_{p,-\mu}^{0}U_p\psi^{\otimes p}\,.$$

Here, from (1.4) and (2.12) follows the conclusion immediately. q.e.d.

Finally we discuss (d, λ)-Hopf algebras. Let A be a quasi (d, λ)-Hopf algebra. We can identify $\Phi_\lambda(A)$ with $\Psi_\lambda(A)$ by the canonical isomorphism κ, [1], (5.11). Then $\Phi_\lambda(A)=\Psi_\lambda(A)$ becomes a quasi Hopf algebra, called the derived quasi Hopf algebra of A, which has multiplication $\Phi_\lambda(\varphi)$ and comultiplication $\Psi_\lambda(\psi)$, [1], (5.16). On the other hand, we introduce another structure of differential quasi Hopf algebra on A, denoted by μA, which has multiplication $_{\mu}\varphi=\varphi(1+\mu d\sigma\otimes d)$ and comultiplication $_{\mu}\psi=(1-\mu d\sigma\otimes d)\psi$. Identifying $\Phi_0(\mu A)$ with $\Psi_0(\mu A)$ by the canonical isomorphism, $\Phi_0(\mu A)=\Psi_0(\mu A)$ gains a structure of quasi Hopf algebra with multiplication $\Phi_0(_{\mu}\varphi)$ and comultiplication $\Psi_0(_{\mu}\psi)$.

Applying (3.2) to a quasi (d, λ)-Hopf algebra A, we obtain

(3.4) **Proposition.** *The map $B_{p,\lambda}$ induces a natural isomorphism*

$$\Phi(B_{p,\lambda}) : \Phi_0(\mu A) \to \Phi_\lambda(A)$$

of quasi derived Hopf algebras.

4. Let L be an extension field of K. We regard L as a G_2-module over K by $L_0=L$ and $L_1=\{0\}$. Let A be a differential algebra (or coalgebra). We can regard $L\underset{K}{\otimes}A$ as a differential algebra (or coalgebra) over L equipped with multiplication

$$(L\underset{K}{\otimes}A)\underset{L}{\otimes}(L\underset{K}{\otimes}A)\cong L\underset{K}{\otimes}(A\underset{K}{\otimes}A)\xrightarrow{1\otimes\varphi}L\underset{K}{\otimes}A$$

(or comultiplication

$$L\underset{K}{\otimes}A\xrightarrow{1\otimes\psi}L\underset{K}{\otimes}(A\underset{K}{\otimes}A)\cong(L\underset{K}{\otimes}A)\underset{L}{\otimes}(L\underset{K}{\otimes}A)).$$

(4.1) **Lemma.** $L\underset{K}{\otimes}F^nA=F^n(L\underset{K}{\otimes}A)$ *for all* $n\geqq 0$

(*or* $L\underset{K}{\otimes}G^nA=G^n(L\underset{K}{\otimes}A)$ *for all* $n\geqq 0$).

The proof is obvious in case of algebras, and can be given by a choice of homogeneous bases of L and A as modules in case of coalgebras.

(4.2) **Lemma.** $Q^n(L\underset{K}{\otimes}A)=L\underset{K}{\otimes}Q^nA$ *for all* $n\geqq 0$

(*or* $P^n(L\underset{K}{\otimes}A)=L\underset{K}{\otimes}P^nA$ *for all* $n\geqq 0$).

Proof. $L\underset{K}{\otimes}$ is an exact functor. Therefore the lemma follows from (4.1).

(4.3) **Proposition.** *A is semi-connected if and only if $L\underset{K}{\otimes}A$ is so.*

Proof. The case of algebras: First suppose that $L\underset{K}{\otimes}A$ is semi-connected, i.e., $\cap_{n\geqq 1}F^n(L\underset{K}{\otimes}A)=\{0\}$ [1], **1.8.**. Take any $x\in\cap_{n\geqq 1}F^nA$, then (4.1) implies

$$1\otimes x\in\cap_{n\geqq 1}F^n(L\underset{K}{\otimes}A).$$

Hence A is semi-connected.

