Kamata, M.
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THE STRUCTURE OF THE BORDISM GROUP $U_*(BZ_p)$

Dedicated to Professor Keizo Asano on his 60th birthday

MASAYOSHI KAMATA

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In this paper, we determine the additive structure of the complex bordism group $U_*(BZ_p)$, where BZ_p is a classifying space for Z_p, p a odd prime. Conner-Floyd [1] computed the case $p=2$, and solved by a goemetric method. Here we use the Mischenko series [4] instead of the geometric method of Conner-Floyd.

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1. The order of the element $[L^{n-1}(p), i]$

We denote by $U_*(X, A)$ and $U^*(X, A)$ the complex bordism group and the complex cobordism group of a CW complex pair (X, A) respectively. Let $L^n(p)$ be a $(2n+1)$-dimensional lens space defined by a rotation Γ which acts on a $(2n+1)$-sphere S^{2n+1} in complex coordinate by $\Gamma(z_0, \cdots, z_n)=(\rho z_0, \cdots, \rho z_n)$ with $\rho=\exp(2\pi i/p)$. BZ_p is a CW complex of which the $(2n+1)$-skeleton is $L^n(p)$. The cell structure of $L^n(p)$ is given as follows:

$$L^n(p)=s^1 \cup_p e^2 \cup e^3 \cup_p \cdots \cup_p e^{2n} \cup e^{2n+1}.$$

Applying the exact sequence of the bordism group to a pair $(L^{n+1}(p), L^n(p))$, it follows immediately that

$$U_k(L^n(p)) \approx U_k(L^{n+1}(p)) \text{ for } k<2n+1.$$

In this section we study the order of the element

$$[L^{n-1}(p), i] \in U_{2n-1}(L^n(p)) \approx \cdots \approx U_{2n-1}(BZ_p),$$

where $i: L^{n-1}(p) \to L^n(p)$ is the inclusion. In order to determine the order of $[L^{n-1}(p), i]$, we use the duality isomorphism between bordism groups and cobordism groups, and the relation between K-theories and cobordism theories.

Theorem 1.1 (Atiyah-Kultze [3]). *If X is an n-dimensional compact U-manifold, there is an isomorphism $D: U_k(X) \to U^{n-k}(X)$.*

$D[M^k, f]$ is given as follows. For a large integer r such that $n+r-k$ is even, there is an embedding map $\tilde{f}: M^k \to S^r X^+ - \{*\}$, which is homotopic to the map $f: M^k \to X \subset S^r X^+ - \{*\}$, where $*$ denotes the base point. Denote by $N(M^k)$ the normal bundle of $M^k \subset S^r X^+ - \{*\}$, and there is a bundle map φ from $N(M^k)$ to the $(n+r-k)/2$-dimensional universal complex bundle $EU((n+r-k)/2)$. Then we can construct the map

$$d(f): S^r X^+ \to T(N(M^k)) \xrightarrow{\phi} MU((n+r-k)/2),$$

where $T(N(M^k))$ and $MU((n+r-k)/2)$ are Thom complexes of $N(M^k)$ and $EU((n+r-k)/2)$ respectively. $D[M^k, f] = [d(f)]$.

The following theorem which connects K-theories with cobordism theories was given by Conner-Floyd [1].

Theorem 1.2 (Conner-Floyd). *If X is a finite connected CW complex, the homomorphism $\rho: \tilde{K}(X) \to U^2(X)$ which maps $\{\xi^n\}-n$ into the 1-st cobordism Chern class $c_1(\xi^n)$ of ξ^n is the monomorphism of $\tilde{K}(X)$ onto a direct summand of $U^2(X)$.*

Let $\pi: L^n(p) \to CP^n$ be a canonical projection. If η is a canonical complex line bundle over CP^n, $\tau_c(CP^n) \oplus 1_c = (n+1)\eta$ and $\tau(L^n(p)) \oplus 1 = \pi^*(\tau(CP^n) \oplus 2)$, where $\tau(L^n(p))$ and $\tau(CP^n)$ are tangent bundles over $L^n(p)$ and CP^n respectively, and lower index c denotes a complex vector bundle. Therefore $L^n(p)$ is a U-manifold. Considering homomorphisms D and ρ of Theorems 1.1 and 1.2 for a space $L^n(p)$, we have the following

Proposition 1.3. $D[L^{n-1}(p), i] = \rho(\pi^*\eta - 1_c)$.

