Ikegami, T.
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DIRICHLET PROBLEM ON GREEN LINES RELATED TO THE COMPACTIFICATIONS OF GREEN SPACES

TERUO IKEGAMI

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Introduction

The Dirichlet problem on Green lines was investigated by Brelot-Choquet [3]. In their work, we can find some results concerning the relations between the Dirichlet problem on Green lines and the completion of Green spaces. Some special compactifications, for instance the Kuramochi's, are becoming more and more important to the study of the potential theory, as the recent researches on the compactifications of Green spaces show. It was shown for instance by F-Y. Maeda [12] that almost every Green line tends to a point of the Kuramochi boundary of a Green space. We discuss in this paper an extension of a radial limit of functions on the unit disc in the classical function theory in connection with the family of Dirichlet solutions. It becomes clear in the course of the discussion that the Dirichlet solutions on Green lines play an important role in the study of harmonic functions.

In §1, we discuss the inclusion relations among the families of Dirichlet solutions related to Green lines. In view of the importance of radial, we have to get a family of functions having radials. To do so, we investigate two sorts of modifications; a lattice modification and a convex modification. They will be stated in §2 and in §3 respectively. As an application of our preceding study, we consider in §4 some classes of holomorphic functions on a hyperbolic Riemann surface and establish the unicity theorem of Riesz type. §5 is devoted to the study of Green spaces where all Dirichlet solutions on Green lines are quasi-bounded. Theorem 10 gives a generalization of a theorem of Gårding-Hörmander [9]. In the last section, we show first that in some measure almost every h-Green line tends to the Kuramochi boundary, by using relative notions introduced by L. Lumer-Naïm [13], [14]. Then we discuss the relative Dirichlet problem on h-Green lines in connection with the Kuramochi's compactification.

For the following notions we refer to Brelot-Choquet [3]. Let Ω be a Green space. We fix a point y_0 in Ω and consider the family $\mathcal{L}$ of all Green lines issuing from y_0. On $\mathcal{L}$, we introduce the topology homeomorphic to the unit sphere and the Green measure g such that $g(\mathcal{L})=1$ and $g(e)=\sigma\times$(area of $[e]$), for each Borel set e on $\mathcal{L}$, where $[e]$ is the set of the unit sphere corresponding to e and σ is a constant depending only on the dimension of Ω. We define

$$D^{\lambda}=\{y\in\Omega;\ G_{y_0}(y)>\lambda\}$$

and

$$\sum{}^{\lambda}=\{y\in\Omega;\ G_{y_0}(y)=\lambda\}$$

for $0<\lambda<G_{y_0}(y_0)$, where G_{y_0} is the Green function of Ω with pole at y_0. A Green line l is called a *regular* Green line if $\inf_{y\in l} G_{y_0}(y)=0$. The set of all regular Green lines will be denoted by $\mathcal{L}'$. It is known that $g(\mathcal{L}-\mathcal{L}')=0$.

1. Dirichlet problem on Green lines

1.1. The Dirichlet problem on Green lines was first considered by Brelot-Choquet. For the following definitions and properties we refer to [3]. Let φ be an extended real valued function on the set of all Green lines $\mathcal{L}$, i. e., φ is a mapping from $\mathcal{L}$ into $[-\infty, +\infty]$. We consider the class:

$$\underline{\mathcal{F}}_{\varphi}=\left\{u;\ \begin{array}{l}\text{subharmonic, bounded from above on } \Omega \text{ and}\\ \overline{\lim_{\lambda\to 0}}\, u(\lambda, l)\leq\varphi(l)\quad dg\text{-}a.e.^{1)}\end{array}\right\}\cup\{-\infty\},$$

where $u(\lambda, l)$ denotes the value of u at the point a on l where $G_{y_0}(a)=\lambda$. The lower solution $\underline{\mathcal{G}}_{\varphi}$ is defined to be an upper envelope of $\underline{\mathcal{F}}_{\varphi}$, i.e.,

$$\underline{\mathcal{G}}_{\varphi}(a)=\sup\{u(a);\ u\in\underline{\mathcal{F}}_{\varphi}\}.$$

$\underline{\mathcal{G}}_{\varphi}$ is either harmonic or $\equiv+\infty$ or $\equiv-\infty$. The upper solution is by definition

$$\overline{\mathcal{G}}_{\varphi}=-\underline{\mathcal{G}}_{(-\varphi)}.$$

It is known that $\underline{\mathcal{G}}_{\varphi}\leq\overline{\mathcal{G}}_{\varphi}$ and

$$\underline{\mathcal{G}}_{\varphi}(y_0)\leq\underline{\int}\varphi dg\leq\overline{\int}\varphi dg\leq\overline{\mathcal{G}}_{\varphi}(y_0).$$

If $\underline{\mathcal{G}}_{\varphi}$ and $\overline{\mathcal{G}}_{\varphi}$ are equal and are harmonic, their common harmonic function will be denoted by $\mathcal{G}_{\varphi}$. We denote by $\mathcal{H}$ the set of all Dirichlet solutions $\mathcal{G}_{\varphi}$.

Next, let $\overline{\Omega}$ be a compactification of Ω which is resolutive[2] and possesses the following property: *except a set of Green lines of Green measure zero, each Green line converges to a point of* $\Delta=\overline{\Omega}-\Omega$. For $l\in\mathcal{L}$ converging to a point of Δ, the limit point of l will be denoted by $x(l)$. Given an extended real valued

1) "dg-a.e." means "except a set of Green lines of Green measure zero".

2) For the resolutive compactification, we refer to [4].

function f on Δ, we define a function on $\mathcal{L}$ as follows:

$$[f](l) = \begin{cases} f(x), & \text{if } l \text{ has a limit point } x=x(l), \\ 0, & \text{otherwise.} \end{cases}$$

Then, we have

$$\underline{H}_f = \underline{\mathcal{G}}_{[f]}{}^{3)} .$$

It was shown by F-Y. Maeda [12] that the Kuramochi's compactification Ω^K of Ω possesses the above property. Since the family of all Dirichlet solutions on Ω^K is MHD(Ω)[4] we have

$$\text{MHD}(\Omega) \subset \mathcal{H} .$$

1.2. A slightly more general formulation of the Dirichlet problem on Green lines was also considered in [3]. Let $\{\lambda_n\}$ be a monotone decreasing sequence of positive numbers tending to zero. We consider the family:

$$\overset{\{\lambda_n\}}{\underline{\mathcal{F}}_\varphi} = \left\{ u; \begin{array}{l} \text{subharmonic, bounded from above on } \Omega \text{ and} \\ \overline{\lim\limits_{n\to\infty}}\, u(\lambda_n, l) \leq \varphi(l) \quad dg\text{-}a.e. \end{array} \right\} \cup \{-\infty\} .$$

As above, the lower solution and the upper solution are defined to be

$$\overset{\{\lambda_n\}}{\underline{\mathcal{G}}_\varphi}(a) = \sup\{u(a);\ u \in \overset{\{\lambda_n\}}{\underline{\mathcal{F}}_\varphi}\}$$

and

$$\overset{\{\lambda_n\}}{\overline{\mathcal{G}}_\varphi} = -\overset{\{\lambda_n\}}{\underline{\mathcal{G}}_{(-\varphi)}} ,$$

respectively. If $\overset{\{\lambda_n\}}{\overline{\mathcal{G}}_\varphi}$ and $\overset{\{\lambda_n\}}{\underline{\mathcal{G}}_\varphi}$ are equal and are harmonic, their common harmonic function is denoted by $\overset{\{\lambda_n\}}{\mathcal{G}_\varphi}$. We denote by $\mathcal{H}_{\{\lambda_n\}}$ the set of all solutions $\overset{\{\lambda_n\}}{\mathcal{G}_\varphi}$. Obviously we have

$$\mathcal{H} \subset \mathcal{H}_{\{\lambda_n\}} .$$

1.3. Another formulation of Dirichlet problem on Green lines was given by M. Brelot [1], In [1], he introduced important notions such as radial, weakly minor, indifferent, etc., which will play a fundamental role in our present paper.

A function v on Ω is said to have a *radial* (resp. *majorant radial*) φ if

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3) Cf. [3], th. 30, p. 253.

4) In the case where Ω is a hyperbolic Reimann surface, this is stated in [4], Hilfssatz 16.1, p. 167. The fact, however, can be easily extended to the general case.

$$\overline{\lim_{\lambda\to 0}}\int |v(\lambda, l)-\varphi(l)|\,dg = 0 \text{ (resp. } \overline{\lim_{\lambda\to 0}}\int [v(\lambda, l)-\varphi(l)]^{+}dg^{5)} = 0) .$$

A radial is a function defined on $\mathcal{L}$ dg-almost everywhere. A harmonic (resp. subharmonic) function u on Ω is called *indifferent* (resp. *weakly minor*) if for each $\lambda>0$, $H_u^{D^\lambda}=u$ (resp. $u\leq H_u^{D^\lambda}$), where $H_u^{D^\lambda}$ is a Dirichlet solution on D^λ with boundary value u; more precisely, we consider the one-point compactification Ω^* of Ω with Alexandroff point $\mathcal{A}$. In a formulation of the Dirichlet problem on D^λ, we agree that all topological notions are refered to this topology and a boundary function is extended as

$$\underset{*}{f}(a) = \begin{cases} f(a) & \text{if } a\in\Omega\,, \\ 0 & \text{if } a=\mathcal{A}\,. \end{cases}$$

In the sequel, instead of the Dirichlet solution $H_{\underset{*}{u}}^{D^\lambda}$ we write it simply $H_u^{D^\lambda}$. The existence of the solution $H_u^{D^\lambda}$ is equivalent to the dg-integrability of $\underset{*}{u}(\lambda, l)$ as a function of l, and then, we have $H_u^{D^\lambda}(y_0)=\int u(\lambda,l)dg^{6)}$.

