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FLOW EQUIVALENCE OF DIFFEOMORPHISMS

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1. Introduction

Two dynamical systems are *equivalent* if there is a diffeomorphism h from one manifold to the other such that h maps every orbit of the dynamical system onto an orbit of the other preserving the natural orientations of orbits. For a diffeomorphism f on X, a dynamical system (M, ϕ) is constructed canonically as follows; $M = R \times X/(t, x) \sim (t+1, f^{-1}(X))$ and the flow ϕ is the one which is induced from the natural flow ψ on $R \times X$. Where, ψ is the 1-parameter group given by $\psi_t(u, x) = (u+t, x)$. This (M, ϕ) is called the *suspension* of f.

Let f and g be diffeomorphisms on X and Y respectively. If the suspensions of f and g are equivalent, the paris (X, f) and (Y, g) will be said to be *flow equivalent*.

By S. Smale ([4], [5]), it is shown that *if f and g are conjugate by a diffeomorphism $X \to Y$, then (X, f) and (Y, g) are flow equivalent.* In [2], the following result is shown. *Suppose that there exists no surjection from the fundamental group of X onto the infinite cyclic group Z. Then f and g are conjugate if and only if (X, f) and (Y, g) are flow equivalent.* If there is a surjection $\pi_1(X) \to Z$, there is an example of (X, f) and (Y, g) such that (X, f) and (Y, g) are flow equivalent but f and g are not conjugate. In [2] this example is shown when $X = Y = S^1$.

In §3 of this paper, we will show a sufficient condition on (X, f) and (Y, g) under which they will be flow equivalent, and will show examples of (X, f) and (Y, g) such that they will be flow equivalent but f and g will be not conjugate.

In §5, some results about flow equivalence of diffeomorphisms are mentioned. Our main result in this paper are Corollary (5.4) and Theorem (5.5), which can be simplified as follows.

Theorem A. *Let X, Y be compact connected manifolds which may possibly have boundaries. If (X, f) and (Y, g) are flow equivalent, then there exist regular-coverings $p: W \to X$ and $q: W \to Y$ with the common connected covrineg space W, such that both covering transformation groups are isomorphic to Z or trivial group* 1.

Theorem B. *Let (X, f) and (Y, g) be as in Theorem A. Then there exist coverings p: $W \to X$ and q: $W \to Y$ as in Theorem A such that for some pairs of* *positive integers* (m, n), *there exist a diffeomorphism h on W such that the following diagram is commutative.*

$$\begin{array}{ccccc} X & \xleftarrow{p} & W & \xrightarrow{p} & Y \\ \downarrow{\scriptstyle f^m} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle g^n} \\ X & \xleftarrow{p} & W & \xrightarrow{p} & Y \end{array}$$

If there exist no surjection $\pi_1(X)\rightarrow Z$, *we can take* m=n=1.

By the above result of *S*. Smale, Theorem *B* is an extension of the above result of [2].

For the prroof of theorems in §5, we will prepare Theorem (4.1) in §4. Let Y be a noncompact manifold in $R\times X$, and let p : $R\times X\rightarrow X$ denote the natural projection. Then, Theorem (4. 1) says that, under certain conditions, $p|Y: Y\rightarrow X$ is a covering map.

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2. Notations and elementary properties

Throuout this paper, all manifolds considered will be assumed to be of class C^r, $r\geqq 0$. In this paper, a C^0-manifold or a C^0-diffeomorphism means a topological manifold or a homeomorphism respectively.

A *dynamical system* or a flow of class C^r, $r\geqq 0$, on a manifold M is a C^r-map $\phi: R^1\times M\rightarrow M$ (R^1; the space of real numbers) such that if we put $\phi_t(t)=\phi(t, x)$, then

(i) $\phi_0(x) = x$

(ii) $\phi_{t+s}(x) = \phi_t\phi_s(x)$,

and ϕ_t is a C^r-diffeomorphism $(M, \partial M)\rightarrow(M, \partial M)$, where ∂M is the boundary of M.

By a pair (M, ϕ) we mean a dynamical aystem ϕ on a manifold M. (M, ϕ) and (M', ϕ') are said to be C^r-equivalent iff there is a C^r-diffeomorphism h : $M\rightarrow M'$ having the property that h maps every orbit of ϕ onto an orbit of ϕ' preserving the orientation. Such a map h will be called an *C^r-equivalence.*

Let R^n denote n-dimensional euclidean space and H^n denote the n-dimensional half space of R^n, i.e.

$$H^n = \{(x_1, \cdots, x_n)\in R^n | x_1\geqq 0\}.$$

Let X and F be C^r-submanifolds, $r\geqq 0$, of a C^r-manifold M such that dim X=n, dim $F=m$ and dim $M=n+m$ and that $F\subset \mathrm{Int}\, M$ or $F\subset\partial M$. Then, an intersection x of X with F is *transversal* iff the following conditions are satisfied; there are neighborhoods of the origins 0, V_1 in R^m and V_2 in R^n or H^n according as $F\subset \mathrm{Int}\, M$ or $F\subset\partial M$, and neighborhood U of x in M, and there is a C^r-diffeomorphism $\alpha : U\to V_1\times V_2$ such that,

$$\alpha(U\cap F) = \mathrm{V}_1\times\{0\}$$
$$\alpha(U\cap X) = \{0\}\times V_2.$$

A cross-section of a dynamical system, (M, ϕ), of class C^r is a compact connected C^r-submanifold X of codimension 1 of a compact manifold M such that $\partial X\subset\partial M$, and that

(i) X intersects every oribt
(ii) the intersection of X with each orbit is transversal
(iii) if $x\in X$, there is a $t>0$ with $\phi_t(x)\in X$, and
(iv) if $x\in X$, there is a $t<0$ with $\phi_t(x)\in X$.

If X is a cross-section of (M, ϕ), X is properly imbedded in M, i.e. $X\cap\partial M=\partial X$.

By $(M, \phi; X)$ we mean a dynamical system ϕ on a manifold M with a crosssection X.

For $(M, \phi; X)$ we can define a map $f : X\to X$ $f(x)=\phi_{t_0}(x)$ where t_0 is the smallest positive t satisfying $\phi_t(x)\in X$. f is a C^r-diffeomorphism; we call f the *associated deffeomorphism* of $(M, \phi; X)$.

Conversely, suppose that a C^r–diffeomorphism f of X onto itself is given. Define a C^r-diffeomorphism $\tau : R\times X\to R\times X$ by $\tau(t, x)=(t+1, \mathrm{f}^{-1}(x))$. Then the infinite cyclic group $\{\tau^m\}=Z$, operates freely on $R\times X$ and the orbit space $(R\times X)/Z$ is a manifold, say M_0. The flow $\psi_t : R\times X\to R\times X$ defined by $\psi_t(u, x)=(u+t, x)$ induces a flow ϕ_t of class C^r on M_0. We call this (M_0, ϕ_t) the *suspension* of f. M_0 has a cross-section $X_0=q(0\times X)\subset M_0$, where $q : R\times X\to M_0$ is the quotient map.

Two C^r–diffeomorphisms $f : M\to M$ and $g : N\to N$ are *C^s–conjugate* iff there exists a C^s–diffeomorphism $h : M\to N$ such that $hf=gh$.

The followings are easily proved, which are shown in [4] or [5] in the version of C^∞.