Proof. Let ν_c be the normal bundle of CP^{n-1} in CP^n. Since $\tau_c(CP^n) | CP^{n-1} = \tau_c(CP^{n-1}) \oplus \nu_c$,

$$r((\tau_c(CP^n) | CP^{n-1}) \oplus 1_c) = r(\tau_c(CP^{n-1}) \oplus 1_c \oplus \nu_c),$$

where r denotes the real restriction. Moreover,

$$\pi^* r(\tau_c(CP^n) \oplus 1_c) | CP^{n-1} = \pi^* r\{(\tau_c(CP^{n-1}) \oplus 1_c) \oplus \nu_c\},$$

which implies that $\pi^* r\nu_c$ is the normal bundle of $L^{n-1}(p)$ in $L^n(p)$. The total space $E(\tau_c(CP^n))$ of $\tau_c(CP^n)$ can be represented as the set of all pairs $[\vec{u}, \vec{v}]$ with $||\vec{u}|| = 1$, $\vec{u} \in C^{n+1}$ and $\langle \vec{u}, \vec{v} \rangle = 0$ by the standard Hermitian metric of C^{n+1}, under the identification $(\vec{u}, \vec{v}) \equiv (\lambda\vec{u}, \lambda\vec{v})$ for all $\lambda \in C^1$, $||\lambda|| = 1$. Now we define the Hermitian metric $F: E(\tau_c(CP^n)) \times E(\tau_c(CP^n)) \to C^1$ by

$$F([\vec{u}_1, \vec{v}_1], [\vec{u}_2, \vec{v}_2]) = \langle \vec{u}_1, \vec{u}_2 \rangle \langle \vec{v}_2, \vec{v}_1 \rangle .$$

Then the total space of ν_c is

$$E(\nu_c)=\{[\vec{u}, \vec{v}] \in E(\tau_c(CP^n)):\vec{u}\in C^n, \text{ and } F([\vec{u}, \vec{v}], [\vec{u}, \vec{v}_1])=0$$
$$\text{for each } [\vec{u}, \vec{v}_1]\in E(\tau_c\, (CP^{n-1}))\}\,,$$

that is, $E(\nu_c)$ consists of the elements $[\vec{u}, \vec{v}]$, where $\vec{v}=(0, \cdots, 0, z_n)$. Therefore $E(\pi^*\nu_c)$ can be represented as the set of all pairs $[\vec{u}, \vec{v}]$ with $||\vec{u}||=1$, $\vec{u}\in C^n$ and and $\vec{v}=(0, \cdots, 0, z_n)$ under the identification $(\vec{u}, \vec{v})\equiv(\rho\vec{u}, \rho\vec{v})$, $\rho=\exp(2\pi i/p)$. Consider the open submanifold

$$\tilde{L}^n(p)=\{[z_0, \cdots, z_n]\in L^n(p);\ |z_n|<1\}$$

of $L^n(p)$; there is a diffeomorphism

$$g\colon \tilde{L}^n(p) \to E(\pi^*\nu_c)$$

given by

$$g([z_0, \cdots, z_n])=[(z_0/\lambda, \cdots z_{n-1}/\lambda), (0, \cdots, 0, z_n/\lambda)],\ \lambda=\sqrt{\sum_{i=0}^{n-1}|z_i|^2}\,,$$

that is, $\tilde{L}^n(p)$ is the tubular neighborhood of $L^{n-1}(p)$ in $L^n(p)$. We define the map

$$f\colon E(\pi^*\nu_c) \to \eta',$$

by $f([z_0, \cdots, z_{n-1}], (0, \cdots, 0, z_n))=([z_0, \cdots, z_{n-1}], z_n)$, where η' is a canonical complex line bundle over CP^{n-1}. Let h be a standard homeomorphism between the Thom complex of η' and CP^n. Then, for $[L^{n-1}(p), i]\in U_{2n-1}(L^n(p))$, we have

$$d(i)([z_0, \cdots, z_n])=\begin{cases} h\circ f\circ g\ ([z_0, \cdots, z_n]), & |z_n|\neq 1 \\ [0, \cdots, 0, 1], & |z^n|= 1. \end{cases}$$

It follows that $\pi=d(i)$. Since $\rho(\pi^*\eta-1_c)=[\pi]$, the proposition follows. q.e.d.