We consider the families:

$$\underline{\mathcal{F}}'_\varphi = \left\{u;\ \begin{matrix}\text{subharmonic on } \Omega, \text{ weakly minor} \\ \text{and has a majorant radial } \varphi\end{matrix}\right\} \cup \{-\infty\}$$

and

$$\overline{\mathcal{F}}'_\varphi = \{-u;\ u\in\underline{\mathcal{F}}'_{(-\varphi)}\}\,.$$

The envelopes

$$\underline{HR}_\varphi(a) = \sup\,\{u(a);\ u\in\underline{\mathcal{F}}'_\varphi\}$$

and

$$\overline{HR}_\varphi(a) = \inf\,\{u(a);\ u\in\overline{\mathcal{F}}'_\varphi\}$$

are either harmonic or $\equiv+\infty$ or $\equiv-\infty$. It is known that

$$\underline{HR}_\varphi \leq \overline{HR}_\varphi = -\underline{HR}_{(-\varphi)}\,.$$

If $\underline{HR}_\varphi$ and $\overline{HR}_\varphi$ are equal and are harmonic, their common harmonic function is called *the Dirichlet solution on Green lines* and will be denoted by HR_φ. It is known that HR_φ is indifferent and has a radial φ. K. Endl [8] discussed the Dirichlet problem on Green lines in this formulation. He called HR_φ the solution for a *principal radial* φ. The set of all Dirichlet solutions on Green lines will be denoted by $\mathcal{K}$.

5) α^+ denotes max $(\alpha, 0)$.
6) Cf. [1], p. 431.

Lemma 1. *If φ is a dg-integrable function on $\mathcal{L}$, then we have*

$$\underline{\mathcal{G}}_\varphi \le \underline{HR}_\varphi \tag{1.1}$$

and

$$\underline{HR}_\varphi(y_0) \le \int \varphi dg\,. \tag{1.2}$$

Proof. Since each member $u \not\equiv -\infty$ of $\underline{\mathcal{F}}_\varphi$ is bounded from above, we have

$$u \le H_u^{D^\lambda}\,,$$

i.e., u is weakly minor. Since $[u(\lambda, l)-\varphi(l)]^+ \le M^+ + |\varphi(l)|$, for every λ and $l \in \mathcal{L}$, where M is an upper bound of u, and since

$$\lim_{\lambda\to 0}[u(\lambda, l)-\varphi(l)]^+ = 0 \qquad dg\text{-}a.e.,$$

we have

$$\lim_{\lambda\to 0}\int[u(\lambda, l)-\varphi(l)]^+ dg = 0\,,$$

i.e., u has a majorant radial φ. Thus $u \in \underline{\mathcal{F}}'_\varphi$, which implies (1.1).

Next, for each $u \in \underline{\mathcal{F}}'_\varphi$ we have

$$\int[u(\lambda, l)-\varphi(l)]^+ dg \ge \int[u(\lambda, l)-\varphi(l)]dg$$
$$= H_u^{D^\lambda}(y_0) - \int\varphi(l)dg \ge u(y_0) - \int\varphi dg\,.$$

By making $\lambda\to 0$, we have $u(y_0) \le \int\varphi dg$, therefore $\underline{HR}_\varphi(y_0) \le \int\varphi dg$, q.e.d..

Since a Dirichlet solution $\mathcal{G}_\varphi$ is associated with dg-integrable φ, we obtain from this lemma relations among the families of Dirichlet solutions relating to Green lines formulated above.

Theorem 1. *We have*

$$\mathcal{H} \subset \mathcal{K} \tag{1.3}$$

and

$$\mathcal{H} \subset \bigcap_{\{\lambda_n\}\in\Lambda} \mathcal{H}_{\{\lambda_n\}} \tag{1.4}$$

where Λ is the family of all monotone sequences of positive numbers tending to zero.

In section 2, we shall see further $\mathcal{K} \subset \bigcup_{\{\lambda_n\}\in\Lambda} \mathcal{H}_{\{\lambda_n\}}$.

Corollary. *If f is a resolutive function on the Kuramochi boundary Δ^K of*

Ω, then we have

$$H_f^K = \mathcal{G}_{[f]} = HR_{[f]}\,,$$

where H_f^K denotes the Dirichlet solution on Ω^K with boundary value f. In particular, the Dirichlet solution H_f^K is indifferent and has a radial $[f]$.

REMARK. Here we state some results on the solutions of $\mathcal{H}_{\{\lambda_n\}}$ which can be easily proved.

1. If $u = \overset{\{\lambda_n\}}{\mathcal{G}_\varphi}$, then we have $u = \overset{\{\lambda_n\}}{\mathcal{G}_{\varphi^+}} - \overset{\{\lambda_n\}}{\mathcal{G}_{\varphi^-}}$, where $\varphi^- = \max(-\varphi, 0)$. If, further, $\varphi \geq 0$ then $u_k = \overset{\{\lambda_n\}}{\mathcal{G}_{\min(\varphi,k)}}$ exists for every k and $\lim_{k\to\infty} u_k = u$. From these facts, we know $\mathcal{H}_{\{\lambda_n\}} \subset MHB(\Omega)$, where $MHB(\Omega)$ is the family of all quasi-bounded harmonic functions.

2. If $u = \overset{\{\lambda_n\}}{\mathcal{G}_\varphi}$ and $v = \overset{\{\lambda_n\}}{\mathcal{G}_\psi}$, then we have $u \vee v = \overset{\{\lambda_n\}}{\mathcal{G}_{\max(\varphi,\psi)}}$ and $u \wedge v = \overset{\{\lambda_n\}}{\mathcal{G}_{\min(\varphi,\psi)}}$, where $u \vee v$ (resp. $u \wedge v$) denotes the least (resp. the greatest) harmonic function which dominates (resp. is dominated by) u and v.

3. Let $\{\varphi_k\}$ be an increasing sequence of functions tending to φ such that $u_k = \overset{\{\lambda_n\}}{\mathcal{G}_{\varphi_k}}$ exists for every k. If $\underline{\overset{\{\lambda_n\}}{\mathcal{G}_\varphi}}$ is harmonic, then $\overset{\{\lambda_n\}}{\mathcal{G}_\varphi}$ exists and coincides with $\lim_{k\to\infty} u_k$.

2. Lattice modifications of radials

2.1. As we have touched on it in the preceding section, the Dirichlet solution on Green lines HR_φ is just the harmonic function which is indifferent and has a radial φ. Thus, given a boundary function φ, in order to find the Dirichlet solution on Green lines HR_φ, we arrive at the problem to seek a harmonic function which is indifferent and has a radial φ. The first point, to find indifferent harmonic functions, can be solved easily since the class of indifferent harmonic functions is fairly large; for example, quasi-bounded harmonic functions are indifferent. In view of the second piont, to get a harmonic function with radial, it is preferable to extend a family of harmonic functions with radials. To this end, we investigate two sorts of modifications: *the lattice modification* and *the convex modification.*

We begin with definitions. A subharmonic function u is said to be *minor,* if it is weakly minor and $\lim_{\lambda\to 0} H_u^{D^\lambda}$ is harmonic. If u is minor, the harmonic function $\lim_{\lambda\to 0} H_u^{D^\lambda}$ is the smallest indifferent harmonic majorant of u. It is called *the best harmonic majorant of u* and is denoted by $\bar{u}$. Denoting by $\hat{u}$ the least harmonic majorant of u, we have $u \leq \hat{u} \leq \bar{u}$.

2.2.

Lemma 2. *If u is subharmonic, minor and has a radial φ, then*

(i) *the best harmonic majorant* $\bar{u}$ *of* u *has a radial* φ, *i.e.*, $\bar{u}=HR_{\varphi}$.
(ii) $u^{+}=max(u, 0)$ *is also minor and has a radial* φ^{+} *and* $\overline{u^{+}}=HR_{\varphi^{+}}$.

Proof. *re* (i). In order to see that $\bar{u}$ has a radial φ, we consider

$$\begin{aligned}\int|\bar{u}(\lambda, l)-\varphi(l)|dg &\leq\int[\bar{u}(\lambda, l)-u(\lambda, l)]dg+\int|u(\lambda, l)-\varphi(l)|dg\\ &\leq[H_{\bar{u}}^{D^{\lambda}}(y_{0})-H_{u}^{D^{\lambda}}(y_{0})]+\int|u(\lambda, l)-\varphi(l)|dg\\ &\leq[\bar{u}(y_{0})-H_{u}^{D^{\lambda}}(y_{0})]+\int|u(\lambda, l)-\varphi(l)|dg\,.\end{aligned}$$

In the above inequalities, the last terms tend to zero as $\lambda\rightarrow 0$. Thus, $\bar{u}$ is indifferent and has a radial φ, that is, $\bar{u}=HR_{\varphi}$.

re (ii). Since u is minor, $u\leq H_{u}^{D^{\lambda}}\leq H_{u^{+}}^{D^{\lambda}}$. Consequently, we have

$$u^{+}\leq H_{u^{+}}^{D^{\lambda}}\,,$$

i.e., u^{+} is weakly minor. On account of the inequality

$$|a^{+}-b^{+}|\leq|a-b|\,,$$

we know that u^{+} has a radial φ^{+}. Since φ is a radial of a minor subharmonic function, it is dg-integrable. Making $\lambda\rightarrow 0$ in

$$H_{u^{+}}^{D^{\lambda}}(y_{0})=\int u^{+}(\lambda, l)dg\leq\int[u^{+}(\lambda, l)-\varphi^{+}(l)]dg+\int\varphi^{+}(l)dg\,,$$

we have

$$\lim_{\lambda\rightarrow 0}H_{u^{+}}^{D^{\lambda}}(y_{0})\leq\int\varphi^{+}(l)dg<+\infty\,,$$

which implies that u^{+} is minor, q.e.d..

Theorem 2. *If* u *and* v *are indifferent harmonic functions with radials* φ *and* ψ *respectively, then there exist indifferent harmonic functions with radials* $max(\varphi, \psi)$ *and* $min(\varphi, \psi)$ *respectively.*

An indifferent harmonic function with radial is quasi-bounded.

Proof. The harmonic function $u-v$ is indifferent and has a radial $\varphi-\psi$. By Lemma 2, there exists an indifferent harmonic function w with radial $(\varphi-\psi)^{+}$. $v+w$ is indifferent and has a radial $\max(\varphi, \psi)$.

Next, it u is an indifferent harmonic function with radial φ, then there exist harmonic functions u_{1} and u_{2} which are indifferent and have radials φ^{+} and

φ^- respectively. Since $u=u_1-u_2$, $u\in HP(\Omega)$[7]. Since a sequence of indifferent harmonic functions w_n with radials $\min(\varphi^+, n)$ is increasing and $w_n\leq u_1$, we see that $w=\lim_{n\to\infty} w_n$ is harmonic and $w=u_1$. Thus u_1, and then u are quasi-bounded, q.e.d..