(2.1) *The associated diffeomorphism of the suspension of C^r–diffeomorphism $f : X\to X$ is C^r–conjugate to f.*

(2.2) *If $(M', \phi'; X')$ is the suspension of the associated diffeomorphism of a dynamical system $(M, \phi; X)$ of class C^r, then (N, ϕ) and (M', ϕ') are C^r–equivalent.*

(2.3) *Let (M, ϕ), (M, ϕ') be the suspensions of C^r–diffeomorphisms $f : X\to X, f' : X'\to X'$ respectively. If f and f' are C^r–conjugate, $s\leqq r$, then (M, ϕ) and (M, ϕ') are C^s–equivalent.*

By the pair (X, f) we mean a manifold X and a C^n–diffeomorphism $f : X\to$

X. (X, f) and (Y, g) are *flow C^s–equivalent* iff the suspensions of f and g are C^s–equivalent.

3. Different cross-sections

In this section, we will show some sufficient conditions under which (X, f) and (Y, g) will be flow equivalent, where f and g will be periodic homeomorphisms. And we will show some examples of (X, f) and (Y, g) which will be flow-equivalent but X and Y will not be homeomorphic or diffeomorphic.

Let X be a compact C^r–manifold, if $r=0$ let X be a polyhedral manifold. Let $f: X \to X$ be a C^r–diffeomorphism such that

(i) $f^n(x) = x, \qquad \forall x \in X;$

(ii) $f^i(x) \neq x$, for any i with $0 < i < n$, $\forall x \in X$.

Let (M, ϕ) be the suspension of f, then

$$M = R \times X/(t, x) \sim (t+1, f^{-1}(x)).$$

Put

$$Y = X/x \sim f(x).$$

Then there exists the following commuting diagram:

$$\begin{tikzcd} R \times X \arrow[r] \arrow[d, "p"] & S^1 \times X \arrow[r] \arrow[d, "p_1"] & M \arrow[d, "p_2"] \\ X \arrow[r, "id"] & X \arrow[r] & Y \end{tikzcd},$$

where, the map $R \times X \to S^1 \times X$ is defined by $(t, x) \mapsto (e^{i2\pi t/n}, x)$; the map $S^1 \times X \to M$ is defined by $(e^{i2\pi t/n}, x) \mapsto [(t, x)]$, and $[(t, x)]$ is the element of M containing (t, x) of $R \times X$; $X \to Y$ is defined by $x \mapsto [x]$; p_2 is defined by $[(t, x)] \mapsto [x]$ and p, p_1 are natural projections. We can simply see that these maps are well defined and the above diagram is commutative.

$S^1 \times X \to M$ and $X \to Y$ are covering maps with fibres consisting of n elements. $p_1 : S^1 \times X \to X$ has natural structure of trivial S^1–bundle. Here, we are using the same definitions about fibre bundles as in [6]. Using certain coordinate neighborhoods and coordinate functions of $p_1 : S^1 \times X \to X$, we can give a structure of coordinate bundle to $p_1 : M \to Y$ with group Z_n acting as orientation preserving rotations of S^1.

Notice that each fibre of $p_2 : M \to Y$ is a orbit of the flow (M, ϕ). If there is a $(C^0$–$)$ cross-section $q : Y \to M$ of the fibre bundle $p_2 : M \to Y$, by a small change of q we get a cross-section q_0 of class C^r, which is transversal to each fibre. $q_0(Y)$ can be considered as a cross-section of the dynamical system (M, ϕ). Now, we use the obstructuion theory to have a cross-section in [6].

Lemma 3.1. *Let $p_2 : M \to Y$ be as above. Then if $H^2(M\,; Z)=0$, p_2 has a cross-section.*

Proof. Since the group of the bundle $p_2 : M \to Y$ acts as the orientation preserving homeomorphism on S^1, by 30.3 and 30.4 in [6], the associated bundle of coefficients $\mathscr{B}(\pi_1(S^1))$ in Theorem 34.2 [6] is the product bundle. Hence, $H^2(Y; \mathscr{B}(\pi_1(S^1))=H^2(Y; Z)$. Since $\pi_n(S^1)=0$ for $n>1$, $H^{n+1}(Y; \mathscr{B}(\pi_n))=0$ for $n>1$. Therefore, by Theorem 34.2 in [6], the only obstruction to the exstence of a cross-section belongs to $H^2(Y; Z)$.

This proves Lemma (3.1).

Now, we show the following theorem.

Theorem 3.2. *Let $f : X \to X$ be a C^r–diffeomorphism on a compact manifold X satisfying $f^n(x)=x$ and $f^i(x) \neq x$ for any $x \in X$ and $0<i<n$. Then if $H^2(X/f : Z)=0$, (X, f) and $(X/f, id)$ are flow C^r–equivalent.*

Proof. Let (M, ϕ) be the suspension of f. Then, as shown above, there is a fibre bundle $p_2 : M \to X/f$ such that each fibre is an orbit of ϕ. By Lemma (3.1), there exists a cross-section of the fibre bundle.

This cross-section is approximated by a cross-section q of class C^r which is transversal to each fibre. This cross-section $q(X/f)$ of fibre bundle can be considered as a cross-section of the dynamical system (M, ϕ). Since the associated diffeomorphism of $(M, \phi; q(X/f))$ is the identity map, (X, f) and $(X/f, id)$ are flow equivalent.

EXAMPLE 1. dynamical systems having two cross-sections which are homeomorphic but not C^∞–diffeomorphic.

Let $\mathrm{Diff}_+(S^n)$ and $\mathrm{Diff}_+(D^{n+1})$ denote the groups of orientation preserving C^∞–diffeomorphisms on a sphere S^n and on a disk D^{n+1} resp., and let $r : \mathrm{Diff}_+(D^{n+1}) \to \mathrm{Diff}_+(S^n)$ denote the homomorphism obtained by the restriction. Then, the group $D(S^n)=\mathrm{Diff}_+(S^n)/\mathrm{Image}\, r$ is finite abelian for $n \geqq 4$ ([3]).

Suppose $f \in \mathrm{Diff}_+(S^n)$ and $[f] \neq 0$ in $D(S^n)$, and let p denote the order of $[f]$. Define a diffomorphism $\tilde{f} : S^1 \times S^n \to S^1 \times S^n$ by $(e^{i2\pi t}, x) \mapsto (e^{i2\pi(t+1/p)}, f(x))$. Then $S^1 \times S^n/\tilde{f}$ is diffeomorphic to the mapping torus of f, $S^n_f = I \times S^n/(0, x) \sim (l, f(x))$. $\tilde{f}$ is a periodic map with period p and $\tilde{f}^i$ has no fixed point for any $0<i<p$. Since $H^2(S^n; Z)=0$, Theorem (3.2) implies that $(S^1 \times S^n, \tilde{f})$ *and* (S^n_f, id) *are flow* C^∞*–equivalent.* $S^1 \times S^n$ and S^n_f *are homeomorphic but not diffeom orphic.* (See Lemma (6.2), [2]).

EXAMPLE 2. Dynamical systems having two cross-sections which are not homeomorphic in 2-dimensional case.

Let $T(m)$ denote a closed connected orientable 2-manifold of genus m.

Assume $m=gi+1$ and $T(m)$ is in R^3 with ig small holes symmetrically about one large hole as pictured in Figure 1 for the case when $g=3$.

Let $f : T(m) \to T(m)$ be the homeomorphism given by a rotation of $2\pi/\mathrm{i}$ degrees about z-axis. Then $T(m)/f = T(g+1)$. Here we set a result of J. L. Tollefson [8].