Kambe [2] showed that the order of $\pi^*(\eta-1_c)\in\tilde{K}(L^n(p))$ is $p^{[(n-1)/(p-1)]+1}$ Then we have the following

Proposition 1.4. *$[L^{n-1}(p), i]\in U_{2n-1}(BZ_p)$ is of order $p^{[(n-1)/(p-1)]+1}$.*

2. The structure of $U_*(BZ_p)$.

We consider the 2n-skeleton $L_0^n(p)$ of $L_n(p)$, that is,

$$L_0^n(p)=s^1\cup_p e^2\cup e^3\cup_p \cdots \cup e^{2n-1}\cup_p e^{2n}.$$

Using the bordism exact sequence for a pair $(L^n(p), L_0^n(p))$, we have $U_k(L_0^n(p)) \approx U_k(L^n(p))$, for $k<2n$. Therefore, for a large n

$$U_{2k+1}(L^n(p))\approx \tilde{U}_{2k+1}(L^n(p))\approx \tilde{U}_{2k+1}(L_0^n(p)),\ U_{2k}(L^n(p))\approx U_{2k}(L_0^n(p)).$$

The bordism spectral sequence $\{E_{s,t}^r\}$ for $L_0^n(p)$ is trivial and if $s+t=2k$, then

$E_{s,t}^2 \approx 0 (s \neq 0)$, $E_{0,2k}^2 \approx U_{2k}$. It follows immediately that $U_{2k}(L^n(p)) \approx U_{2k}$.

Lemma 2.1. *If $\alpha[L^0(p), i]=0$ in $U_{2j+1}(L^n(p))$ for a large n and $\alpha \in U_{2j}$, then $\alpha \in pU_{2j}$.*

Proof. Since $U_{2j+1}(L^n(p)) \approx \tilde{U}_{2j+1}(L_0^n(p))$, we can assume that $\alpha[L^0(p), i] \in \tilde{U}_{2j+1}(L_0^n(p))$. Consider the reduced bordism spectral sequence $\{\tilde{E}_{t,s}^r\}$ for $L_0^n(p)$, which is trivial. There is a filtration $0 \subset J_{0,k} \subset J_{1,k-1} \subset \cdots \subset J_{k,0} \approx \tilde{U}_k(L_0^n(p))$ with $J_{s,t}/J_{s-1,t+1} \approx \tilde{H}_s(L_0^n(p), U_t)$. The multiplication

$$m: \tilde{U}_s(L_0^n(p)) \otimes U_t \to \tilde{U}_{s+t}(L_0^n(p))$$

induces the following commutative diagram

$$\begin{array}{ccc}
\tilde{U}_1(L_0^n(p)) \otimes U_{2j} = J_{1,0} \otimes U_{2j} & \xrightarrow{\mu \otimes id} & \tilde{H}_1(L_0^n(p)) \otimes U_{2j} \\
\downarrow m_1 & & \downarrow m_2 \\
J_{1,2j} & \xrightarrow{\mu'} & \tilde{H}_1(L_0^n(p), U_{2j})
\end{array}$$

where μ is the edge homomorphism.

$\alpha\mu[L^0(p), i] = m_2(\mu \otimes id)([L^0(p), i] \otimes \alpha) = \mu' \circ m_1([L^0(p), i] \otimes \alpha) = \mu' \alpha[L^0(p), i]) = 0$. On the other hand $\mu[L^0(p), i]$ is a generator of $\tilde{H}_1(L_0^n(p))$. Since $\tilde{H}_1(L_0^n(p))$ is p-torsion group, $\alpha \in pU_*$. q.e.d.

Lemma 2.2. *Suppose that X is an n-dimensional U-manifold. If $[M_1, f_1]$, $[M_2, f_2] \in U_*(X)$ are the elements represented by embedding maps $f_k: M_k \to X$ ($k=1, 2$). If the two embeddings are transversal to each other, then $D[M_1, f_1]D[M_2, f_2] = D[M_1 \cdot M_2, f_1 | M_1 \cdot M_2]$, where $M_1 \cdot M_2$ is intersection manifold of M_1 and M_2 in X.*

Proof. We can suppose that $M_1 \cdot M_2$ is a submanifold satisfying $N(M_1 \cdot M_2) = i^*_1 N(M_1) \oplus i^*_2 N(M_2)$, where $i_k: M_1 \cdot M_2 \to M_k (k=1, 2)$ is the inclusion map and $N(M)$ is the normal bundle of M in X. $D[M_1 \cdot M_2, f_1 | M_1 \cdot M_2]$ is constructed by the bundle map

$$\begin{array}{ccccccc}
N(M_1 \cdot M_2) & \xrightarrow{\tilde{\Delta}} & N(M_1) \times N(M_2) & \longrightarrow & EU(s) \times EU(t) & \longrightarrow & EU(s+t) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
M_1 \cdot M_2 & \xrightarrow{\Delta} & M_1 \times M_2 & \longrightarrow & BU(s) \times BU(t) & \longrightarrow & BU(s+t),
\end{array}$$

where Δ is a diagonal map, and s and t are the dimensions of $N(M_1)$ and $N(M_2)$ respectively. In view of the definition of multiplication in the cobordism group, we complete the proof.