Corollary 1. *If u is subharmonic, minor and has a radial φ, then the best harmonic majorant $\overline{u^+}$ coincides with the least harmonic majorant $\widehat{u^+}$ of u^+, i.e., $\overline{u^+}=\widehat{u^+}=HR_{\varphi^+}$.*

In fact, on account of Theorem 2, HR_{φ^+} is quasi-bounded. Therefore

$$0\leq\widehat{u^+}\leq\overline{u^+} = HR_{\varphi^+}$$

implies that $\widehat{u^+}$ is quasi-bounded. Thus, $\widehat{u^+}$ is indifferent and we conclude $\widehat{u^+}=\overline{u^+}$.

Corollary 2. $\mathcal{K}\subset \bigcup_{\{\lambda_n\}\in\Lambda} \mathcal{H}_{\{\lambda_n\}}$.

In order to prove this, it would be enough to show that for any positive $u=HR_\varphi$, there exists a decreasing sequence of positive numbers $\{\lambda_n\}$ tending to zero such that $u=\overset{\{\lambda_n\}}{\mathcal{G}_\varphi}$. Since u is indifferent and has a radial φ, we can find an indifferent harmonic function u_k with radial $\min(\varphi, k)$. Then, there exists a sequence of positive numbers $\{\lambda_n^k\}$ tending to zero such that $\lim_{n\to\infty} u_k(\lambda_n^k, l)=\min(\varphi(l), k)$ dg-*a.e.*. From this and the boundedness of u_k, we conclude $u_k=\overset{\{\lambda_n^k\}}{\mathcal{G}}_{\min(\varphi, k)}$. By choosing a subsequence, we can assume that $\{\lambda_n^k\}$ is a subsequence of $\{\lambda_n^{k-1}\}$. We set $\lambda_n=\lambda_n^n$. Since it is clear that $u_k=\overset{\{\lambda_n\}}{\mathcal{G}}_{\min(\varphi, k)}$ for all k, by making $k\to\infty$, we have $u=\overset{\{\lambda_n\}}{\mathcal{G}_\varphi}$, i.e., $u\in\mathcal{H}_{\{\lambda_n\}}$.

REMARK. By Corollary 2, the Dirichlet solution on Green lines $u=HR_\varphi$ has an experssion $u=\overset{\{\lambda_n\}}{\mathcal{G}_\varphi}$ for suitable $\{\lambda_n\}$. Then, $\overline{u^+}=HR_{\varphi^+}=\overset{\{\lambda_n\}}{\mathcal{G}_{\varphi^+}}=\widehat{u^+}=u\vee 0$.

2.3.

Theorem 3. *Let u be an indifferent harmonic function with radial φ, and let F be a bounded continuous function defined on $[-\infty, +\infty]$. Then, there exists an indifferent harmonic function with radial $F\circ\varphi$.*

Proof. Let $\mathcal{M}$ be the family of bounded continuous function f on $[-\infty, +\infty]$ such that $f\circ\varphi$ is a principal radial, i.e., there exists an indifferent harmonic function with radial $f\circ\varphi$. Obviously $\mathcal{M}$ is a vector space. By Theorem 2, we observe that $\mathcal{M}$ forms a lattice by the usual maximum and mini-

7) HP (Ω) denotes the family of all harmonic functions each of which is expressed by a difference of two non-negative harmonic functions.

mum operations. Constant functions are contained in $\mathcal{M}$. Finally, any two distinct points t_1, t_2 of $[-\infty, +\infty]$ are separated by a function of $\mathcal{M}$. Indeed, assuming $t_1 < t_2$, we take τ_1 and τ_2 so that $t_1 < \tau_1 < \tau_2 < t_2$. If we set $f_{\tau_1, \tau_2}(t) = \min\{\max(\tau_1, t), \tau_2\}$, then $f_{\tau_1, \tau_2}(t_1) = \tau_1 \neq \tau_2 = f_{\tau_1, \tau_2}(t_2)$. Since $f_{\tau_1, \tau_2} \circ \varphi = \min\{\max(\tau_1, \varphi), \tau_2\}$, we have $f_{\tau_1, \tau_2} \in \mathcal{M}$. By the Stone's theorem[8], $\mathcal{M}$ is dense in the space of all bounded continuous functions on $[-\infty, +\infty]$ with respect to the topology ot the uniform convergence. From this we conclude easily that, for each bounded continuous function F on $[-\infty, +\infty]$, $F \circ \varphi$ is a principal radial.

REMARK. If $u = HR_\varphi$, then u is quasi-bounded; therefore u is a Dirichlet solution on the Martin space Ω^M. Thus, u is expressed by H_f^M for suitable resolutive function f on the Martin boundary Δ^M. Then, by a similar way as in the proof of Theorem 3, we may prove that if F is a bounded continuous function on $[-\infty, +\infty]$, then $H_{F \circ f}^M = HR_{F \circ \varphi}$. Further, when F is a Baire function on $(-\infty, +\infty)$, $F \circ \varphi$ is a principal radial if and only if $F \circ f$ is integrable with respect to the harmonic measure ω^M on Δ^M. For HD-function[9], this is stated in [6][10].

Corollary. *Let u be an indifferent harmonic function with radial φ, and let S be a continuous function on $(-\infty, +\infty)$ such that $\lim_{t \to -\infty} S(t)$ and $\lim_{t \to \infty} S(t)$ exist (which may be $\pm\infty$). If*

$$\int |S[\varphi(l)]| dg < +\infty ,$$

then $S \circ \varphi$ is a principal radial.

Proof. Without loss of generality, we may suppose $S \geq 0$. Since $S_n = \min(S, n)$ is bounded and continuous, $S_n \circ \varphi$ is a principal radial. If we set $w_n = HR_{S_n \circ \varphi}$, $\{w_n\}$ is an increasing sequence.

$$w_n(y_0) = \int S_n[\varphi(l)] dg \leq \int S[\varphi(l)] dg < +\infty$$

shows that $\lim_{n \to \infty} w_n$ is indifferent and has a radial $S \circ \varphi$, q.e.d. .

3. Convex modifications of radials

3.1. In the following, $\Psi(t)$ denotes always a function defined on $(-\infty, +\infty)$ which is non-negative, increasing and convex. We set $\Psi(-\infty) = \lim_{t \to -\infty} \Psi(t)$.

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8) Cf. [4], Hilfssatz 0.1, p. 5.

9) An HD-function is a harmonic function with finite Dirichlet integral.

10) Cf. [6], pp. 580–581.

Lemma 3. *If a non-negative Borel function u has a quasi-bounded harmonic majorant V, then $u(\lambda, l)$ is uniformly integrable with respect to dg, i.e., for any positive number ε there exists a positive number δ such that if e is a set of Green lines whose Green measure is less than δ, then $\int_e u(\lambda, l)dg \leq \varepsilon$ holds for every λ.*

Proof. We follow the Doob's idea [5]. Let e be a Borel subset of $\mathcal{L}$. For $\lambda>0$ we set $[e]_\lambda=\{l\cap\sum^\lambda;\ l\in e\}$. It is known that $\omega_{y_0}^\lambda([e]_\lambda)=g(e)$, where $\omega_{y_0}^\lambda$ is a harmonic measure on $\sum^\lambda$ with respect to y_0. To prove the lemma it is enough to show $V(\lambda, l)$ is uniformly integrable with respect to dg.

In the first place, we shall show that $V(x)$ is uniformly integrable with respect to $d\omega_{y_0}^\lambda$. In fact, since $V=H_f^M$ for some resolutive function f on Δ^M, given any positive ε we can find subharmonic functions s_1 and $-s_2$ such that

(3. 1) $\qquad s_1\leq M_0$ and $-s_2\leq M_0$, where M_0 is a constant,

(3. 2) $\qquad s_1\leq V\leq s_2$,

(3. 3) $\qquad 0\leq s_2(y_0)-s_1(y_0)<\varepsilon/4$.

Let us write $A_{\alpha,\lambda}=\{x\in\sum^\lambda;\ V(x)\geq\alpha\}$ for $\alpha>0$. We have

$$\begin{aligned}\int_{A_{\alpha,\lambda}} Vd\omega_{y_0}^\lambda &\leq \int_{A_{\alpha,\lambda}}[s_2-s_1]d\omega_{y_0}^\lambda+\int_{A_{\alpha,\lambda}} s_1\, d\omega_{y_0}^\lambda \\ &\leq \int_{\Sigma\lambda}[s_2-s_1]d\omega_{y_0}^\lambda+\int_{A_{\alpha,\lambda}} s_1\, d\omega_{y_0}^\lambda \\ &\leq H_{(s_2-s_1)}^{D^\lambda}(y_0)+M_0\cdot\omega_{y_0}^\lambda(A_{\alpha,\lambda}) \\ &\leq [s_2(y_0)-s_1(y_0)]+M_0\cdot\omega_{y_0}^\lambda(A_{\alpha,\lambda}) \\ &\leq \varepsilon/4+M_0V(y_0)/\alpha\ .\end{aligned}$$

The last inequality follows from

$$\sup_\lambda\int_{\Sigma\lambda} Vd\omega_0^\lambda=\sup_\lambda H_V^{D^\lambda}(y_0)\leq V(y_0)\ .$$

Next, we set $\tilde{A}_{\alpha,\lambda}=\{l\in\mathcal{L};\ V(\lambda, l)\geq\alpha\}$. Then $[\tilde{A}_{\alpha,\lambda}]_\lambda=A_{\alpha,\lambda}$. We select $\alpha_0>0$ so large that $M_0V(y_0)/\alpha_0<\varepsilon/4$. We set $\delta=\varepsilon/2\alpha_0$. Then, $g(e)<\delta$ implies

$$\begin{aligned}\int_e V(\lambda, l)dg &=\int_{e\cap\tilde{A}_{\alpha_0,\lambda}} V(\lambda, l)dg+\int_{e-\tilde{A}_{\alpha_0,\lambda}} V(\lambda, l)dg \\ &\leq \int_{\tilde{A}_{\alpha_0,\lambda}} V(\lambda, l)dg+\int_e \alpha_0\, dg \\ &\leq \int_{A_{\alpha_0,\lambda}} V(x)d\omega_{y_0}^\lambda+\alpha_0\, g(e) \\ &\leq \varepsilon/4+\varepsilon/4+\varepsilon/2=\varepsilon\ .\end{aligned}$$

Thus, the lemma is proved.

Lemma 4. *Let v be a subharmonic minor function with radial φ, and let Θ be an increasing continuous function on $(-\infty, +\infty)$. Then $\Theta[v(\lambda, l)]$ converges to $\Theta(\varphi)$ in dg-measure as $\lambda \to 0$, i.e.,*

$$\lim_{\lambda\to 0} g(\{l\in \mathcal{L};\ |\Theta[v(\lambda, l)]-\Theta[\varphi(l)]|\geq\sigma\})=0$$

for all $\sigma>0$.