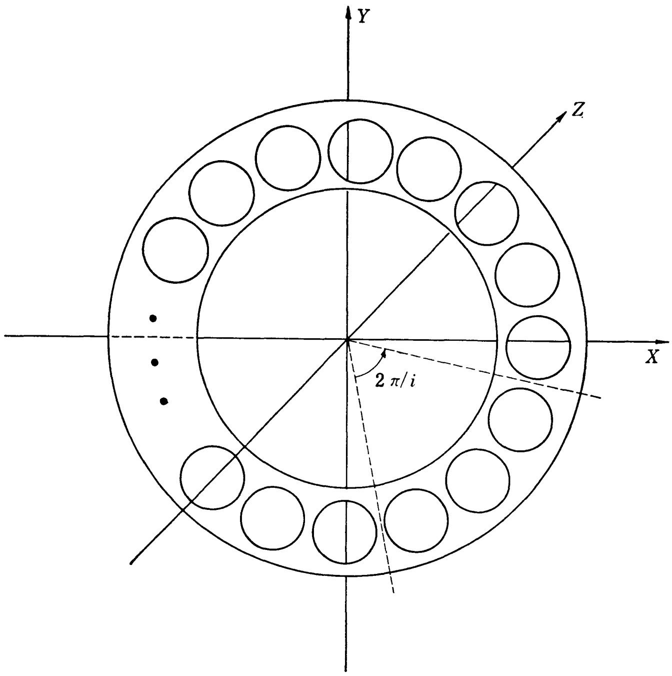

Fig. 1

(3.3) *For any integers* $i \geqq 1$ *and* $g \geqq 1$, f : $T(gi+1) \rightarrow T(gi+1)$ *be as above. Then,* $(T(gi+1), f)$ *and* $(T(g+1), id)$ *are flow equivalent.*

We can show another proof of (3.2) by using obstruction theory [6], it is as follows. Denote $M = R \times T(gi+1)/(t, x) \sim (t+1, f^{-1}(x))$. Then we have a S^1-bundle $p_2 : M \rightarrow T(g+1)$, as above. Let K^2 be a triangulation of $T(g+1)$. Over the 1-skeleton K^1, we have easily a cross-section of p_2, $h : K^1 \rightarrow M$. 2-dimentional obstruction cocycle $C(h)$, defined in [6], is in $H^2(K^2 ; Z)$. We can show that $C(h)$ is a coboundary. Hence, by Theorem 34.2 [6], there is a cross-section over K^2. Therefore, as the proof of Theorem (3.2), $(T(gi+1), f)$ and $(T(g+1), \text{id})$ are flow equivalent.

By (3.3), we have

Theorem 3.4. *For any two integers* m, n $\geqq 2$, *there exists a periodic dynamical system having two cross-sections* $T(m)$ *and* $T(n)$.

Proof. Let g be the g.c.m. of $m-1$ and $n-1$. If we put

$$\frac{m-1}{g} - 1 = p, \quad \frac{n-1}{g} - 1 = q,$$

we have p, $q \geqq 0$ and

$$m = g(p+1)+1$$
$$n = g(q+1)+1.$$

Let $(S^1 \times T(g+1), \phi)$ be the suspension of the identification map on $T(g+1)$. Then by (3.3), $(S^1 \times T(g+1), \phi)$ has cross-sections $T(m)$ and $T(n)$.

4. Covering theorem

The purpose of this section is to prove theorem (4.1).

Let X be an n-dimensional C^r-manifold not necessarily compact and Y be an n-dimensional C^r-submanifold (may be nonconnected) of $R \times X$. We set the following conditions on Y.

$C(1)$. Y is a closed subset of $R \times X$.

$C(2)$. Let $p : R \times X \to X$ be the projection on the second factor, then $p \mid Y : Y \to X$ is locally a C^r-diffeomorphism, that is, for any $y \in Y$ there are neighborhoods V of y and U of $p(y)$ such that V and U are C^r-diffeomorphic by p.

$C(3)$. For any $x \in X$, $p^{-1}(x) \cap Y$ is a discrete subset of $p^{-1}(x)$.

$C(4)$. For any $x \in X$ and any $y \in p^{-1}(x) \cap Y$, there exist y_i, y_j in $p^{-1}(x) \cap Y$ such that

$$\pi(\mathrm{y}_i) < \pi(\mathrm{y}) < \pi(\mathrm{y}_j).$$

Where π denotes the projection $R \times X \to R$ on the first factor.

Theorem 4.1. *Suppose X, Y and p be as above satisfying conditions $C(1), \cdots, C(4)$. Then $p \mid Y : Y \to X$ is a covering map of class C^r. Furthermore, this covering is a regular covering with covering transformation group isomorphic to Z.*

$p \mid Y : Y \to X$ is a submersion by $C(2)$. It is known that if a submersion from Y to X is a proper map then it fibres Y over X. (See[7] or, if Y is compact, [1].) But $C(4)$ implies that in our case $p \mid Y$ is not proper.

To prove Theorem (4.1) we will prepare some lemmas. X, Y and p in this section will be assumed to be the same as these in Theorem (4.1).

For any subset Z of X, let $\tilde{Z}$ denote a connceted component of $p^{-1}(Z) \cap Y$.

Lemma 4.2. *For any subset Z of X, $p(\tilde{Z})$ is an open subset of Z.*

Proof. Let z be any point in $p(\tilde{Z})$ and $\tilde{z}$ be any point in $p^{-1}(z) \cap \tilde{Z}$. As $p \mid Y$ is locally a homeomorphism, there exists a neighborhood U of $\tilde{z}$ in Y such that U and $p(U)$ are homeomorphic by p.

Since $\tilde{Z}$ is a component of $p^{-1}(Z)$, there exists a neighborhood V of $\tilde{z}$ in Y such that

$$V \cap p^{-1}(Z) = V \cap \tilde{Z}.$$

If we take V so that $V \subset U$, $V \cap p^{-1}(Z)$ and $p(V) \cap Z$ are homeomorphic by p.

Put $p(V) = W$. Then we have

$$\begin{aligned} W \cap Z &= p(V \cap p^{-1}(Z)) \\ &= \mathrm{p}(V \cap \tilde{Z}) \subset p(\tilde{Z}). \end{aligned}$$

Therefore, we have a neighborhood $W \cap Z$ of Z such that

$$z \in W \cap Z \subset p(\tilde{Z}).$$

This implies that $p(\tilde{Z})$ is open in Z.

Let C be a simple arc in X, i.e. C is the image of a C^r–diffeomorphism γ from $I = [0, 1]$ into X.

Lemma 4.3. *$p \mid \tilde{C} : \tilde{C} \to C$ is a C^r–diffeomorphism.*

Since $p \mid \tilde{C}$ is locally a C^r–diffeomorphism by $C(2)$, it is sufficient to prove that $p \mid \tilde{C} : \tilde{C} \to C$ is a bijection.

Sublemma 4.4. *$p \mid \tilde{C} : \tilde{C} \to C$ is an injection.*

Proof. Supppose that $p \mid \tilde{C}$ is not an injection. Then, there exist (t_0, x), $(t, x) \in \tilde{C} \to R \times X$ with $t_0 \neq t_1$. As $\tilde{C}$ is connected there exists a simple arc C' in $\tilde{C}$ with (t_0, x) and (t_1, x) as the ends. That is, C' is the image of a homeomorphism $\gamma' : I \to \tilde{C}$ such that $\gamma'(0) = (t_0, x)$ and $\gamma'(1) = (t_1, x)$.

Consider a function $\gamma^{-1} p \gamma' : I \to I$. We have $\gamma^{-1} p \gamma'(0) = \gamma^{-1} p \gamma'(1)$. If $\gamma^{-1} p \gamma'(0) = \gamma^{-1} p \gamma'(1)$ is not the maximal number of $\gamma^{-1} p \gamma'(I)$, let $s \in I$ be a number such that $\gamma^{-1} p \gamma'(s)$ is the maximal number of $\gamma^{-1} p \gamma'(I)$; if $\gamma^{-1} p \gamma'(0) = \gamma^{-1} p \gamma'(1)$ is the maximal number, let $s \in I$ be a number such that $\gamma^{-1} p \gamma'(s)$ is the minimal number.