Conversely, suppose that A is semi-connected. Take any $y\in\cap_{n\geqq 1}F^n(L\underset{K}{\otimes}A)$. Choosing a homogeneous basis $T=\{x_\iota\}_{\iota\in J}$ of A, we may put $y=\sum_{1\leqq j\leqq n}l_j\otimes x_j$ where $l_j\in L$ and $x_j\in T$. Since A is semi-connected there exists an integer $m>0$ such that

$$K\{x_1,\cdots,x_n\}\cap F^mA=\{0\}$$

where $K\{x_1,\cdots,x_n\}$ denotes the submodule of A generated by $x_1,\cdots,x_n$. Moreover this means by (4.1) that

$$L\underset{K}{\otimes}K\ \{x_1,\cdots,x_n\}\cap F^m(L\underset{K}{\otimes}A)=L\underset{K}{\otimes}(K\{x_1,\cdots,x_n\}\cap F^mA)=\{0\}$$

for some $m>0$. However

$$y \in L \otimes_K K\{x_1, \cdots, x_n\} \cap F^m(L \otimes_K A), \text{ hence } y = 0.$$

Therefore $L \otimes_K A$ is semi-connected.

The case of coalgebras can be proved by a routine discussion. q.e.d.

Let A be a quasi (d, λ)-Hopf algebra. Then $L \otimes_K A$ becomes a quasi (d, λ)-Hopf algebra over L where $\lambda = \lambda \otimes 1 \in L = K \otimes_K L$.

(4.4) **Proposition.** *A is coprimitive (or primitive) if and only if $L \otimes_K A$ is so.*

Proof. From (4.1) and (4.2) it follows that

$$P(L \otimes_K A) \cap F^2(L \otimes_K A) = L \otimes_K (P(A) \cap F^2 A)$$

and

$$P(L \otimes_K A) + F^2(L \otimes_K A) = L \otimes_K (P(A) + F^2 A).$$

These prove the proposition.

5. Let K^p be the subfield of K generated by elements k^p, $k \in K$ and $\theta_K : K \to K$ be the monomorphism defined by $\theta_K(k) = k^p$. $\theta_K(K) = K^p$. Let M and N be modules. We say that a map $\theta : M \to N$ is *θ_K-linear* if

$$\theta(kx) = \theta_K(k)\theta(x) \quad \text{and} \quad \theta(x+y) = \theta(x) + \theta(y)$$

for all $x, y \in M$ and all $k \in K$. If $\theta : M \to N$ is θ_K-linear, then $\theta(M)$ is a module over K^p. In particular we say that a θ_K-linear map $\theta : M \to N$ is a *θ_K-isomorphism* if θ is injective and $N = K \otimes_{K^p} \theta(M)$. Remark that a θ_K-isomorphism θ is bijective if K is a perfect field.

Let A be a differential algebra (or coalgebra).

Define a map $\varepsilon_m : A^{\otimes m} \to A^{\otimes m}$ by

$$\varepsilon_m(x_1 \cdots x_m) = \begin{cases} (-1)^n x_1 \otimes \cdots \otimes x_m & p \equiv 3 \bmod 4 \\ x_1 \otimes \cdots \otimes x_m & \text{others} \end{cases}$$

where $n = \sum_{1 \leq i < j \leq m} \sigma(x_i)\sigma(x_j)$ and σ is the canonical involution [1], (1.1). By an induction on m we have the following relation

(5.1) $(\varphi\varepsilon_2)^{w_m}_m = \varphi^{w_m}_m \varepsilon_{m+1}$ (or $(\varepsilon_2\psi)^{w_m}_m = \varepsilon_{m+1}\psi^{w_m}_m$) *for each $w_n \in W_n$.*

The diagonal map $\Delta : A \to A^{\otimes p}$, $\Delta(x) = x^{\otimes p}$ for a homogeneous element $x \in A$, induces a map

(5.2) $\theta_p : A \to \Phi_0(A)$(or $\Psi_0(A)$).