Proof. Suppose, on the contrary, there exists $\sigma_0>0$ such that

$$\overline{\lim_{\lambda\to 0}}\, g(\{l\in \mathcal{L};\ |\Theta[v(\lambda, l)]-\Theta[\varphi(l)]|\geq\sigma_0\})>0\,.$$

Then, we could seek $\alpha>0$ and a sequence of positive numbers $\{\lambda_n\}$ tending to zero such that

$$(3.4)\qquad g(\{l\in \mathcal{L};|\Theta[v(\lambda_n, l)]-\Theta[\varphi(l)]|\geq\sigma_0\})\geq\alpha \text{ for } n=1, 2,\cdots,$$

$$(3.5)\qquad \lim_{n\to\infty} v(\lambda_n, l)=\varphi(l) \qquad dg\text{-}a.e.\,.$$

By Egoroff's theorem, there exists dg-measurable subset e_1 of $\mathcal{L}$ such that $v(\lambda_n, l)$ converges to $\varphi(l)$ uniformly on $\mathcal{L}-e_1$ and

$$(3.6)\qquad g(e_1)<\alpha/4.$$

We select N so large that

$$(3.7)\qquad g(e_2)<\alpha/4, \text{ where } e_2=\{l;\ |\varphi(l)|>N\}\,.$$

For $\varepsilon=\sigma_0/2$ there exists $\eta<N$ such that

$$(3.8)\qquad |\Theta(t_1)-\Theta(t_2)|<\varepsilon \qquad \text{whenever } |t_1-t_2|\leq\eta \text{ and } t_1, t_2\in[-2N, 2N].$$

On account of the uniform convergence of $v(\lambda_n, l)$ on $\mathcal{L}-e_1$ there exists a number n_0 such that

$$(3.9)\qquad \sup_{l\in\mathcal{L}-(e_1\cup e_2)} |v(\lambda_n, l)-\varphi(l)|<\eta \qquad \text{for all } n\geq n_0\,.$$

From (3.9), if $n\geq n_0$ and $l\notin(e_1\cup e_2)$, then

$$-2N<-N-\eta\leq\varphi(l)-\eta\leq v(\lambda_n, l)\leq\varphi(l)+\eta\leq N+\eta<2N\,,$$

therefore in view of (3.8)

$$\Theta[\varphi(l)]-\varepsilon\leq\Theta[\varphi(l)-\eta]\leq\Theta[v(\lambda_n, l)]\leq\Theta[\varphi(l)+\eta]\leq\Theta[\varphi(l)]+\varepsilon\,,$$

that is,

$$|\Theta[v(\lambda_n, l)]-\Theta[\varphi(l)]| \leq \varepsilon = \sigma_0/2 \text{ whenever } n \geq n_0 \text{ and } l \in \mathcal{L}-(e_1 \cup e_2)\,.$$

In other words

$$\{l \in \mathcal{L};\ |\Theta[v(\lambda_n, l)]-\Theta[\varphi(l)]| \geq \sigma_0\} \subset e_1 \cup e_2 \qquad \text{for all } n \geq n_0\,.$$

From (3.6) and (3.7), we have for all $n \geq n_0$

$$g(\{l \in \mathcal{L};\ |\Theta[v(\lambda_n, l)]-\Theta[\varphi(l)]| \geq \sigma_0\}) \leq \alpha/2\,,$$

which contradicts (3.4), q.e.d..

Theorem 4. *Let u be subharmonic, minor and have a radial φ. If $\Psi[u]$ has a harmonic majorant, then $\Psi[u]$ has a quasi-bounded harmonic majorant and it is minor with radial $\Psi[\varphi]$. The least harmonic majorant $\widehat{\Psi[u]}$ of $\Psi[u]$ is an indifferent harmonic function with radial $\Psi[\varphi]$, i.e., $\widehat{\Psi[u]}=\widehat{\Psi[\bar{u}]}=HR_{\Psi[\varphi]}$.*

Proof. We write $\Psi_n(t) = \min(\Psi(t), n)$. Since $\bar{u} = HR_\varphi$, there exists $w_n=HR_{\Psi_n\circ\varphi}$. Let $\{\lambda_n\}$ be a decreasing sequence of positive numbers tending to zero such that

$$\lim_{n\to\infty} u(\lambda_n, l) = \varphi(l) \qquad dg\text{-}a.e.\,.$$

Denoting by V the harmonic majorant of $\Psi[u]$, we have

$$\int\Psi[\varphi(l)]dg = \int\lim_{n\to\infty}\Psi[u(\lambda_n, l)]dg \leq \varliminf_{n\to\infty}\int\Psi[u(\lambda_n, l)]dg$$
$$\leq \varliminf_{n\to\infty}\int V(\lambda_n, l)dg = \lim_{n\to\infty} H_V^{D^{\lambda_n}}(y_0) \leq V(y_0) < +\infty\,,$$

which implies $\Psi\circ\varphi$ is dg-integrable. From

$$w_n(y_0) = HR_{\Psi_n\circ\varphi}(y_0) = \int\Psi_n[\varphi(l)]dg \leq \int\Psi[\varphi(l)]dg\,,$$

we conclude $\lim_{n\to\infty} w_n=HR_{\Psi\circ\varphi}$.

Since $\bar{u}=HR_\varphi=H_f^M$, by the remark in 2.3, $HR_{\Psi\circ\varphi}=H_{\Psi\circ f}^M$. By M. Parreau [17][11] we have $H_{\Psi\circ f}^M=\widehat{\Psi[\bar{u}]}$. Hence $\Psi[u]$ has a quasi-bounded harmonic majorant.

By the properties of Ψ and the Jensen's inequality, we can derive from

$$u(y) \leq H_u^{D^\lambda}(y) = \int_{\Sigma\lambda} u\, d\omega_y^\lambda$$

$$\Psi[u(y)] \leq \Psi\left[\int_{\Sigma\lambda} u\, d\omega_y^\lambda\right] \leq \int_{\Sigma\lambda}\Psi[u]d\omega_y^\lambda = H_{\Psi\circ u}^{D^\lambda}(y) \leq V(y)\,,$$

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11) In [17] it is not considered a Green space. We have, however, an obvious extension to our general case.

which means that $\Psi(u)$ is minor.

Finally, by the above lemmas, $\Psi[u(\lambda, l)]$ is uniformly integrable with respect to dg and $\Psi[u(\lambda, l)]$ converges to $\Psi[\varphi(l)]$ in dg-measure. Therefore

$$\lim_{\lambda\to 0}\int |\Psi[u(\lambda, l)]-\Psi[\varphi(l)]|\,dg = 0\,,$$

i.e., $\Psi[u]$ has a radial $\Psi\circ\varphi$. Thus the proof is completed.

The following example shows that the condition "u is minor" is essential in Theorem 4.

EXAMPLE. Let a and b be two distinct non-zero complex numbers in the unit disc. We consider the Green space $\Omega=\{|z|<1\}-\{a, b\}$. We denote by G_a and G_b Green functions of Ω with poles at a and b respectively. We take the reference point y_0 at the origin. $u=G_a-G_b$ is harmonic and has the radial zero. Since $u\leq G_a$, u has a harmonic majorant. The function

$$\Psi(t)=\begin{cases} t & \text{if } t>0 \\ 0 & \text{otherwise}\end{cases}$$

is non-negative, increasing and convex. Then, $\Psi[u]$ is not minor, since the indifferent harmonic function with radial $\Psi(0)$ is identically zero and is not the best harmonic majorant of $\Psi[u]$.

When a subharmonic function u has a radial, the property "u is minor" is closely connected with the property "$\Psi[u]$ has a quasi-bounded harmonic majorant". We shall state this fact precisely in the following theorem.

Theorem 5. *Let u be a subharmonic function with radial φ, and let $\Psi(t)$ be a non-negative non-constant increasing convex function. Suppose that $\Psi[u]$ has a harmonic majorant. In order that u be minor, it is necessary and sufficient that $\Psi[u]$ have a quasi-bounded harmonic majorant. In this case, we have $\widehat{\Psi[u]}=\overline{\Psi[u]}$ $=HR_{\Psi\circ\varphi}$.*

Proof. The necessary part is already proved in Theorem 4. Suppose that $\Psi[u]$ has a quasi-bounded harmonic majorant W. We can find numbers $t_0>0$, $a>0$ and b such that $\Psi(t)\geq at+b$ whenever $t\geq t_0$. Denoting by $u^{t_0}=\max(u, t_0)$, we have

$$au+b\leq au^{t_0}+b\leq\Psi(u^{t_0})\leq\Psi(u)+\Psi(t_0)\leq W+\Psi(t_0)\,,$$

therefore

$$u\leq 1/a\cdot[W+\Psi(t_0)-b] = W_1\,.$$

Since W_1 is a quasi-bounded harmonic majorant of u and W_1-u is positive superharmonic, we have for every $\lambda>0$,

$$u=W_1-(W_1-u)\leq H_{W_1}^{D^\lambda}-H_{(W_1-u)}^{D^\lambda}=H_u^{D^\lambda}\leq H_{W_1}^{D^\lambda}=W_1\,,$$

which proves that u is minor.

3.2. Here we shall mention the results related to the work of K. Endl [7].

For $y\in D^\lambda$, we set $K_\lambda\ (l, y)=\dfrac{\partial G_y^\lambda(x)/\partial n}{\partial G_{y_0}^\lambda(x)/\partial n}$, where $G_\cdot^\lambda$ denotes the Green function of D^λ with pole at $\cdot$, $\partial/\partial n$ is the outer normal derivative on $\sum^\lambda$ and x is the point of l on $\sum^\lambda$, namely $x=(\lambda, l)$. Let $\{\lambda_n\}$ be a monotone decreasing sequence of positive numbers tending to zero. In [7], the followings are established:

[1°] there exist a set $\mathcal{L}^*$ of Green lines of Green measure 1 and a function $K(l, y)$ defined on $\mathcal{L}^*\times\Omega$ such that for fixed l, $K(l, y)$ is harmonic as a function of y and $K(l, y_0)=1$ for all l.