Then, $s \in (0, 1)$.

As p maps Y locally homeomorphic into X, there exist neighborhoods U of $\gamma'(s)$ in Y and V of $p\gamma'(s)$ in X such that U and V are homeomorphic by p. Hence, $s \in (0, 1)$ implies that there exists a neighborhood $(s - \varepsilon, s + \varepsilon)$ of s in I such that $\gamma'(s - \varepsilon, s + \varepsilon) \subset U$. Hence, $\gamma^{-1} p \gamma'(s - \varepsilon, s + \varepsilon)$ is homeomorphic to $(s - \varepsilon, s + \varepsilon)$. But this contradict the fact that $\gamma^{-1} p \gamma'(s)$ is the maximal or the minimal.

Therefore, $p \mid \tilde{C}$ is an injection.

Suppose that $p \mid \tilde{C} : \tilde{C} \to C$ is not a surjection. Since $p \mid (\tilde{C})$ is an open subset of C by Lemma (4.2), there exist a and b such that

$$\begin{gathered} 0 \leqq a < b \leqq 1 \\ \gamma([a, b)) \subset p(\tilde{C}) \\ \gamma(b) \notin p(\tilde{C}) \end{gathered} \tag{4.5}$$

or

$$(4.5)' \qquad \begin{aligned} &0 \leqq b < a \leqq 1 \\ &\gamma((b, a]) \subset p(\tilde{C}) \\ &\gamma(b) \notin p(\tilde{C}). \end{aligned}$$

Since $p \mid \tilde{C}$ is an injection,

$$p^{-1}\gamma(t) \cap \tilde{C} = R \times \{\gamma(t)\} \cap \tilde{C}$$

is one point for any $t \in [a, b)$. Furthermore, by the condition $C(2)$, the map

$$[a, b) \to R \times X,$$

given by

$$t \mapsto p^{-1}\gamma(t) \cap \tilde{C},$$

is continuous.

Let $\pi : R \times X \to R$ be the projection on the first factor as before.

Sublemma 4.6. *Let C be a simple arc given by $\gamma : I \to X$, and suppose a and b satisfy* (4.5) *or* (4.5)$'$. *Then*

$$\lim_{t \to b} \pi(p^{-1}\gamma(t) \cap \tilde{C}) = \infty \qquad \text{or } -\infty.$$

Proof. We will prove in the case that a and b satisfy (4.5). It is sufficient, to prove

$$Cl(p^{-1}\gamma([a, b)) \cap \tilde{C}) \subset p^{-1}\gamma(b) = \phi.$$

Suppose

$$Cl(p^{-1}\gamma([a, b)) \cap \tilde{C}) \cap p^{-1}\gamma(b) \ni y.$$

Then, $y \in p^{-1}(C)$ and $y \in Y$ by the condition $C(2)$. Since $\tilde{C}$ is a connected component of $p^{-1}(C) \cap Y$, $y \in Cl(p^{-1}\gamma([a, b)) \cap \tilde{C})$ implies $y \in \tilde{C}$. Hence,

$$\gamma(b) = p(y) \in p(\tilde{C}).$$

This contradicts the above assumption that $\gamma(b) \notin p(\tilde{C})$. This proves (4.6).

Sublemma 4.7. $p \mid \tilde{C} : \tilde{C} \to C$ *is a surjection.*

Proof. Suppose that $p \mid \tilde{C}$ is not a surjection onto C. Then there exist a, b satisfying (4.5) or (4.5)$'$. Hence by (4.6),

$$\lim_{t \to b} \pi(p^{-1}\gamma(t) \cap \tilde{C}) = \pm\infty.$$

Suppose now $a < b$ and

$$\lim_{t\to b} \pi(p^{-1}\gamma(t)\cap \tilde{C}) = \infty.$$

In the other cases, we can do similarly.

For any point y_i in $p^{-1}\gamma(b)$, let $\tilde{C}_i$ be the connected component of $p^{-1}(C)\cap Y$ containing y_i. For a point y_0 in $p^{-1}\gamma(b)$, there is $c\in R$ satisfying

$$a\leqq c<b,$$
$$\gamma([c, b])\subset \mathrm{p}(\tilde{C}_0).$$

By $C(5)$, there exist infinite points $y_1, y_2, y_3, \cdots$ in $p^{-1}\gamma(b)$ such that

$$\pi(y_i)>\pi(y_0),\ i= 1, 2, 3, \cdots .$$

By (4.4) $p|\tilde{C}_i : \tilde{C}_i\to X$ is injection, $i=1, 2, 3, \cdots$. We put here the following assertion.

ASSERTION 4.8. $\gamma([c, b])\subset p(C_i)$, $i=1, 2, 3, \cdots$.

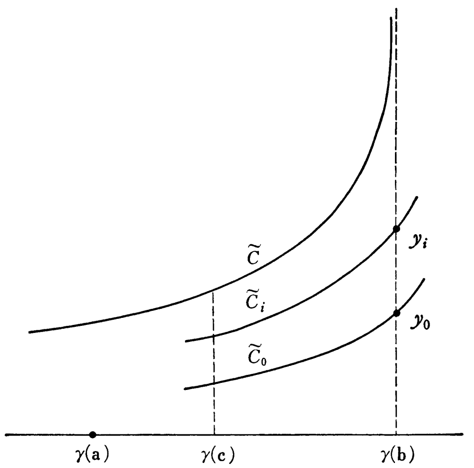

Fig. 2

If (4.8) is shown, the map

$$[c, b] \to \tilde{C}_i$$

given by

$$t \mapsto p^{-1}\gamma(t)\cap \tilde{C}_i$$

is into homeomorphism for any integer $i\geqq 0$.
But, (4.4) and (4.6) imply that the map

$$[c, b) \to \tilde{C}$$

given by

$$t \to p^{-1}\gamma(t) \cap \tilde{C}$$

is an into homeomorphism. $\tilde{C}$ and $\tilde{C}_i$ are contained in a plane $R \times \gamma(c)$. Therefore

$$p^{-1}\gamma(c) \cap \tilde{C}_0 < p^{-1}\gamma(c) \cap \tilde{C}_i < p^{-1}\gamma(c) \cap \tilde{C}$$

for any integer $i \geqq l$ (see Fig. 2). But, this contradicts condition $C(3)$.

Therefore, if (4.8) is proved the proof of Sublemma (4.7) is completed.

Proof of 4.8. Suppose that there exists $s \in [c, b)$ such that

$$\gamma(s) \notin p(\tilde{C}_i).$$

We have

$$b = \gamma^{-1}p(y_i) \in \gamma^{-1}p(\tilde{C}_i),$$

and that $\gamma^{-1}p(\tilde{C}_i) \cap [c, b]$ is an open set of $[c, b]$. Hence, the above assumption implies that there exists d such that

$$c \leqq d < b,$$

and that the connected component of $\gamma^{-1}p(\tilde{C}_i) \cap [c, b]$ containing b is $(d, b]$.

Then, by (4.6),

$$\lim_{t \to d} \pi(p^{-1}\gamma(t) \cap \tilde{C}_i) = \pm\infty, \qquad d < t < b.$$

But, this contradicts the fact that for any t with $d < t \leqq b$,

$$\pi(p^{-1}\gamma(t) \cap \tilde{C}_0) < \pi(p^{-1}\gamma(t) \cap \tilde{C}_i) < \pi(p^{-1}\gamma(t) \cap \tilde{C}).$$

This proves (4.8).