(5.3) **Lemma.** *The above map θ_p satisfies the following properties :*

i) *θ_p is a θ_K-isomorphism,*

ii) *"multiplicative up to signs"*, i.e.,

$$\theta_p \varphi_n^{w_n} \varepsilon_{n+1} = \Phi_0(\varphi)_n^{w_n} \theta_p^{\otimes n+1} \qquad \textit{for each } w_n \in W_n$$

(or ii)* *"comultiplicative up to signs"*, i.e.,

$$\theta_p^{\otimes n+1} \varepsilon_{n+1} \psi_n^{w_n} = \Psi_0(\psi)_n^{w_n} \theta_p \qquad \text{for each } w_n \in W_n) ,$$

iii) *compatible with η and ε*, i.e.,

$$\theta_p \eta = \eta \theta_K \quad \text{and} \quad \varepsilon \theta_p = \theta_K \varepsilon, \quad \text{and}$$

iv) *natural*, i.e.,

$$\theta_p f = \Phi_0(f) \theta_p \text{ (or } \Psi_0(f) \theta_p)$$

for any morphism $f : A \to B$ of algebras (or coalgebras).

Proof. θ_p is θ_K-linear because $(kx)^{\otimes p} = k^p x^{\otimes p}$ and $(x+y)^{\otimes p} \equiv x^{\otimes p} + y^{\otimes p}$ mod $\operatorname{Im} \Sigma_0$. Choosing a homogeneous basis $T = \{x_\iota\}_{\iota \in I}$ of A, we see by [1], **5.3.** that $\Phi_0(A) \cong \Psi_0(A)$ is generated by $\{x_\iota^{\otimes p} ; x_\iota \in T\}$. Hence θ_p is injective and $K \underset{K^p}{\otimes} \theta_p(A) = \Phi_0(A)$ (or $\Psi_0(A)$). Thus θ_p is a θ_K-isomorphism. Since iii) and iv) are obvious by the definition of θ_p it remains to prove ii) and ii)*.

Remark that $U_p(x \otimes y)^{\otimes p} = \varepsilon_2(x^{\otimes p} \otimes y^{\otimes p})$. Then we obtain

$$\varphi^{\otimes p} U_p^{-1}(x^{\otimes p} \otimes y^{\otimes p}) = (\varphi \varepsilon_2(x \otimes y))^{\otimes p}$$

and

$$U_p \psi^{\otimes p}(x^{\otimes p}) = U_p(\textstyle\sum_i x_i \otimes x_i')^{\otimes p} \equiv U_p(\sum_i (x_i \otimes x_i')^{\otimes p}) \equiv \sum_i \varepsilon_2(x_i^{\otimes p} \otimes x_i'^{\otimes p}) \quad \text{mod } \operatorname{Im} \tilde{\Sigma}_0 ,$$

where $\psi(x) = \sum_i x_i \otimes x_i'$, and

$$\operatorname{Im} \tilde{\Sigma}_0 \subset \operatorname{Im} \Delta_0 \otimes (A^{\otimes p}) + (A^{\otimes p}) \otimes \operatorname{Im} \Delta_0 .$$

Thus

$$\Phi_0(\varphi) \theta_p \otimes \theta_p = \theta_p(\varphi \varepsilon_2) \quad \text{and} \quad \Psi_0(\psi) \theta_p = \theta_p \otimes \theta_p(\varepsilon_2 \psi) .$$

Using an induction on n we can easily verify that

$$\Phi_0(\varphi)_n^{w_n} \theta_p^{\otimes n+1} = \theta_p(\varphi \varepsilon_2)_n^{w_n} \quad \text{and} \quad \Psi_0(\psi)_n^{w_n} \theta_p = \theta_p^{\otimes n+1}(\varepsilon_2 \psi)_n^{w_n}$$

for all $w_n \in W_n$. Now by (5.1) we obtain ii) and ii)*. q.e.d.

Since θ_p is multiplicative (or comultiplicative) up to signs $\theta_p(A)$ becomes an algebra (or coalgebra) over K^p with multiplication (or comultiplication) induced by that of $\Phi_0(A)$ (or $\Psi_0(A)$). We see by (5.3) that

(5.4) $\theta_p(F^n A) = F^n(\theta_p(A))$ (*or* $\theta_p(G^n A) = G^n(\theta_p(A))$) *for all* $n \geqq 0$.