[2°] there exists a subsequence $\{\alpha_n\}$ of $\{\lambda_n\}$ such that

$$\lim_{n\to\infty}\int K_{\alpha_n}(l, y)\varphi(l)dg=\int K(l, y)\varphi(l)dg$$

holds for every dg-integrable φ. Further, if a harmonic function u has a radial φ, then $H_u^{D^{\alpha_n}}(y)=\int K_{\alpha_n}(l, y)u(\alpha_n, l)dg$ tends to $\int K(l, y)\varphi(l)dg$ as $n\to\infty$.

[3°] if φ is dg-integrable, then $\int K(l, y)\varphi(l)dg$ is a harmonic function of y.

[4°] if u is an indifferent harmonic function and has a radial φ, then

$$u(y)=\int K(l, y)\varphi(l)dg\,.$$

In view of our preceding study, we conclude:

(1) *If v is subharmonic, minor and has a radial φ, then the best harmonic majorant $\tilde{v}$ of v is expressed by*

$$\tilde{v}(y)=\int K(l, y)\varphi(l)dg\,.$$

This is an immediate consequence of [4°].

(2) *If φ is dg-integrable, then*

$$\underline{HR}_\varphi(y)\leq\int K(l, y)\varphi(l)dg\leq\overline{HR}_\varphi(y)\,.$$

Therefore, if both $\underline{HR}_\varphi$ and $\overline{HR}_\varphi$ are harmonic, then $\int K(l, y)\varphi(l)dg$ is an indifferent harmonic function.

In fact, let $v \in \underline{\mathcal{F}}_\varphi'$ and $v \not\equiv -\infty$. Since v is weakly minor

$$v(y) \leq H_v^{D^{\alpha_n}}(y) = \int K_{\alpha_n}(l, y) v(\alpha_n, l) dg .$$

On the other hand,

$$(3.10) \quad \begin{cases} \int K_{\alpha_n}(l, y) v(\alpha_n, l) dg - \int K(l, y) \varphi(l) dg \\ \quad = \int K_{\alpha_n}(l, y) [v(\alpha_n, l) - \varphi(l)] dg + \int [K_{\alpha_n}(l, y) - K(l, y)] \varphi(l) dg \\ \quad \leq \int K_{\alpha_n}(l, y) [v(\alpha_n, l) - \varphi(l)]^+ dg + \int [K_{\alpha_n}(l, y) - K(l, y)] \varphi(l) dg . \end{cases}$$

Since the family of positive harmonic functions $K_{\alpha_n}(l, y)$ is normalized at y_0, i.e., $K_{\alpha_n}(l, y_0) = 1$ for all $l \in \mathcal{L}^*$, by the Harnack's inequality, $\{K_{\alpha_n}(l, y);\ n \geq n_0\}$ is uniformly bounded as a function of $l \in \mathcal{L}^*$ whenever y is fixed and n_0 is sufficiently large. Therefore, if n is so large that $y \in D^{\alpha_n}$, then $K_{\alpha_n}(l, y)$ does not exceed a constant C for all such n and $l \in \mathcal{L}^*$. The first term of the last integrals of (3.10) does not exceed $C \cdot \int [v(\alpha_n, l) - \varphi(l)]^+ dg$ for sufficiently large n and therefore tends to zero as $n \to \infty$. This is also true for the second term in view of [2°]. Thus,

$$v(y) \leq \lim_{n \to \infty} H_v^{D^{\alpha_n}}(y) = \lim_{n \to \infty} \int K_{\alpha_n}(l, y) v(\alpha_n, l) dg \leq \int K(l, y) \varphi(l) dg ,$$

which proves

$$\sup\ v(y) = \underline{HR}_\varphi(y) \leq \int K(l, y) \varphi(l) dg .$$

If both $\underline{HR}_\varphi$ and $\overline{HR}_\varphi$ are harmonic, then they are indifferent and $\int K(l, y) \varphi(l) dg$ is also indifferent.

We restate Theorem 4 in

(3) *Let v be a subharmonic function with radial φ. If v is minor and $\Psi(v)$ has a harmonic majorant, then*

$$\Psi[\widehat{v(y)}] = \overline{\Psi[v(y)]} = \int K(l, y) \Psi[\varphi(l)] dg .$$

4. An application to the theory of Riemann surfaces

In connection with our previous investigation, we mention here the unicity theorem of Riesz type for holomorphic functions on a Riemann surface.

Theorem 6. *Let Ω be a hyperbolic Riemann surface, and let f be holomorphic on Ω belonging to the Smirnov class $S(\Omega)$, i.e., $\log^+ |f|$ has a quasi-bounded harmonic majorant. For a subset α of $\mathcal{L}$ of positive Green measure, if we have*

$$\lim_{\lambda\to 0}\int_\alpha |f(\lambda, l)|\,dg = 0\,,$$

then $f\equiv 0$.

Proof. Suppose, on the contrary, $f\not\equiv 0$. Then, $\log |f|$ is minor. In fact, let U be a quasi-bounded harmonic majorant of $\log^+|f|$. Since $v=U-\log|f|$ is a positive superharmonic, $v\geq \bar{H}_v^{D^\lambda}\geq 0$ in D^λ, which means that v is $d\omega_{y_0}^\lambda$-integrable. Since U is $d\omega_{y_0}^\lambda$-integrable, $\log |f|$ is $d\omega_{y_0}^\lambda$-integrable, therefore

$$U-\log|f|\geq H_{U-\log|f|}^{D^\lambda} = U-H_{\log|f|}^{D^\lambda}\,,$$

that is, $\log |f|\leq H_{\log|f|}^{D^\lambda}\leq U$, which means that $\log |f|$ is minor.

$$\begin{aligned}
&\int\log |f(\lambda, l)|\,dg = \int_\alpha \log|f(\lambda, l)|\,dg+\int_{\mathcal{L}-\alpha}\log|f(\lambda, l)|\,dg\\
&\leq\int_\alpha \log|f(\lambda, l)|\,dg+\int_{\mathcal{L}-\alpha}\log^+|f(\lambda, l)|\,dg \leq\int_\alpha \log|f(\lambda, l)|\,dg+\int U(\lambda, l)dg\\
&=\int_\alpha \log|f(\lambda, l)|\,dg+H_U^{D^\lambda}(y_0) = \int_\alpha \log|f(\lambda, l)|\,dg+U(y_0)\\
&\leq g(\alpha)\log\int_\alpha |f(\lambda, l)|\,dg+\log\frac{1}{g(\alpha)}+U(y_0)\,.
\end{aligned}$$

By making $\lambda\to 0$, the right hand side tends to $-\infty$, therefore $\lim_{\lambda\to 0} H_{\log|f|}^{D^\lambda}(y_0) = -\infty$. This implies that the best harmonic majorant of $\log |f|$ is $-\infty$ at y_0, which is a contradiction, q.e.d..

Since the Hardy class $H_p(\Omega)$ $(p>0)$ is contained in the Smirnov class we have

Corollary. *Let Ω be a hyperbolic Riemann surface, and let f be holomorphic on Ω belonging to the Hardy class $H_p(\Omega)$ for $p>0$, i.e., $|f|^p$ has a harmonic majorant. For a subset α of $\mathcal{L}$ of positive Green measure, if we have*

$$\lim_{\lambda\to 0}\int_\alpha |f(\lambda, l)|\,dg = 0\,,$$

then $f\equiv 0$.

M. Brelot [1] considered the unicity theorem for bounded holomorphic functions. The theorem of Riesz type for holomorphic functions with finite

Dirichlet integrals was established by M. Nakai [15] by means of the Royden's compactification and the *radial limits* along Green lines.

5. Green spaces of type MHB(Ω) = MHD(Ω)

5.1. In this section, we consider strongly subharmonic functions of which importance in the theory of harmonic functions is clarified by Gårding-Hörmander [9]. A generalization of the results of Gårding-Hörmander is given in Theorem 10 below in connection with Green lines. A research for generalizing Gårding-Hörmander's results was done by S. Yamashita [19] in terms of the Martin's compactification not containing a notion of "radial limit".

Throughout this section, we suppose MHB(Ω) = MHD(Ω). In [11], it is shown that in the case where Ω is a hyperbolic Riemann surface above condition is equivalent to the following: *the Martin boundary Δ^M and the Kuramochi boundary Δ^K of Ω correspond each other in the one-to-one manner almost everywhere, that is, there exist two sets E of $d\omega^M$-measure zero and E' of $d\omega^K$-measure zero and the bijection Φ form $\Delta^M - E$ to $\Delta^K - E'$.* This fact is also valid for a Green space Ω. Using above bijection Φ, we have a surjection Θ from a subset of $\mathcal{L}$ to a subset of Δ^M; more precisely, to every regular Green line l tending to a point $x'(l)$ of $\Delta^K - E'$ we assign $\Theta(l) = \Phi^{-1}[x'(l)]$. Θ is defined dg-almost everywhere on $\mathcal{L}$ and has the image of $d\omega^M$-measure 1. We note Θ is not a bijection in general.

Any $u \in$ MHB(Ω) is expressed by

$$u = H_f^M = H_{f'}^K = HR_\varphi\,,$$

where $f(x) = f'[\Phi(x)]$ and $\varphi(l) = f'[x'(l)] = f[\Theta(l)]$. From what we have shown in §1, it follows that all families of Dirichlet solutions on Green lines considered in §1 coincide with those of the Dirichlet solutions with respect to Martin's and Kuramochi's compactifications.

5.2.

Lemma 5. *Let u be a positive singular harmonic function on Ω. Given any decreasing sequence of positive numbers $\{\lambda_n\}$ tending to zero, then there exists a subsequence $\{\lambda'_n\}$ such that $e' = \{l \in \mathcal{L}'; \overline{\lim_{n\to\infty}}\, u(\lambda'_n, l) > 0\}$ is of Green measure zero.*

Proof. For every positive integer k, the function $u_k = \min(u, 1/k)$ is a potential, therefore u_k has the radial zero. Hence we can choose a subsequence $\{\lambda_{n_\nu}\}$ of $\{\lambda_n\}$ such that

$$\lim_{\nu\to\infty} u_k(\lambda_{n_\nu}, l) = 0\,, \qquad dg\text{-}a.e..$$

By taking subsequences successively, we can find a diagonal subsequence $\{\lambda'_n\}$ of $\{\lambda_n\}$ and a subset e of $\mathcal{L}$ of dg-measure zero such that

(5. 1) $\lim_{n\to\infty} u_k(\lambda'_n, l) = 0$ for every $l \in \mathcal{L}-e$ and $k = 1, 2, \cdots$.