Therefore, the proof of Lemma (4.3) is completed.

Let I^m denote the closed m-cube in R^m as follows,

$$I^m = \{(x_1, \cdots, x_m) \in R^m \mid 0 \leqq x_i \leqq 1, \forall i\}.$$

And, let

$$I_t^{m-1} = \{(x_1, \cdots, x_m) \in I^m \mid x_1 = t\}.$$

Let Q^m denote *a* C^r m-cube in X^n, that is the image of *a* C^r-embedding j from I^m into X^n, and Q_t^{m-1} denote the image of I_t^{m-1} by j.

Lemma 4.9. *For any connected component $\tilde{Q}^m$ of $p^{-1}(Q^m) \cap Y$, $p \mid \tilde{Q}^m : \tilde{Q}^m \to Q^m$ is a C^r-diffeomorphism, $m=1, \cdots, n$.*

Proof. We will prove this lemma by using an induction. For $m=1$ the lemma is true, by Lemma (4.3).

Here, we suppose that for any connected component $\tilde{Q}^{m-1}$ of $p^{-1}(Q^{m-1})\cap Y$, $p|\tilde{Q}^{m-1} : \tilde{Q}^{m-1}\to Q^{m-1}$ is a C^r-diffeomorphism.

Let $\tilde{Q}^m$ be a fixed connected component of $p^{-1}(Q^m)\cap Y$.

First, we prove that $p|\tilde{Q}^m$ is an injection. Let

$$I^1 = \{(x_1, \cdots, x_m)\in I^m \,|\, x_i = 0,\ i\geqq 2\} \text{ and } U\subset I^1.$$

Denote

$$I^m_U = \{(x_1, \cdots, x_m)\in I^m \,|\, x_1\in U\}$$

and $Q^m_U=j(I^m_U)$. Let $Q^{m-1}_t=j(I^{m-1}_t)$ and $\tilde{Q}^{m-1}_t$ be a connected component of $p^{-1}(Q^{m-1}_t)\cap Y$ such that $\tilde{Q}^{m-1}_t\subset\tilde{Q}^m$. By the assumption of the induction, $\tilde{Q}^{m-1}_t$ and Q^{m-1}_t are homeomorphic by p. Then the following property follows from condition $C(2)$.

(*). For any $t\in I^1$, there exists a neighborhood $U\subset I^1$ of t such that

$$p|\tilde{Q}^m_U : \tilde{Q}^m_U \to Q^m_U$$

is a homeomorphism, where $\tilde{Q}^m_U$ is the connected component of $p^{-1}(Q^m_U)\cap Y$ such that $\tilde{Q}^m_U\supset\tilde{Q}^{m-1}_t$.

We define a map

$$\eta : \tilde{Q}^m \to p^{-1}(Q^1)\cap\tilde{Q}^m$$

as follows, where $Q^1=j(I^1)\subset Q^m$. For each $y\in\tilde{Q}^m$, there exists $t\in I^1$ such that $p(y)\in Q^{m-1}_t$. Let $\tilde{Q}^{m-1}_t$ be the connected component of $p^{-1}(Q^{m-1}_t)\cap Y$ containing y. Then, we define η by

$$\eta(y) = p^{-1}(Q^1)\cap\tilde{Q}^{m-1}_t.$$

Since the mapping $\tilde{Q}^{m-1}_t \to Q^{m-1}_t$, given by p, is a homeomorphism, and since $Q^{m-1}_t\cap Q^1$ is one point, η is well defined.

By using (*) we can easily see that η is continuous.

Suppose that $p|\tilde{Q}^m$ is not an injection. There exist $y_1, y_2\in\tilde{Q}^m$ such that $y_1\neq y_2$ and $p(y_1)=p(y_2)$. Then, by the assumption of the induction,

$$(**).\qquad \eta(y_1)\neq\eta(y_2),\ p\eta(y_1)\neq p(y_2).$$

Let C be an arc which joins y_1 with y_2 in $\tilde{Q}^m$. Since η is continuous, $\eta(C)$ is a connected subset of $p^{-1}(Q^1)\cap\tilde{Q}^m$. Hence, $\eta(C)$ is included in a connected component $\tilde{Q}^1$ of $p^{-1}(Q^1)\cap Y$. This contradicts Lemma (4.3).

This proves that $p\ |\tilde{Q}^m$ is an injection.

Next, we prove that $p|\tilde{Q}^m : \tilde{Q}^m\to Q^m$ is a surjection.

(***). $p^{-1}(Q^1)\cap\tilde{Q}^m\to Q^1$ is a homeomorphism by the map p.

To prove this, let $y\in\tilde{Q}^m$ be any point and let $\tilde{Q}^1$ be the connected component of $p^{-1}(Q^1)\cap Y$ including $\eta(y)$. Since $\tilde{Q}^1$ is connected,

$$\tilde{Q}^m \supset \tilde{Q}^1$$

Hence,

$$p^{-1}(Q^1) \cap \tilde{Q}^m \supset \tilde{Q}^1.$$

But $p|\tilde{Q}^1 : \tilde{Q}^1 \to Q^1$ is a homeomorphism by Lemma (4.3), and $p|(p^{-1}(Q^1) \cap \tilde{Q}^m)$ is an injection, as above. Hence, $p^{-1}(Q^1) \cap \tilde{Q}^m = \tilde{Q}^1$. Therefore $p|(p^{-1}(Q^1) \cap \tilde{Q}^m)$ is a homeomorphism.

Next, recall that

$$Q^m = \bigcup_{t \in I^1} Q_t^{m-1}.$$

To each Q_t^{m-1}, let $\tilde{Q}_t^{m-1}$ be the connected component of $p^{-1}(Q_t^{m-1}) \cap Y$ such that

$$\tilde{Q}_t^{m-1} \ni p^{-1}(Q_t^{m-1} \cap Q^1) \cap \tilde{Q}^m.$$

By assumption, $p(\tilde{Q}_t^{m-1}) = Q_t^{m-1}$.

Therefore, $p|\tilde{Q}^m : \tilde{Q}^m \to Q^m$ is a surjection.

Since $p|\tilde{Q}^m : \tilde{Q}^m \to Q^m$ is locally a C^r-diffeomorphism and is 1-1 and onto map, this is a C^r-diffeomorphism.

This completes the proof of Lemma (4.9).

Proof of Theorem 4.1.

1°. First, we show that $p|Y^n : Y^n \to X^n$ is a covering map. We can take an open covering of X^n

$$X^n = \bigcup_{\alpha} D_\alpha$$

such that each D_α is Int Q_α^n, where Q_α^n is a C^r-embedded image of I^n and the interior is considered in the topology of X^n. By Lemma (4.9),

$$p|p^{-1}(D_\alpha) \cap Y^n : p^{-1}(D_\alpha) \cap Y^n \to D_\alpha$$

is a trivial covering for any α. Therefore, $p|Y^n : Y^n \to X^n$ is a trival covering.

2°. Next, we show that this covering is a regular covering with transform ation group Z. Define a map

$$\sigma : Y^n \to Y^n$$

by $\sigma(t, x) = (t + t_0, x)$ for any (t, x) in Y, where t_0 is the smallest positive one satisfying $(t + t_0, x) \in Y \subset R \times X$. By condition $C(4)$, σ is wel defined.
Clearly σ is a C^r-covering transformation of the covering $Y^n \to Y^n$.