6. Here we consider a similar map to (5.2) when $p=2$ and $\lambda \neq 0$. Suppose $p=2$ and $\lambda \neq 0$. The diagonal map $\Delta : Z(A) \to Z(A) \otimes Z(A)$ given by $\Delta(x) = x \otimes x$, induces a map

(6.1) $\theta_{2,\lambda} : H(A) \to \Phi_\lambda(A)$ (or $\Psi_\lambda(A)$).

(6.2) **Lemma.** *The above map* $\theta_{2,\lambda}$ *satisfies the following properties* :
i) $\theta_{2,\lambda}$ *is a* θ_K*-isomorphism,*
ii) "*multiplicative*", i.e.,

$$\theta_{2,\lambda} H(\varphi)_n^{wn} = \Phi_\lambda(\varphi)_n^{wn} \theta_{2,\lambda}^{\otimes n+1}$$

(*or* ii)* "*comultiplicative*", i.e.,

$$\theta_{2,\lambda}^{\otimes n+1} H(\psi)_n^{wn} = \Psi_\lambda(\psi)_n^{wn} \theta_{2,\lambda}) ,$$

iii) *compatible with* η *and* ε, i.e.,

$$\theta_{2,\lambda}\eta = \eta\theta_K \quad \textit{and} \quad \varepsilon\theta_{2,\lambda} = \theta_K \varepsilon, \quad \text{and}$$

iv) *natural*, i.e.,

$$\theta_{2,\lambda} H(f) = \Phi_\lambda(f)\theta_{2,\lambda} \text{ (or } \Psi_\lambda(f)\theta_{2,\lambda})$$

for any morphism $f : A \to B$ *of differential algebras* (*or coalgebras*).

Proof. Choose a d-stable homogeneous basis $\{x_\iota, dx_\iota, y_\kappa\}_{\iota \in I, \kappa \in J}$ of A where $dy_\kappa = 0$. Then $\Phi_\lambda(A) = \Psi_\lambda(A)$ is generated by $\{y_\kappa^{\otimes 2}\}_{\kappa \in J}$, [1], (5.8) and (5.9.2). Hence proofs are easy except ii)*.

As is well known

(6.3) $\psi(Z(A)) \subset Z(A) \otimes Z(A) + d(A \otimes A)$.

Put

$$\psi(x) = \textstyle\sum_i z_i \otimes z_i' + \sum_k (du_k \otimes u_k' + u_k \otimes du_k')$$

for $x \in Z(A)$ where z_i, $z_i' \in Z(A)$. Routine computations show :

$$(1 \otimes T_\lambda \otimes 1)(\textstyle\sum_{i,j} z_i \otimes z_i' \otimes z_j \otimes z_j') \equiv \sum_i z_i^{\otimes 2} \otimes z_i'^{\otimes 2} ,$$
$$(1 \otimes T_\lambda \otimes 1)(\textstyle\sum_{i,k} (z_i \otimes z_i' \otimes u_k \otimes du_k' + u_k \otimes du_k' \otimes z_i \otimes z_i')) \equiv 0 ,$$
$$(1 \otimes T_\lambda \otimes 1)(\textstyle\sum_{i,k} (z_i \otimes z_i' \otimes du_k \otimes u_k' + du_k \otimes u_k' \otimes z_i \otimes z_i')) \equiv 0 ,$$
$$(1 \otimes T_\lambda \otimes 1)(\textstyle\sum_{k,l} (u_k \otimes du_k' \otimes u_l \otimes du_l' + du_k \otimes u_k' \otimes du_l \otimes u_l'))$$

$$\equiv \sum_k (u_k^{\otimes 2} \otimes (du'_k)^{\otimes 2} + (du_k)^{\otimes 2} \otimes u'^{\otimes 2}_k),$$
$$(1 \otimes T_\lambda \otimes 1)(\sum_{k,l}(u_k \otimes du'_k \otimes du_l \otimes u'_l + du_k \otimes u'_k \otimes u_l \otimes du'_l)) \equiv 0$$
$$\mod (A^{\otimes 2})_\lambda \otimes \operatorname{Im} \Delta_\lambda + \operatorname{Im} \Delta_\lambda \otimes (A^{\otimes 2})_\lambda .$$