We assert that $e'=\{l\in\mathcal{L}'; \overline{\lim}_{n\to\infty} u(\lambda'_n, l)>0\}$ is of Green measure zero. In fact, we suppose, on the contrary, that the outer Green measure of e' is positive. Then, there would exist $l\in e'\cap[\mathcal{L}-e]$ and a subsequence $\{\lambda''_n\}$ of $\{\lambda'_n\}$ satisfying $\lim_{n\to\infty} u(\lambda''_n, l)=a>0$. From this we deduce $u(\lambda''_n, l)>a/2$ for all sufficiently large n. If we take k_0 so large that $a/2>1/k_0$, then we have

$$\lim_{n\to\infty} u_{k_0}(\lambda''_n, l)=1/k_0 ,$$

which contradicts (5.1), q.e.d..

5.3. A subharmonic function is called *strongly subharmonic* if it is the composed function of Ψ and u, i.e., $\Psi\circ u$, where u is subharmonic and Ψ is a real valued real variable function satisfying the conditions:

1) Ψ is non-negative, increasing and convex,
2) $\lim_{t\to\infty} \Psi(t)/t = \infty$.

We define $\Psi(-\infty)=\lim_{t\to-\infty}\Psi(t)$.

If a strongly subharmonic function $\Psi[u]$ has a harmonic majorant, then it is known that u has the least harmonic majorant $\hat{u}$ which is decomposed into a quasi-bounded $\hat{u}_B$ and a non-positive singular harmonic part $\hat{u}_S$[12]. Under our assumption MHB(Ω)=MHD(Ω), $\hat{u}_B$ is a Dirichlet solution HR_φ. Hence

$$u = HR_\varphi+\hat{u}_S-p ,$$

where p is a potential.

Lemma 6. *Under the assumption MHB*(Ω)=*MHD*(Ω), *if* $\Psi\,[u]$ *is strongly subharmonic and has a harmonic majorant, then* $u=HR_\varphi+\hat{u}_S-p$ *for some potential* p *and a singular harmonic function* $\hat{u}_S$. *Moreover, for every* c, *we have* $\overline{u^c}=\widehat{u^c}$ $=HR_{\varphi^c}$, *where* f^c *denotes the function max*(f, c).

Proof. Only the latter half of the lemma needs to be proved. Write $v=\max(HR_\varphi, c)$. Since $\hat{u}_S\leq 0$, we have $u^c\leq v$. v has a radial φ^c, and $v\leq HR_\varphi\vee c$ implies v is minor. Thus $v\leq HR_{\varphi^c}$, which proves $\widehat{u^c}\leq\overline{u^c}\leq HR_{\varphi^c}$. Since $\widehat{u^c}$ is quasi-bounded, we can write $\widehat{u^c}=HR_\psi$. By Lemma 5, there exists $\{\lambda_n\}$ tending to zero such that

$$\lim_{n\to\infty}|\hat{u}_S(\lambda_n, l)| = 0 , \qquad dg\text{-}a.e..$$

Replacing, if necessary, $\{\lambda_n\}$ by a suitable subsequence, we have

12) Cf. [19], Lemma 3, p. 64.

$$\lim_{n\to\infty}|u^c(\lambda_n, l)-\varphi^c(l)| = 0, \qquad dg\text{-}a.e..$$

and

$$\lim_{n\to\infty}|u^c(\lambda_n, l)-\psi(l)| = 0, \qquad dg\text{-}a.e..$$

These show that $\psi=\varphi^c$ *dg-a.e.*, q.e.d..

Theorem 7. *Under the assumption $MHB(\Omega)=MHD(\Omega)$, every strongly subharmonic function possessing a harmonic majorant has a radial. More precisely, if a strongly subharmonic function $\Psi[u]$ possesses a harmonic majorant, then*

$$u = HR_\varphi+\hat{u}_S-p$$

and $\Psi[u]$ has a radial $\Psi[\varphi]$.

Proof. Let us consider the subharmonic function $u^c=\max(u, c)$, where c is a real number. Since, by Lemma 6, $\overline{u^c}$ is a Dirichlet solution on Green lines and $\overline{u^c}-u^c$ is a potential, u^c is minor and has a radial φ^c. Since $\Psi[u^c]$ has a harmonic majorant also, we deduce form Theorem 4 that $\Psi[u^c]$ is minor with radial $\Psi[\varphi^c]$, and therefore $\widehat{\Psi[u^c]}=\overline{\Psi[u^c]}=HR_{\Psi[\varphi^c]}$. When $c\to-\infty$, $\Psi[u^c]$ decreases and $\Psi[\varphi^c]\to\Psi[\varphi]$. Hence $0\leq\widehat{\Psi[u]}\leq\lim_{c\to-\infty}\widehat{\Psi[u^c]}=\lim_{c\to-\infty}\overline{\Psi[u^c]}=HR_{\Psi[\varphi]}$. This means that $\widehat{\Psi[u]}$ is quasi-bounded and $\widehat{\Psi[u]}=\overline{\Psi[u]}=HR_\psi$ for some ψ. We clearly have $\psi\leq\Psi[\varphi]$ *dg-a.e.*. The converse inequality is also valid. Indeed, let σ be a non-negative bounded dg-measurable function on $\mathcal{L}$. By Lemma 5, there exists $\{\lambda_n\}$ tending to zero so that $u(\lambda_n, l)\to\varphi(l)$ *dg-a.e.*. Then, we have

$$\begin{aligned}\int\Psi[\varphi(l)]\sigma(l)dg(l) &= \int\lim_{n\to\infty}\Psi[u(\lambda_n, l)]\sigma(l)dg(l)\\ &\leq\varliminf_{n\to\infty}\int\Psi[u(\lambda_n, l)]\sigma(l)dg(l)\\ &\leq\int\psi(l)\sigma(l)dg(l).\end{aligned}$$

From this we conclude that $\Psi[\varphi]\leq\psi$ *dg-a.e.*, thus, we complete the proof.

Corollary. *Let Ω be a hyperbolic Riemann surface satisfying $MHB(\Omega)=MHD(\Omega)$, and let $p\geq1$. If $f\in H_p(\Omega)$, that is, f is holomorphic on Ω and $|f|^p$ has a harmonic majorant, then $f=HR_\varphi$ for some $\varphi=\varphi_1+i\varphi_2$ and*

$$(5.2) \qquad \lim_{\lambda\to0}\int|f(\lambda, l)-\varphi(l)|^p dg = 0.$$

In particular, f has a radial φ and φdg is a boundary measure of f, that is, $f(\lambda, l)dg$ tends to $\varphi(l)dg$ vaguely as $\lambda\to0$.

Proof. First we consider the case $p=1$. Since $|f|=\exp(\log|f|)$, $|f|$ is strongly subharmonic and has a harmonic majorant, therefore, by Theorem 7, $\widehat{|f|}=\overline{|f|}=HR_\psi$ for some ψ. Since the modulus of the real part u of f and that of the imaginary part v of f are both bounded by $\widehat{|f|}$, it follows that u and v are quasi-bounded, and therefore $u=HR_{\varphi_1}$ (resp. $v=HR_{\varphi_2}$) for some φ_1 (resp. φ_2). Thus,

$$f=HR_{\varphi_1}+i\,HR_{\varphi_2}$$

and

$$\lim_{\lambda\to 0}\int|f(\lambda,l)-\varphi(l)|\,dg=0\,,$$

where $\varphi=\varphi_1+i\,\varphi_2$.

Next, let $p>1$. If we set

$$\Psi(t)=\begin{cases} t^p & \text{if } t\geq 0\,,\\ 0 & \text{otherwise,}\end{cases}$$

then $|f|^p=\Psi(|f|)$, which implies $|f|^p$ is strongly subharmonic and has a harmonic majorant. We have seen that $|f|$ has a quasi-bounded harmonic majorant, therefore both the real part u and the imaginary part v of f are quasi-bounded. Hence $u=HR_{\varphi_1}$ (resp. $v=HR_{\varphi_2}$) for some φ_1 (resp. φ_2). Since

$$|f(\lambda,l)-[\varphi_1(l)+i\,\varphi_2(l)]|^p\leq 2^p(|u(\lambda,l)-\varphi_1(l)|^p+|v(\lambda,l)-\varphi_2(l)|^p)\,,$$

to prove (5.2), it is sufficient to show

$$\lim_{\lambda\to 0}\int|u(\lambda,l)-\varphi_1(l)|^p dg=0\,.$$

Since strongly subharmonic function $|u|^p$ has a harmonic majorant, $|u|^p$ has a quasi-bounded harmonic majorant. It follows from Lemma 3 that $\{|u(\lambda,l)|^p\}$ are uniformly integrable with respect to dg. The inequality

$$|u(\lambda,l)-\varphi_1(l)|^p\leq 2^p(|u(\lambda,l)|^p+|\varphi_1(l)|^p)$$

implies that $\{|u(\lambda,l)-\varphi_1(l)|^p\}$ are also uniformly integrable. From this and the fact that $u(\lambda,l)$ converges in measure to $\varphi_1(l)$ as $\lambda\to 0$, we derive the result, q.e.d..

REMARK. For $0<p<1$, a function f in $H_p(\Omega)$ can not be expressed by the form $f=HR_\varphi$ in general. For, if $f\notin H_1(\Omega)$, either the real part or the imaginary part of f does not belong to $HP(\Omega)$.

5.4. In §4, we have discussed the unicity theorem for some function classes. Here we state, under the assumption $\mathrm{MHB}(\Omega)=\mathrm{MHD}(\Omega)$, the unicity theorem for a wider function class.

Theorem 8. *Let Ω be a hyperbolic Riemann surface. Assume $MHB(\Omega)=MHD(\Omega)$. Let $f\in AL(\Omega)$, that is, f is holomorphic on Ω and $\log^+|f|$ has a harmonic majorant. For a set $\alpha\subset\mathcal{L}$ of positive dg-measure, we suppose*

$$\lim_{\lambda\to 0}\int_\alpha |f(\lambda, l)|\,dg=0\,,$$

or more generally, there exists a sequence $\{\lambda_n\}$ of positive numbers tending to zero such that

$$\lim_{n\to\infty} f(\lambda_n, l)=0 \qquad \text{for all } l\in\alpha\,,$$

then $f\equiv 0$.

Proof. Suppose, on the contrary, $f\not\equiv 0$,. Then, the least harmonic majorant U of $\log|f|$ could be written in the form $U=HR_\varphi+U_S$, where U_S is singular. Therefore, $\log|f|=HR_\varphi+U_S-p$, where p is a potential. Then, there exists a subsequence $\{\lambda_n'\}$ of $\{\lambda_n\}$ so that $\log|f(\lambda_n', l)|\to\varphi(l)$ *dg-a.e.*. This means $\varphi=-\infty$ on α of positive dg-measure, which is a contradiction since φ is dg-summable.