Let (t_1, x), (t_2, x) be any two elements in a fiber of the covering, and suppose that $t_1 < t_2$. Let r be the number of the elements of the set

$$\{(t, x) \in Y^n \mid t_1 < t \leqq t_2\}.$$

Then,

$$\sigma^r(t_1, x) = (t_2, x).$$

This implies that $p\mid Y^{n}$: $Y^{n}\rightarrow X^{n}$ is a regular covering and that the covering transformation group is isomorphic to Z having σ as a generator.

This completes the proof of Theorem (4.1).

5. Cross-section theorem

Throughout this section, X and Y will be compact C^{s}–manifolds, f and g will be C^{s}–diffeomorphism on X and Y respectively, and we suppose that the suspensions of f and g are C^{r}–equivalent, $s\geqq r\geqq 0$.

Let (M, ϕ) and (M', ϕ') be the suspensions of f and g respectively, and let $h: M'\rightarrow M$ be a C^{r}–equivalence. Then, X and $h(Y)$ are cross–sections of (M, ϕ) of class C^{s} and C^{r} respectively.

Let $\tau: R\times X\rightarrow R\times X$ be a C^{s}–diffeomorphism defined by $(t, x)\mapsto(t+1, f^{-1}(x))$. Then $\{\tau^{m}\}=Z$ operates freely on $R\times X$ and the orbit space $(R\times X)/Z$ is M. The quotient map $q: R\times X\rightarrow M$ is a regular covering with covering transformation group Z.

Denote $\tilde{Y}=q^{-1}h(Y)$, it is a C^{r}–submanifold of $R\times X$. Let $p: R\times X\rightarrow X$ be the projection on the second factor.

Lemma 5.1. *X, $\tilde{Y}$ satisfy the conditions $C(1), \cdots, C(4)$ in §4.*

Proof. $C(1)$. Since Y is compact, Y is a closed subset in M. $q: (R\times X, \tilde{Y})\rightarrow(M, Y)$ is locally a C^{r}–diffeomorphism. Hence, $\tilde{Y}$ is a closed subset in $R\times X$.

$C(2)$. The flow of $R\times X$, $\psi_{t}: R\times X\rightarrow R\times X$ is defined by $\psi_{t}(u, x)\times(u+t, x)$. $q: (R\times X, \tilde{Y})\rightarrow(M, Y)$ is locally a C^{r}–diffeomorphism having the property that q maps every orbit of ψ onto an orbit of ϕ. Since the intersection of Y with any orbit of ϕ is transversal, the above facts implies that $\tilde{Y}$ has transversal intersections with each orbit of ψ, $R\times\{x\}$. Hence, as $\tilde{Y}$ is a C^{r}–submanifold of $R\times X$, $p\mid\tilde{Y}: \tilde{Y}\rightarrow X$ is locally a C^{r}–diffeomorphism.

$C(3)$. Since $\tilde{Y}$ has transversal intersections with each orbit $R\times\{x\}$ of ψ as above, each element in $p^{-1}(x)\cap\tilde{Y}$ is an isolated point of $p^{-1}(x)\cap\tilde{Y}$. This proves $C(3)$.

$C(4)$. Let $y\in p^{-1}(x)\cap\tilde{Y}$. Since Y is a cross-section of (M, ϕ), there are $t>0$ and $t'<0$ such that $\phi_{tq}(y), \phi_{t'q}(y)\in Y$. Put, $\psi_{t}(y)=y_{i}$ and $\psi_{t'}(y)=y_{i}$. Then, y_{i} and y_{j} satisfy the condition $C(4)$.

These complete the proof of Lemma (5.1).

Theorem 5.2. *Let $f: X\rightarrow X$ and $g: Y\rightarrow Y$ be C^{s}–diffeomorphisms, where X, Y are compact connected manifolds which may possibly have boundaries.*

If (X, f) and (Y, g) are flow C^{r}–equivalent $(\infty\geqq s\geqq r\geqq 0)$, then there exist C^{r}–coverings $p: W\rightarrow X$ and $q: W\rightarrow Y$ with the common covering space W (which may be nonconnected) satisfying that

(i) *p and q are regular coverings with covering transformation groups isomorphic to infinite cyclic group Z,*

(ii) *there exists a C^r-diffeomorphism h on W such that*

$$p \circ h = f \circ p \text{ and } q \circ h = g \circ q$$

Before the proof of Theorem (5.2), we set a corollary of this theorem.

Corollary 5.3. *Let $f: X \to X$ and $g: Y \to Y$ be as in Theorem (5.2). If (X, f) and (Y, g) are flow C^r-equivalent, then the universal covering spaces of X and Y are C^r-diffeomorphic.*

Proof. Since X, Y are C^s-manifolds, the universal covering spaces $\tilde{X}$, $\tilde{Y}$ are considered to be of class C^s. Let $p: W \to X$, $q: W \to Y$ be the C^r-coverings obtained by Theorem (5.2), and let $r: \tilde{W} \to W$ be the universal covering (of class C^r) of W. Then $p \circ r: \tilde{W} \to X$ and $q \circ r: \tilde{W} \to Y$ are universal coverings of class C^r. By usual covering theory, we get C^r-diffeomorphisms $\tilde{W} \to \tilde{X}$ and $\tilde{W} \to \tilde{Y}$. Therefore $\tilde{X}$ and $\tilde{Y}$ are C^r-diffeomorphic.

Proof of Theorem 5.2. Let (M, ϕ) and (M', ϕ') be the suspensions of f and g respectively, $h: M' \to M$ be a C^r-equivalence and $q: R \times X \to M$ be the regular covering as above.

Put $W = q^{-1}h(Y)$. Since the covering transformation group of $q: R \times X \to M$ is isomorphic to Z, $h^{-1}q \mid W: W \to Y$ is a regular covering of class C^r with covering transformation group isomorphic to Z. Put $h^{-1}q \mid W = q$ also. We will show that $q: W \to Y$ is a desirable covering.

Let $p: R \times X \to X$ be the projection on the second factor, as above. By Lemma (5.1), X and W satisfy the conditions $C(1), \cdots, C(4)$ in §4. Therefore, by Theorem (4.1), $p \mid W: W \to X$ is a regular covering of class C^r with covering transformation group isomorphic to Z. Denoting $p \mid W$ by p also, we will show that this is another desirable covering.

The C^s-diffeomorphism $\tau: R \times X \to R \times X$ defined by $(t, x) \mapsto (t+1, f^{-1}(x))$ is a generator of the covering transformation group of the covering $R \times X \to M$. Hence, $\tau \mid W: W \to W$ is a C^r-diffeomorphism and is a generator of the covering transformation group of $q: W \to Y$. Next, as in the proof of Theorem (4.1), the C^r-diffeomorphism $\sigma: W \to W$ is a generator of the covering transformation group of $p: W \to X$. σ is defined by $(t, x) \mapsto (t+t_0, x)$, where t_0 is the smallest positive real number satisfying $(t+t_0, x) \in W$. By the definitions of τ and σ, we have the following commutative diagram.

$$\begin{tikzcd}
X \arrow[d, "f"] & W \arrow[l, "p"'] \arrow[r, "q"] \arrow[d, "\tau^{-1}"] & Y \arrow[d, "id"] \\
X \arrow[d, "id"] & W \arrow[l, "p"'] \arrow[r, "q"] \arrow[d, "\sigma"] & Y \arrow[d, "g"] \\
X & W \arrow[l, "p"'] \arrow[r, "q"] & Y
\end{tikzcd}$$

Hence, if we put $h=\sigma\circ\tau^{-1}$, we have $p\circ h=f\circ p$ and $q\circ h=g\circ q$. Therefore the proof of Theorem (5.2) is completed.