Therefore we obtain that

$$(6.4) \quad (1 \otimes T_\lambda \otimes 1)\psi(x)^{\otimes 2} \equiv \sum_i z_i^{\otimes 2} \otimes z'^{\otimes 2}_i + \sum_k ((du_k)^{\otimes 2} \otimes u'^{\otimes 2}_k + u_k^{\otimes 2} \otimes (du'_k)^{\otimes 2})$$
$$\mod (A^{\otimes 2})_\lambda \otimes \operatorname{Im} \Delta_\lambda + \operatorname{Im} \Delta_\lambda \otimes (A^{\otimes 2})_\lambda .$$

Thus we have

$$\Psi_\lambda(\psi)\theta_{2,\lambda} = (\theta_{2,\lambda} \otimes \theta_{2,\lambda})H(\psi) .$$

General case is obtained immediately by an induction on n. q.e.d.

(6.2) means that $\theta_{2,\lambda} : H(A) \to \Phi_\lambda(A)$ (or $\Psi_\lambda(A)$) is a *θ_K-isomorphism of algebras (or coalgebras)*. Hence $\theta_{2,\lambda}(H(A))$ is an algebra (or coalgebra) over K^2 with multiplication (or comultiplication) induced by that of $\Phi_\lambda(A)$(or $\Psi_\lambda(A)$). And we see by (6.2) that

$$(6.5) \quad \theta_{2,\lambda}(F^n H(A)) = F^n(\theta_{2,\lambda}(H(A))) \qquad \text{for all } n \geqq 0$$
$$(\text{or} \quad \theta_{2,\lambda}(G^n H(A)) = G^n(\theta_{2,\lambda}(H(A))) \qquad \text{for all } n \geqq 0) .$$

7. Now we study properties of $\Phi_\lambda(A)$ (or $\Psi_\lambda(A)$) making use of maps θ_p and $\theta_{2,\lambda}$.

First we examine semi-connectedness of an algebra $\Phi_\lambda(A)$ (or coalgebra $\Psi_\lambda(A)$). Putting (3.2), (4.3), (5.3) and (6.2) together we have

(7.1) **Theorem.** *Let A be a differential algebra (or coalgebra) over a field K of characteristic $p \neq 0$ and $\lambda \in K$.*

i) *When p is odd or $p=2$ and $\lambda d=0$, A is semi-connected if and only if $\Phi_\lambda(A)$ (or $\Psi_\lambda(A)$) is so.*

ii) *When $p=2$ and $\lambda \neq 0$, $H(A)$ is semi-connected if and only if $\Phi_\lambda(A)$ (or $\Psi_\lambda(A)$) is so.*

Proof. First we shall prove the theorem in case $\lambda d=0$. Remark that $\Phi_\lambda(A)=\Phi_0(A)$ (or $\Psi_\lambda(A)=\Psi_0(A)$) in case $\lambda d=0$. By (5.4) and the injectivity of θ_p, A is semi-connected if and only if $\theta_p(A)$ is so. Since multiplication (or comultiplication) of $\Phi_0(A)$ (or $\Psi_0(A)$) induces that of $\theta_p(A)$, $K \underset{K^p}{\otimes} \theta_p(A)$ coincides with $\Phi_0(A)$ as an algebra (or coalgebra). Now (4.3) proves the theorem in case $\lambda d=0$.

Similarly (4.3), (6.2) and (6.5) prove that the theorem is true in case $p=2$ and $\lambda \neq 0$.

In case p odd we prove the theorem, combining the theorem in case $\lambda=0$

with (3.1) and (3.2). q.e.d.

Next we examine coprimitivity and primitivity of the derived Hopf algebra $\Phi_\lambda(A)=\Psi_\lambda(A)$.

(7.2) **Theorem.** *Let A be a quasi (d, λ)-Hopf algebra over a field K of characteristic $p\neq 0$.*

i) *When p is odd or $p=2$ and $\lambda d=0$, A is coprimitive (or primitive) if and only if $\Phi_\lambda(A)=\Psi_\lambda(A)$ is so.*

ii) *When $p=2$ and $\lambda\neq 0$, $H(A)$ is coprimitive (or primitive) if and only if $\Phi_\lambda(A)=\Psi_\lambda(A)$ is so.*

Proof. Making use of (3.4), (4.4), (5.3), (5.4), (6.2) and (6.5) the theorem is proved in a parallel way to (7.1).