5.5. Relations between the boundary measure of a subharmonic function u and its radial seem to be not so simple, when the least harmonic majorant $\hat{u}$ of u has a singular part. On the other hand, if $\hat{u}$ is quasi-bounded, we can write $\hat{u}=\bar{u}=HR_\varphi$ for some φ. Then, we have

$$\begin{aligned}\overline{\lim_{\lambda\to 0}}\int|u(\lambda, l)-\varphi(l)|\,dg &\le \lim_{\lambda\to 0}\int[\bar{u}(\lambda, l)-u(\lambda, l)]dg\\ &\quad+\lim_{\lambda\to 0}\int|\bar{u}(\lambda, l)-\varphi(l)|\,dg\\ &\le\lim_{\lambda\to 0}[H_{\bar{u}}^{D^\lambda}(y_0)-H_u^{D^\lambda}(y_0)]\\ &\le\lim_{\lambda\to 0}[\bar{u}(y_0)-H_u^{D^\lambda}(y_0)]=0\,.\end{aligned}$$

This shows that $\varphi(l)dg$ is a boundary measure of u. Thus, we have

Theorem 9. *Under the assumption $MHB(\Omega)=MHD(\Omega)$, if the least harmonic majorant of a subharmonic function u is quasi-bounded, then the boundary measure of u exists and is absolutely continuous with respect to the Green measure.*

Now we state a generalization of Gårding-Hörmander's theorem [9].

Theorem 10. *We assume $MHB(\Omega)=MHD(\Omega)$. Let $\Psi[u]$ be a strongly* *subharmonic function possessing a harmonic majorant. Then*

(i) *for every sequence* $\{\alpha_n\}$ *of positive numbers tending to zero, there exists a subsequence* $\{\lambda_n\}$ *so that* $u(\lambda_n, l)dg$ *tends to a measure* $d\mu$ *on* $\mathcal{L}$ *vaguely.*

(ii) *if* $d\mu=\varphi dg+d\mu_S$ *is the Lebsgue decomposition of* $d\mu$ *into an absolutely continuous and a singular measure with respect to dg, then we have*

(1) $\mu_S\leq 0$.

(2) *denoting by* $\hat{u}_B$ *the quasi-bounded part of the least harmonic majorant* $\hat{u}$ *of u, if we write* $\hat{u}_B=HR_{\psi}$, *then* $\varphi\leq\psi$.

(3) *the boundary measure of* $\Psi[u]$ *is* $\Psi[\varphi]\, dg$.

(4) ψ *is the least majorant radial of u.*

Proof. In the decomposition of

$$u = \hat{u}_B+\hat{u}_S-p\ ,$$

where $\hat{u}_B$ and $\hat{u}_S$ are the quasi-bounded and the singular part of the least harmonic majorant $\hat{u}$ of u respectively and p is a potential, $\hat{u}_B$ and $-p$ have radials respectively, therefore they have boundary measures. Since $\hat{u}_S\leq 0$,

$$\int|\hat{u}_S(\lambda, l)|dg = \int[-\hat{u}_S(\lambda, l)]dg = -H^{D^\lambda}_{\hat{u}_s}(y_0)\leq -\hat{u}_S(y_0)\ ,$$

which means that the total mass of $|\hat{u}_S(\lambda, l)|dg$ is bounded. Hence, for a suitable $\{\lambda_n\}$, the vague limits of $\hat{u}_B(\lambda_n, l)dg$, $\hat{u}_S(\lambda_n, l)dg$ and $p(\lambda_n, l)dg$ exist. We denote them by $\psi\, dg$, $d\nu$ and 0 respectively. Now $\{\lambda_n\}$ and $d\mu=\psi\, dg+d\nu$ fulfill the proposition (i). Since $\hat{u}_S\leq 0$, in the Lebesgue decomposition of $d\nu =\varphi_0 dg+d\nu_S$, we see that $\varphi_0\leq 0$ and $\nu_S\leq 0$. Thus, if $d\mu=\varphi dg+d\mu_S$ is the Lebesgue decomposition, then we have $\varphi=\psi+\varphi_0$ and $\nu_S=\mu_S$. The propositions (1) and (2) in (ii) are now easily derived, and (3) is an immediate consequence of Theorem 7. To prove (4), we note first that ψ is a majorant radial of u. This is a consequence of relations:

$$u\leq\hat{u},\ \hat{u}-\hat{u}_B = \hat{u}_S\leq 0 \text{ and } \hat{u}_B = HR_{\psi}\,.$$

Indeed, in the following inequality

$$\begin{aligned}\int[u(\lambda, l)-\psi(l)]^+dg\leq&\int[u(\lambda, l)-\hat{u}(\lambda, l)]^+dg+\\&\int[\hat{u}(\lambda, l)-\hat{u}_B(\lambda, l)]^+dg+\\&\int[\hat{u}_B(\lambda, l)-\psi(l)]^+dg\ ,\end{aligned}$$

the first two terms of the right-hand side vanish and the remainder tends to zero as $\lambda\to 0$. Next, let ψ_1 be an arbitrary majorant radial of u, i.e.,

$$\lim_{\lambda\to 0}\int [u(\lambda, l)-\psi_1(l)]^+ dg = 0 .$$

We have clearly

$$\lim_{\lambda\to 0}\int [u^c(\lambda, l)-\psi_1^c(l)]^+ dg = 0 .$$

By Lemma 6, $\widehat{u^c}=\overline{u^c}=HR_{\psi^c}$ for all real numbers c. By making $\lambda\to 0$ in the following inequalities

$$\begin{aligned}\int [\psi^c-\psi_1^c]^+ dg &\le \int [\psi^c-u^c(\lambda, l)]^+ dg+\int [u^c(\lambda, l)-\psi_1^c]^+ dg \\ &\le \int |u^c(\lambda, l)-\psi^c|\, dg+\int [u^c(\lambda, l)-\psi_1^c]^+ dg ,\end{aligned}$$

we have

$$\int [\psi^c-\psi_1^c]^+ dg = 0 ,$$

i.e., $\psi^c \le \psi_1^c$ *dg-a.e.*. Since c is arbitrary, we get $\psi\le\psi_1$ *dg-a.e.*. Thus the proof is completed.

6. Relative Dirichlet problem on Green lines

6.1. The relative notions such as h-radial, h-indifferent etc. were introduced by L. Lumer-Naïm. We refer for them to [13] and [14]. Let h be a positive harmonic function on Ω. For simplicity, we assume that y_0 is not an infinity point. Throughout this section, we shall fix h. We introduce the notations

$$D^{\lambda,h} = \{y\in\Omega;\ G_{y_0}(y)/h(y)>\lambda\}$$

and

$$\Sigma^{\lambda,h} = \{y\in\Omega;\ G_{y_0}(y)/h(y) = \lambda\} .$$

The maximal orthogonal trajectories of a curve family $\{\Sigma^{\lambda,h}\}$ are termed *h-Green lines*. All h-Green lines issue from y_0. On the set of all h-Green lines $\mathcal{L}_h$, we can give a topology homeomorphic to the unit sphere. When we refer to the topological notions on $\mathcal{L}_h$, we agree that they are always considered with respect to this topology. On $\mathcal{L}_h$, we can also define a positive Radon measure, which will be called *h-Green measure* and is denoted by g_h, such that

1) the total mass of g_h is $h(y_0)$,

2) $g_h(e)=(\text{area of } e')\times h(y_0)/\sigma_\tau$ for every Borel set e on $\mathcal{L}_h$, where e' is the set on the unit sphere corresponding to e and σ_τ is a constant depending only on the dimension τ of Ω.

An h-Green line l is called *h-regular* if the infimum of G_{y_0}/h on l is zero. It is known that dg_h-almost all h-Green lines are h-regular.

Let f be a function defined quasi-everywhere[13] on $\check{\Omega}=\Omega-\{\text{infinity points}\}$. We denote by $[\lambda, l]$ the point where an h-Green line l intersects $\sum^{\lambda,h}$. f is termed to have an *h-radial* φ, if there exists a dg_h-integrable function $\varphi(l)$ on $\mathcal{L}_h$ such that

$$\overline{\lim_{\lambda\to 0}}\int |f([\lambda, l])-\varphi(l)|\, dg_h(l)=0 .$$

A harmonic function u is said to be *h-indifferent* if $\mathfrak{D}^{D^{\lambda,h}}_{\frac{u}{h},h}=u$ for all $\lambda>0$, where $\mathfrak{D}^{D^{\lambda,h}}_{\frac{u}{h},h}$ is an h-Dirichlet solution on $D^{\lambda,h}$ for a boundary function u/h[14].

6.2. An extension of the notion of BLD-functions was given by L. Lumer-Naïm as follows [13]: a function u is an *h-BLD function* if it is a quasi-everywhere finite limit of $u_n\in C^\infty(\check{\Omega})$ such that $||u_n-u||_h\to 0$, where

$$||u||_h^2=\int_{\check{\Omega}} h^2\left|\text{grad}\,\frac{u}{h}\right|^2 dV .$$

We obtain an extension of Godefroid's result [10] to h-BLD fuctions.

Theorem 11. *For any h-BLD function u, u/h has a finite limit along dg_h-almost every h-Green line.*

The proof is quite simlar to that of Godefroid's [10]. We shall omit the proof.

More interesting case is that where, instead of h-BLD functions, *BLD-functions* are connected with h-Green lines. By applying Ohtsuka's method of extremal length [16], we have the following theorem under the restriction that *h is bounded.*

Theorem 12. *Let h be a bounded positive harmonic function on Ω. Every BLD-function f has a finite limit along dg_h-almost every h-Green line.*

Proof. It is known that a BLD-function has a finite limit along each open curve except a family of curves of extremal length $+\infty$[15]. It is proved by a similar argument in [16] that the set of h-regular h-Green lines, along each of which f has a limit is dg_h-measurable.

If Γ is a set of h-regular h-Green lines of dg_h-measure positive, the module

13) f is said to be defined quasi-everywhere, if f is defined except a polar set.

14) For a relative Dirichlet solution $\mathfrak{D}_{f,h}$ we may refer to [2].