Corollary 5.4. *Let $f: X\to X$ and $g: Y\to Y$ be as above.*

If (X,f) and (Y,g) are flow C^r-equivalent, then there exist C^r-coverings $p: W_0\to X$ and $q: W_0\to Y$ with the common connected covering space W_0 such that p and q are both trivial coverings or both regular coverings with the covering transformation groups isomorphic to Z.

Proof. Let $p: W\to X$ and $q: W\to Y$ be the covering obtained by Theorem (5.2) and W_0 be a connected component of W. Let γ be the number of connected component of W, if it is finite. Then, $p|W_0: W_0\to X$ and $q|W_0: W_0\to Y$ are regular subcovering of p and q respectively. The covering transformation groups of $p|W_0$ and $q|W_0$ are subgroups of Z that is the covering transformation group of p and q. The subgroups are γZ, if γ is finite, or 0, if the number of connected components of W is infinite. Therefore, the covering transformation groups of $p|W_0$ and $q|W_0$ are isomorphic to Z or 0. This proves Corollary (5.4).

For the purpose of obtaining a result similar to Theorem (5.2) for connected covering spaces, we can show the next theorem, which involves Corollary (5.4).

Theorem 5.5. *Suppose that (X,f) and (Y,g) satisfy the same conditions as in Theorem* (5.2).

Then, there exist C^r-coverings $p: W_0\to X$ and $q: W_0\to Y$ with the common connected covering space W_0 satisfying the next conditions.

(i) *p and q are both trivial covering or both regular covering with covering transformation groups isomorphic to Z.*

(ii) *There exist C^r-diffeomorphisms h_i on W_0 $(i=1, 2, 3)$ such that for certain non-negative integers α, β, γ,*

$$p\circ h_1=f\circ p \text{ and } g\circ h_1=g^{\alpha}\circ q,$$
$$p\circ h_2=f^{\beta}\circ p \text{ and } g\circ h_2=g\circ q,$$
$$p\circ h_3=f^{\gamma}\circ p \text{ and } q\circ h_3=g^{\gamma}\circ q.$$

Furthermore, if p and q are the trivial covering, we can take $\alpha=\beta=\gamma=1$.

As a corollary of this theorem, using (2.3), we obtain the following result which is shown in a previous paper ([2], Theorem (4. 1)).

Corollary 5.6. *Suppose that there exists no surjection of $\pi_1(X)$ onto Z. Then (X,f) and (Y,g) are flow C^r-equivalent if and only if they are C^r-conjugate.*

Before proving Theorem (5.5) we prepare the following lemmas.

Let W, p, q, σ and τ be as in the proof of Theorem (5.2). Let γ be the number of connected components of W, if it is finite, and W_0 be a connected component of W.

If the number of the connected components of W is finite, $p|W_0$ and $q|W_0$ are regular subcoverings of p and q respectively. The covering transformation groups of $p|W_0$ and $q|W_0$ are $\{\sigma^{\gamma i}|i : \text{integer}\}$ and $\{\tau^{\gamma i}|i : \text{integer}\}$ respectively. Denote $W_i=\sigma^i(W_0)$, $i=0, \cdots, \gamma-1$. Then, $\sigma^k(W_0)=W_i$, where $i\equiv k$ (mod. γ). Define a map $\tau : Z_\gamma\to Z_\gamma$ by

$$W_{\tau(i)} = \tau(W_i),$$

where $Z_\gamma = \{0, 1, \cdots, \gamma-1\}$.

Lemma 5.7. *If the number of connected components of W is finite, then*
(i) $\tau : (0, 1, 2, \cdots, \gamma-1)\mapsto(\tau(0), \tau(1), \tau(2), \cdots, \tau(\gamma-1))$ *is a cyclic permutation.*
(ii) γ *and* $\tau(0)$ *are relatively prime.*

Proof. (*i*) Let $\tau(t, x)=(s, y)$ and $\tau(t', x)=(s', y)$. Then $t\leqq t'$ if and only if $s\leqq s'$. Hence we get easily the following commutative diagram.

$$\begin{array}{ccc} W_i & \xrightarrow{\sigma} & W_{i+1}(\text{mod. } \gamma) \\ \downarrow{\scriptstyle\tau} & & \downarrow{\scriptstyle\tau} \\ W_j & \xrightarrow{\sigma} & W_{j+1}(\text{mod. } \gamma) \end{array}$$

This implies (i) in Lemma (5.7), for $W_{\tau(i)}=\tau(W_i)$.

(ii) $q : W\to Y$ is a regular covering and τ is a generator of the covering transformation group Z. Since W_0 and $W_1=\sigma(W_0)$ is connected components of W, there is a positive integer a such that $\tau^a(W_0)=W_1$. This implies

$$W_{\tau^a(0)} = W_1$$

Hence

$$a\cdot\tau(0)\equiv\tau^a(0)\equiv 1 \text{ (mod. } \gamma).$$

Therefore, there exists a integer b such that

$$a\cdot\tau(0)+b\cdot\gamma = 1.$$

This implies that γ and $\tau(0)$ are relatively prime.

Lemma 5.8 *If the number of connected components of W is infinite, then for any connected component W_0 of W,*

$$\sigma(W_0) = \tau(W_0).$$

Proof. Since $p : W\to X$ is a regular covering with covering transformation group Z, the assumption of this lemma implies that this covering is trivial.

Hence $p : W_0\to X$ is a homeomorphism.

Therefore, the proof of this lemma is trivial, if we use Lemma (4.5) of [2] which says as follows.

(5.9). *suppose that $\tau^i W_0$ is homeomorphic to X by the map p and that $(s, x) \in \tau^i W_0$, $(t, x) \in \tau^j W_0$. Then, $s<t$ if and only if $i<j$.*

Proof of Theorem 5.5. Let p, q, σ, τ and W be as in the proof of Theorem (5.2). Let W_0 be a connected component of W. Then, Corollary (5.4) implies (*i*) of this theorem.

Let γ be the number of connected components of W. Denote $W_i=\sigma^i(W_0)$, $i=0, \cdots, \gamma-1$, and define a map $\tau: Z_\gamma \to Z_\gamma$ by $W_{\tau(i)}=\tau(W_i)$ as above.

We have following two commutative diagrams.

$$\begin{tikzcd} X \arrow[d, "f"] & W \arrow[l, "p"'] \arrow[r, "q"] \arrow[d, "\tau^{-1}"] & Y \arrow[d, "id"] \\ X & W \arrow[l, "p"'] \arrow[r, "q"] & Y \end{tikzcd} \qquad \begin{tikzcd} X \arrow[d, "id"] & W \arrow[l, "p"'] \arrow[r, "q"] \arrow[d, "\sigma"] & Y \arrow[d, "g"] \\ X & W \arrow[l, "p"'] \arrow[r, "q"] & Y \end{tikzcd}$$

Put $\alpha=\tau(0)$. Then, $\sigma^\alpha\tau^{-1}(W_0)=W_0$. We remark that, by Lemma (5.7), (i), the integer α is unique for any selection of a connected component W_0. If we put $h_1=\sigma^\alpha\tau^{-1}$, we have next commutative diagram which implies the first part in (ii) of this theorem;

$$\begin{tikzcd} X \arrow[d, "f"] & W_0 \arrow[l, "p"'] \arrow[r, "q"] \arrow[d, "h_1"] & Y \arrow[d, "g^\alpha"] \\ X & W_0 \arrow[l, "p"'] \arrow[r, "q"] & Y \end{tikzcd}$$

By Lemma (5.7), (ii), there are integers β and k such that

$$1=\beta\alpha+k\gamma.$$

Here we can take β in non-negative integers. In fact if $\beta<0$,

$$\begin{aligned} 1 &= \beta\alpha(\beta\alpha+k\gamma)+k\gamma \\ &= (\beta^2\alpha)\alpha+(1+\beta\alpha)k\gamma. \end{aligned}$$