8. Let A be a differential algebra (or coalgebra) which is associative and λ-commutative. Suppose p is odd and $\mu=\lambda/2\in K$. By (3.1) i) $_\mu A$ is associative and commutative. Therefore we can consider maps

$$\xi_\lambda : \Phi_\lambda(A)\to A \quad \text{and} \quad \xi_0 : \Phi_0({}_\mu A)\to {}_\mu A$$

(or

$$\eta_\lambda : A\to \Psi_\lambda(A) \quad \text{and} \quad \eta_0 : A\to \Psi_0({}_\mu A))$$

induced by φ_{p-1} and ${}_\mu\varphi_{p-1}$ (or ψ_{p-1} and ${}_\mu\psi_{p-1}$) respectively [1], **6.3.**. ξ_λ and ξ_0 (or η_λ and η_0) become morphisms of differential algebras (or coalgebras) by the λ-commutativity of A and the commutativity of $_\mu A$.

(8.1) **Proposition.** *The following diagram*

$$\begin{array}{ccc} \Phi_0({}_\mu A) & \xrightarrow{\xi_0} & {}_\mu A \\ {\scriptstyle \Phi(B_{p,\lambda})}\downarrow & & \| \\ \Phi_\lambda(A) & \xrightarrow[\xi_\lambda]{} & A \end{array} \qquad \left(\text{or}\ \begin{array}{ccc} {}_\mu A & \xrightarrow{\eta_0} & \Psi_0({}_\mu A) \\ \| & & \downarrow{\scriptstyle \Psi(B_{p,\lambda})} \\ A & \xrightarrow[\eta_\lambda]{} & \Psi_\lambda(A) \end{array}\right)$$

is commutative.

Proof. The case of algebras: It is sufficient to show that

$$\varphi_{p-1}B_{p,\lambda}\,|\,\mathrm{Ker}\,\Delta_0=\varphi_{p-1}\textstyle\prod_{1\leqq j<k\leqq p}(1+\mu d_j d_k)\,|\,\mathrm{Ker}\,\Delta_0$$

because ${}_\mu\varphi_{p-1}=\varphi_{p-1}\prod_{1\leqq j<k\leqq p}(1+\mu d_j d_k)$ by (3.1). Using (1.1), (1.2) and (2.7) we compute

(8.2) $B_{p,-\lambda}\prod_{1\leqq j<k\leqq p}(1+\mu d_j d_k)$
$\quad = \prod_{1\leqq j<k\leqq p, k-j\geqq (p+1)/2}(1-2\mu d_j d_k)\prod_{1\leqq j<k\leqq p}(1+\mu d_j d_k)$

$$=\prod_{1\leqq j<k\leqq p}(\prod_{k-j\leqq(p-1)/2}(1+\mu d_j d_k)\prod_{k-j\geqq(p+1)/2}(1-\mu d_j d_k))$$
$$=\prod_{1\leqq s\leqq(p-1)/2}(\prod_{1\leqq j\leqq p-s}(1+\mu d_j d_{s+j})\prod_{1\leqq j\leqq s}(1+\mu d_{p-s+j}d_j))$$
$$=\prod_{1\leqq s\leqq(p-1)/2}D^s_{p,\mu}\,.$$

Making use of (1.4) and (2.11) it follows that

$$\begin{aligned}&\varphi_{p-1}(\prod_{1\leqq j<k\leqq p}(1+\mu d_j d_k)-B_{p,\lambda})(\mathrm{Ker}\,\Delta_0)\\ &=\varphi_{p-1}B_{p,\lambda}(B_{p,-\lambda}\prod_{1\leqq j<k\leqq p}(1+\mu d_j d_k)-1)(\mathrm{Ker}\,\Delta_0)\\ &=\varphi_{p-1}B_{p,\lambda}(\prod_{1\leqq s\leqq(p-1)/2}D^s_{p,\mu}-1)(\mathrm{Ker}\,\Delta_0)\\ &\subset\varphi_{p-1}B_{p,\lambda}(\mathrm{Im}\,\Sigma_0)=0\,.\end{aligned}$$