15) Cf. [16], th. 2, p. 68.

of $\Gamma'=\Gamma-D^{\lambda_0,h}$[16] is positive for sufficiently large λ_0. In fact, let ρ be admissible for Γ', that is, ρ is a measurable function in Ω such that

$$\int_l \rho\, ds \geq 1 \qquad \text{for all } l\in\Gamma' .$$

For $l\in\Gamma'$, we have

$$1\leq\left(\int_l \rho\, ds\right)^2\leq\left(\int_l \rho^2 |\operatorname{grad} G_{y_0}/h|^{-1} ds\right)\cdot\left(\int_l |\operatorname{grad} G_{y_0}/h|\, ds\right).$$

Since l is an h-Green line, on l locally by using a local coordinate $x_1, \cdots, x_\tau$ we have

$$\frac{dx_1}{h\cdot(G)_{x_1}-G\cdot(h)_{x_1}}=\cdots=\frac{dx_\tau}{h\cdot(G)_{x_\tau}-G\cdot(h)_{x_\tau}}=\frac{ds}{[\sum_{i=1}^{\tau}\{h\cdot(G)_{x_i}-G\cdot(h)_{x_i}\}^2]^{1/2}},$$

where we abbreviate G_{y_0}, $\partial h/\partial x_i$ etc. to G, $(h)_{x_i}$ etc., respectively. From

$$h^2|\operatorname{grad} G/h|^2=(1/h^2)\sum_{i=1}^{\tau}\{h\cdot(G)_{x_i}-G\cdot(h)_{x_i}\}^2$$

we get

$$\begin{aligned} d\lambda = d(G/h) &= \left|\sum_{i=1}^{\tau}\frac{h\cdot(G)_{x_i}-G\cdot(h)_{x_i}}{h^2}dx_i\right| \\ &= [\sum_{i=1}^{\tau}\{h\cdot(G)_{x_i}-G\cdot(h)_{x_i}\}^2]^{1/2}\frac{ds}{h^2} = |\operatorname{grad} G/h|\, ds. \end{aligned}$$

Therefore

$$1\leq\left(\int_l \rho^2|\operatorname{grad} G/h|^{-1}ds\right)\left(\int_l d\lambda\right)\leq\lambda_0\cdot\int_l \rho^2|\operatorname{grad} G/h|^{-1}\, ds,$$

from which we derive

$$g_h(\Gamma)/\lambda_0\leq\int_{\Gamma'}\rho^2|\operatorname{grad} G/h|^{-1}\, ds\, dg_h . \tag{6.1}$$

Since $ds\, dg_h=(h^2/\varphi_\tau)\ |\operatorname{grad} G/h|\, dV$, where φ_τ is a constant depending only on the dimension τ of Ω, the second member of (6.1) is equal to

$$1/\varphi_\tau\cdot\int_{\Gamma'}\rho^2 h^2\, dV .$$

Consequently, denoting by M the supremum of h^2, we have the estimate

$$g_h(\Gamma)/\lambda_0\leq(M/\varphi_\tau)\cdot\int_{\Gamma'}\rho^2\, dV,$$

16) We use Γ' to denote a curve family $\{l-D^{\lambda_0,h};\, l\in\Gamma\}$, i.e., consisting of curves in Γ taken off a neighbourhood of y_0. We shall also denote the set $\{x\in l-D^{\lambda_0,h},\ l\in\Gamma\}$ by the same Γ'.

which means that the module of Γ' is not less than $g_h(\Gamma)\cdot\varphi_\tau/(\lambda_0 M)>0$. Thus, we have proved that the exceptional family of curves of extremal length $+\infty$ is of dg_h-measure zero, which completes the proof.

Corollary. *Let h be a bounded positive harmonic function. Then, dg_h-almost every h-Green line tends to a point of the Kuramochi boundary Δ^K.*

This is an easy consequence of the above theorem and those of Maeda[17].

6.3. In the sequel, we shall always suppose that *h is bounded*. Under this assumption the axiom $(\mathcal{A}_h)$, that is, *all bounded continuous functions are h-resolutive*, is valid for the Kuramochi's compactification Ω^K. When Ω is a hyperbolic Riemann surface this is proved by H. Tanaka[18].

In view of the corollary to Theorem 12, we may consider the relative Dirichlet problem on h-Green lines in connection with compactifications.

Let φ be an extended real valued function on $\mathcal{L}_h$. The upper envelope of the class in which each function u is identically $-\infty$ or subharmonic satifying the conditions that u/h is bounded from above and

$$\overline{\lim_{\lambda\to 0}}\frac{u([\lambda, l])}{h([\lambda, l])}\leq\varphi(l) \qquad dg_h\text{-}a.e.,$$

is denoted by $\underline{\mathcal{G}}_{\varphi,h}$, where $[\lambda, h]$ is the point on l at which $G_{y_0}/h=\lambda$. We define $\overline{\mathcal{G}}_{\varphi,h}=-\underline{\mathcal{G}}_{(-\varphi),h}$.

Let f be a function on Δ^K. For every h-regular h-Green line tending to $x'\in\Delta^K$, we shall set $[f]_h(l)=f(x')$. The function $[f]_h$ is defined dg_h-almost everywhere. By giving the value zero for the other l, we consider $[f]_h$ is to be defined on $\mathcal{L}_h$. As in §1, we have

$$(6.2) \qquad \underline{\mathcal{D}}^K_{f,h}\leq\underline{\mathcal{G}}_{[f]_h,h}\leq\overline{\mathcal{G}}_{[f]_h,h}\leq\overline{\mathcal{D}}^K_{f,h},$$

where $\underline{\mathcal{D}}^K_{f,h}$ and $\overline{\mathcal{D}}^K_{f,h}$ are lower and upper relative Dirichlet solutions for a boundary function f with respect to the Kuramochi's compactification respectively.

A subharmonic function u is called to be *weakly h-minor* if $\mathcal{D}^{D^{\lambda,h}}_{\frac{u}{h},h}\geq u$ for all $\lambda>0$. Also, if $\lim_{\lambda\to 0}\int\left[\frac{u([\lambda, l])}{h([\lambda, l])}-\varphi(l)\right]^+ dg_h=0$ for some φ, then u/h is termed to have *a majorant h-radial φ*. $\underline{HR}_{\varphi,h}$ is defined to be the upper envelope of the class consisting of functions, each of which is identically $-\infty$ or weakly h-minor subharmonic u such that u/h has a majorant h-radial φ. We define $\overline{HR}_{\varphi,h}=-\underline{HR}_{(-\varphi),h}$ also.

17) Cf. [12] th. 1, p, 60 and th. 5, p. 64.

18) Cf. [18], Cor. 3, p. 55.

If φ is dg_h-integrable, then we can prove as in §1

$$\underline{\mathcal{G}}_{\varphi,h} \leq \underline{HR}_{\varphi,h} \leq \overline{HR}_{\varphi,h} \leq \overline{\mathcal{G}}_{\varphi,h}, \tag{6.3}$$

$$\underline{HR}_{\varphi,h}(y_0) \leq \int \varphi(l) dg_h \leq \overline{HR}_{\varphi,h}(y_0). \tag{6.4}$$

From (6.2), (6.3) and (6.4) we obtain the following theorem which is an extension of Maeda's result[19].

Theorem 13. *Let ω_h^K be an h-harmonic measure on Δ^K and let Λ be a family of all h-regular h-Green lines tending to a point of Δ^K. Denoting by $x'(l)$ the limit point of l on Δ^K for $l \in \Lambda$, we have for any $d\omega_h^K$-measurable set $e \subset \Delta^K$, the set $[e] = \{l \in \Lambda;\ x'(l) \in e\}$ is dg_h-measurable and $\omega_h^K(e) = g_h([e])$.*

By the above theorem we conclude in particular, if e is of $d\omega_h^K$-measure zero, then an h-Green measure of $[e]$ vanishes. As is shown in the following examples, the converse of this is not ture in general.

EXAMPLE 1. We consider $\Omega = R^3$ and the one-point compactification Ω^*. The Alexandroff point $\mathcal{A}$ is of harmonic measure 1, while all Green lines issuing from the origin tend to this point.

EXAMPLE 2. Let h be bounded minimal. Then $h = c \cdot K_{x_0}$, where K_{x_0} is the Martin kernel corresponding to a minimal Martin boundary point x_0. In this case, dg_h-almost every h-Green line tends to the same point $\Phi(x_0)$, the pole of x_0 on Δ^K[20]. In fact, denoting by f_0 the characteristic function of the set e_0 consisting of a single point $\Phi(x_0)$, we have

$$g_h(\mathcal{L}_h) = h(y_0) = \mathcal{D}_{f_0,h}^K(y_0) = \mathcal{G}_{[f_0]_h,h}(y_0) = \int [f_0]_h dg_h = g_h([e_0]).$$

6.4. The solution $u = HR_{\varphi,h}$ of the relative Dirichlet problem on h-Green lines for boundary function φ is characterized as a harmonic function such that

1) u is h-indifferent,
2) u/h has an h-radial φ.

In the ordinary Dirichlet problem on Green lines, to get the solution, we only find a quasi-bounded harmonic function having a given boundary function as a radial. The situation is however more difficult in the relative Dirichlet problem on h-Green lines. To solve the problem, following Naïm, we set

$$\Delta_h = \{x \in \Delta^M;\ \lim_{y \to x} G_{y_0}(y)/h(y) = 0\}.$$

It is known that $\Delta^M - \Delta_h$ is of $d\omega_h^M$-harmonic measure zero. Every positive

19) Cf. [12], th. 8, p. 65.
20) Cf. [11], p. 286.
21) Cf. [14], pp. 85–86.

harmonic function u can be decomposed into

$$u = u_{\Delta_h} + u_{\Delta^M - \Delta_h},$$

where u_{Δ_h} and $u_{\Delta^M - \Delta_h}$ are reduced functions of u on Δ_h and on $\Delta^M - \Delta_h$ respectively. u_{Δ_h} is h-indifferent and $(u_{\Delta^M - \Delta_h})/h$ has the h-radial zero[21]. Therefore, u is h-indifferent if and only if $u_{\Delta^M - \Delta_h}$ is so and u/h has an h-radial φ if and only if u_{Δ_h}/h has an h-radial φ. Thus we have

Theorem 14. *Let φ be a positive function on $\mathcal{L}_h$. If we can obtain a positive harmonic function u such that u/h has an h-radial φ, then u_{Δ_h} in the decomposition*

$$u = u_{\Delta_h} + u_{\Delta^M - \Delta_h},$$

is the solution $HR_{\varphi,h}$.

OSAKA CITY UNIVERSITY

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