Hence $\sigma\tau^{-\beta}(W_0)=W_0$. If we put $h_2=\sigma\tau^{-\beta}$, we have the following commutative diagram which implies the second part in (ii) of this theorem;

$$\begin{tikzcd} X \arrow[d, "f^\beta"] & W_0 \arrow[l, "p"'] \arrow[r, "q"] \arrow[d, "h_2"] & Y \arrow[d, "g"] \\ X & W_0 \arrow[l, "p"'] \arrow[r, "q"] & Y \end{tikzcd}$$

$p: W_0 \to X$ and $q: W_0 \to Y$ are regular subcoverings of $p: W \to X$ and $q: W \to Y$ respectively and the covering transformation group is γZ. Hence,

$$\sigma^\gamma(W_0) = W_0 \text{ and } \tau^\gamma(W_0) = W_0.$$

Therefore, $(\sigma\tau^{-1})^\gamma(W_0)=W_0$. If we put $h_3=(\sigma\tau^{-1})^\gamma$, we have the following commutative diagram which implies the third part in (ii) of this theorem;

$$\begin{tikzcd} X \arrow[d, "f^\gamma"] & W_0 \arrow[l, "p"'] \arrow[r, "q"] \arrow[d, "h_3"] & Y \arrow[d, "g^\gamma"] \\ X & W_0 \arrow[l, "p"'] \arrow[r, "q"] & Y \end{tikzcd}$$

If p and q are the trivial coverings, Lemma (5.8) implies that we can $\alpha=\beta=\gamma=1$.

These completes the proof of Theorem (5.5).

REMARK. Let (M, ϕ) and (M', ϕ') be the suspensions of $f : X\to X$ and $g : Y\to Y$, and let $h : M'\to M$ be the C^r-equivalence, as above. Let $R\to S^1$ be a covering defined by $t\mapsto e^{i2\pi t}$, and let $\tilde{\pi} : R\times X\to R$ denote by the natural projection. Since $W=q^{-1}h(Y)$, we have the following commutative diagram;

$$\begin{tikzcd} W \arrow[r, "\tilde{h}"] \arrow[d, "q"] & R\times X \arrow[r, "\tilde{\pi}"] \arrow[d, "q"] & R \arrow[d] \\ Y \arrow[r, "h"] & M \arrow[r, "\pi"] & S^1 \end{tikzcd}$$

where $\tilde{h}$ and π are naturally induced maps from h and $\tilde{\pi}$ respectively. Then we can show

(5.10). *The integer γ in Theorem (5.5) may be given as the number of connected components of W if it is finite, which is equal to the order of the group $\pi_1(S^1)/(\pi h)_*\pi_1(Y)$.*

In fact, we have the following commutative diagram;

$$\begin{tikzcd} & \pi_1(Y) \arrow[r] \arrow[d, "(\pi h)_*"] & \pi_0(F_1) \arrow[r] \arrow[d, "(\pi h)_*"] & \pi_0(W) \arrow[r] \arrow[d, "(\pi h)_*"] & \pi_0(Y) \\ \pi_1(R) \arrow[r] & \pi_1(S^1) \arrow[r] & \pi_0(F_2) \arrow[r] & \pi_0(R) & \end{tikzcd},$$

where, F_1 and F_2 are the fibres of $q : W\to Y$ and $R\to S^1$ respectively, and horizontal sequences are exact. Since $\pi_1(S^1)\to\pi_0(F_2)$ and $\pi_0(F_1)\to\pi_0(F_2)$ are isomorphisms, $\pi_0(W)$ is isomorphic to $\pi_1(S^1)/(\pi h)_*\pi_1(Y)$. This implies that the order of $\pi_1(S^1)/(\pi h)_*\pi(Y)$ is equal to the number of connected components of W.

A dynamical system is said to be *periodic* if any orbit is closed.

Theorem 5.11. *Let X and Y be two cross-sections of a periodic dynamical system (M, ϕ) such that there exists no surjection from $\pi_1(X)$ onto Z.*

Then, for each orbit C of ϕ, $X\cap C$ and $Y\cap C$ have the same number of elements.

Proof. Let f be the associated diffeomorphism of $(M, \phi; X)$. We may consider (M, ϕ) as the suspension of (X, f). Let $q : R\times X\to M$ be the covering map as above. Let $\tilde{Y}$ denote a connected component of $q^{-1}(Y)$. By Corollary (5.4) and the proof, $p\,|\,\tilde{Y} : \tilde{Y}\to X$ and $g\,|\,\tilde{Y} : \tilde{Y}\to Y$ are homeomorphisms, where p is the natural projection $R\times X\to X$. Put $\tilde{X}=\{0\}\times X\subset R\times X$. Let C be an orbit of ϕ. And let $X\cap C=\{x_1, \cdots, x_n\}$. Then,

$$q^{-1}(C) = R\times\{x_1, \cdots, x_n\}\subset R\times X.$$

And

$$q(\tilde{Y}\cap q^{-1}(C)) = Y\cap C.$$

Since $p\,|\,\tilde{Y}$ and $q\,|\,\tilde{Y}$ are homeomorphisms, we obtain that $X\cap\gamma$ and $Y\cap\gamma$ have the same number of elements.

This proves Theorem (5.11).

The manifolds and maps in the following corollaries should be considered in polyhedral category.

Corollary 5.12. *Let X be a compact manifolds with $H^2(X\,;Z)=0$. Then, there exists no nontrivial regular covering $\tilde{X}\to X$ such that the covering transformation group is isomorphic to a finite cyclic group and there is no epimorphism $\pi_1(\tilde{X})\to Z$.*

Corollary 5.13. *Let X be a compact manifold such that $H^2(X\,;Z)=0$ and there exists no epimorphism $\pi_1(X)\to Z$. Then, there exists no nontrivial regular covering with covering transformation group isomorphic to a finite cyclic group.*

Proof of (5.12) and (5.13). Suppose that there is a regular covering $\tilde{X}\to X$ with covering transformation group isomorphic to Z_p, $1<p<\infty$. Let $f:\tilde{X}\to\tilde{X}$ be a generator of the covering transformation group. Let (M, ϕ) denote the suspension of f. Since $H^2(X\,;Z)=0$, as in the proof of Theorem (3.2), X is a cross-section of (M, ϕ) such that, for any orbit C, $X\cap C$ is one point. (M, ϕ) is periodic. The assumptions on $\pi_1(\tilde{X})$ or $\pi_1(X)$ imply, by Theorem (5.11), that $\tilde{X}\cap C$ is one point for any orbit C of ϕ. But $f:\tilde{X}\to\tilde{X}$ is a periodic map with period $p>1$. This implies that $\tilde{X}\subset C$ consists of p elements. This is a contradiction. This proves (5.12) and (5.13).

Corollary 5.14. *Let X be a compact manifold. If $H^2(X\,;Z)=0$ and $\pi_1(X)\neq 1$, then $\pi_1(X)$ can not be a finite abelian group.*

Proof. Snppose that

$$\pi_1(X)\cong Z_{p_0}\oplus Z_{p_1}\oplus\cdots\oplus Z_{p_n}.$$

Let $\tilde{X}$ be the covering over X associated with $\pi_1(X)\supset Z_{p_1}\oplus\cdots\oplus Z_{p_n}$. Then, $\tilde{X}\to X$ is a regular covering with covering transformation group isomorphic to Z_{p_0}. Since there is no surjection $\pi_1(X) \to Z$, by Corollary (5.13), we get a contradiction. This proves (5.14).

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