The case of coalgebras: Since

$$\prod_{1\leqq j<k\leqq p}(1-\mu d_j d_k)B_{p,\lambda}=\prod_{1\leqq s\leqq(p-1)/2}D^s_{p,-\mu}$$

by (8.2) and

$$B_{p,\lambda\mu}\psi_{p-1}-\psi_{p-1}=B_{p,\lambda}(\prod_{1\leqq j<k\leqq p}(1-\mu d_j d_k)B_{p,\lambda}-1)B_{p,-\lambda}\psi_{p-1}$$

by (3.1), we see by (1.4) and (2.11) that

$$\begin{aligned}&\mathrm{Im}(B_{p,\lambda\mu}\psi_{p-1}-\psi_{p-1})\subset B_{p,\lambda}(\prod_{1\leqq s\leqq(p-1)/2}D^s_{p,-\mu}-1)(\mathrm{Ker}\,\Sigma_0)\\ &\subset B_{p,\lambda}(\mathrm{Im}\,\Delta_0)\subset\mathrm{Im}\,\Delta_\lambda\,.\end{aligned}$$

Thus

$$B_{p,\lambda\mu}\psi_{p-1}\equiv\psi_{p-1}\qquad\mathrm{mod}\ \mathrm{Im}\,\Delta_\lambda\,.$$

Hence the proof is complete.

Finally we discuss $\mathrm{Im}\,\xi_\lambda$. Let A be a quasi (d,λ)-Hopf algebra whose multiplication φ is associative and λ-commutative.

Define a map

$$\xi_p : A\to A$$

by $\xi_p(x)=x^p$ [2], **4.19.**, and by A_0 (or A_1) we denote the submodule of A of even (or odd) type.

First suppose p is odd and $\mu=\lambda/2\in K$. We have

(8.3) **Lemma.** *The map ξ_p satisfies the following properties*:
i) *$\xi_p|A_0$ is θ_K-linear,*
ii) *$\xi_p(xy)=\xi_p(x)\xi_p(y)$ for $x, y\in A_0$, and*
iii) *for $x\in A_0$, putting $\psi(x)=\sum_i y_i\otimes y'_i+\sum_j z_j\otimes z'_j$, $y_i, y'_i\in A_0$ and $z_j, z'_j\in A_1$, we obtain*

$$\psi\xi_p(x)=\xi_p\otimes\xi_p(\textstyle\sum_i y_i\otimes y'_i+\mu\sum_j dz_j\otimes dz'_j)\,.$$

Proof. By [1],(6.10) we can easily verify i) and ii).

iii) is proved as follows. By (3.1) i) a differential quasi Hopf algebra μA has associative and commutative multiplication $_\mu\varphi$. Hence, as in classical case, we obtain

$$_\mu\psi\, _\mu\varphi_{p-1}(x^{\otimes p}) = {}_\mu\varphi_{p-1}\otimes{}_\mu\varphi_{p-1}(\sum_i(y_i^{\otimes p}\otimes y_i'^{\otimes p}+\mu^p(dz_j)^{\otimes p}\otimes(dz_j')^{\otimes p}))$$

because we can express as

$$_\mu\psi(x) = \sum_i(y_i\otimes y_i'-\mu(dy_i\otimes dy_i'))+\sum_j(z_j\otimes z_j'+\mu(dz_j\otimes dz_j'))\,.$$

By (3.1) and (8.1) we have

$$(8.4)\quad \varphi_{p-1}B_{p,\lambda}(w^{\otimes p}) = {}_\mu\varphi_{p-1}(w^{\otimes p}) = \begin{cases} w^p & \text{if } w\in A_0 \\ 0 & \text{if } w\in A_1 \end{cases}$$

because by λ-commutativity of φ

$$(dw)^2 = 0 \quad \text{for } w\in A_0 \quad \text{and} \quad {}_\mu\varphi(w\otimes w) = 0 \quad \text{for } w\in A_1$$

[1], (6.9). Therefore we have

$$\begin{aligned}\psi(x^p) &= (1+\mu d\sigma\otimes d)(\sum_i y_i^p\otimes y_i'^p+\mu^p\sum_j(dz_j)^p\otimes(dz_j')^p) \\ &= \sum_i y_i^p\otimes y_i'^p+\mu^p\sum_j(dz_j)^p\otimes(dz_j')^p\end{aligned}$$

using the fact $d(y^p)=0$ for $y\in A_0$, [1], (6.9). q.e.d.

The above lemma says that

(8.5) $K\underset{K^p}{\otimes}\xi_p(A_0)$ *becomes a quasi sub Hopf algebra of A when p is odd.*

Next suppose $p=2$. Then we have

(8.6) **Lemma.** *The map ξ_2 satisfies the following properties*:

i) $\xi_2|\operatorname{Ker}\lambda d$ *is θ_K-linear,*

ii) $\xi_2(xy) = \xi_2(x)\xi_2(y)$ *for $x, y\in \operatorname{Ker}\lambda d$, and*

iii) *for $x\in\operatorname{Ker}\lambda d$, putting $\psi(x)\equiv\sum_i y_i\otimes y_i' \bmod \operatorname{Im}\lambda d$, $y_i\ y_i'\in\operatorname{Ker}\lambda d$, we obtain*

$$\psi\xi_2(x) = \xi_2\otimes\xi_2(\sum_i y_i\otimes y_i')\,.$$

Proof. i) and ii) is obvious by [1], (6.10).

By (6.3) we may put

$$\psi(x) = \sum_i z_i\otimes z_i'+\sum_k(du_k\otimes u_k'+u_k\otimes du_k') \text{ with } z_i, z_i'\in Z(A)\,.$$

Then by (6.4) we get

$$\psi(x^2) = \sum_i z_i^2 \otimes z_i'^2 + \sum_k (du_k)^2 \otimes u_k'^2 + u_k^2 \otimes (du_k')^2) .$$

When $\lambda=0$ this completes the proof of iii). When $\lambda \neq 0$ remark that $(du)^2=0$ by [1], (6.9), hence also the proof is complete.

The above lemma says that

(8.7) $K \otimes_{K^2} \xi_2(\mathrm{Ker}\ \lambda d)$ *becomes a quasi sub Hopf algebra of A when* $p=2$.

On the other hand we know that Im ξ_λ is a quasi sub Hopf algebra of A because $\xi_\lambda : \Phi_\lambda(A) = \Psi_\lambda(A) \to A$ is a morphism of quasi (d, λ)-Hopf algebras and $\Phi_\lambda(A) = \Psi_\lambda(A)$ has a trivial differential, [1], (6.5).

Here we have

(8.8) **Proposition.** i) *When* $p=2$, Im $\xi_\lambda = K \otimes_{K^2} \xi_2(\mathrm{Ker}\ \lambda d)$ *and it is a quasi sub Hopf algebra of A.*

ii) *When p is odd,* Im $\xi_\lambda = K \otimes_{K^p} \xi_p(A_0)$ *and it is a quasi sub Hopf algebra of A.*

Proof. When p is odd, we consider the following composition map

$$\xi_p' : A = \mu A \xrightarrow{\theta_p} \Phi_0(\mu A) \xrightarrow{B_{p,\lambda}} \Phi_\lambda(A) \xrightarrow{\xi_\lambda} A$$

where $\mu = \lambda/2 \in K$. By (3.4) and (5.3) i) we see that Im $\xi_\lambda = K \otimes_{K^p} \xi_p'(A)$. Since (8.4) is equivalent to say that

$$\xi_p' | A_0 = \xi_p | A_0 \quad \text{and} \quad \xi_p' | A_1 = 0 .$$

we get the proposition in case p odd.

Next suppose $p=2$. We see easily that

$$\xi_2 = \xi_\lambda \theta_2 \qquad \text{when } \lambda d = 0 ,$$

and

$$\xi_2 | \mathrm{Ker}\ \lambda d = \xi_\lambda \theta_{2,\lambda} \pi \qquad \text{when } \lambda \neq 0$$

where $\pi : Z(A) \to H(A)$ is the natural projection. Therefore it follows from (5.3) i) and (6.2) i) that

$$\mathrm{Im}\ \xi_\lambda = K \otimes_{K^2} \xi_2(\mathrm{Ker}\ \lambda d) .$$

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