Suppose that η is the canonical line bundle over CP^n, it follows from Lemma 2.2 that $\{c_1(\pi^*\eta)\}^k = D[L^{n-k}(p), i]$.

Mischenko obtained the following theorem [4], which plays an important role to deduce some relations of the elements of $U_{2k-1}(BZ_p)$.

Theorem 2.3 (Mischenko). *For a complex line bundle ξ over a CW complex X, define a series $g(c_1(\xi))$ by*

$$g(c_1(\xi))=\sum_{k\geq 0}\frac{x_k}{k+1}c_1(\xi)^{k+1}\in U^2(X)\otimes Q,\ x_k=[CP^k].$$

This satisfies, for line bundles ξ and η, the relation

$$g(c_1(\xi\otimes\eta))=g(c_1(\xi))+g(c_1(\eta)).$$

Proposition 2.4. *There exists $\alpha_a\not\equiv 0$ mod p such that*

$$p^a[L^{a(p-1)}(p),i]=\alpha_a[CP^{p-1}]^a[L^0(p),i].$$

Proof. The proof is by induction on a. Let η be the canonical complex line bundle over CP^p. By Theorem 2.3

$$g(c_1(\eta^p))=pg(c_1(\eta))=p\left\{c_1(\eta)+\frac{x_1}{2}c_1(\eta)^2+\cdots+\frac{x_{p-1}}{p}c_1(\eta)^p\right\}$$

and

$$(p-1)!g(c_1(\eta^p))=p!c_1(\eta)+p(p-1)\cdots\check{2}\cdot 1x_1c_1(\eta)^2+\cdots+(p-1)!x_{p-1}c_1(\eta)^p.$$

Since $U^*(CP^p)$ is torsion free, the above relation is an integral relation. Then, by the naturality of g and $(\pi^*\eta)^p=1$,

$$p!c_1(\pi^*\eta)+p(p-1)\cdots\check{2}\cdot 1x_1c_1(\pi^*\eta)^2+\cdots+(p-1)!x_{p-1}c_1(\pi^*\eta)^p=0.$$

Using Lemma 2.2,

$$p![L^{p-1}(p),i]+p(p-1)\cdots\check{2}\cdot 1x_1[L^{p-2}(p),i]+\cdots+(p-1)!x_{p-1}[L^0(p),i]=0.$$

Since the order of $[L^j(p),i]$ is p for $j<p-1$ and the order of $[L^{p-1}(p),i]$ is p^2 by Proposition 1.4,

$$p![L^{p-1}(p),i]+(p-1)!x_{p-1}[L^0(p),i]=0.$$

Since p is prime, the case $a=1$ follows. Suppose our assertion is true for $b<a$. Let ξ be the canonical line bundle over $CP^{a(p-1)+1}$. By Theorem 2.3,

$$g(c_1(\xi^p))=p\left\{c_1(\xi)+\frac{x_1}{2}c_1(\xi)^2+\cdots+\frac{x_{a(p-1)}}{a(p-1)+1}c_1(\xi)^{a(p-1)+1}\right\}.$$

Put $\{a(p-1)+1\}!=p^sm$, $m\not\equiv 0$ mod p. If $n!=p^un'$, $n'\not\equiv 0$ mod p then $u=\sum_{k\geq 1}[n/p^k]$. Hence

$$s=a\quad\text{if}\ a=p^r+p^{r-1}+\cdots+1,\quad s<a\quad\text{otherwise.}$$

Consider the following equation

$$Ag(c_1(\xi^p)) = Ap\left\{c_1(\xi)+\frac{x_1}{2}c_1(\xi)^2+\cdots+\frac{x_{a(p-1)}}{a(p-1)+1}c_1(\xi)^{a(p-1)+1}\right\},$$

where $A=\{a(p-1)+1\}!p^{a-s-1}$.
This is an integral relation. Therefore, using $(\pi^*\xi)^p=1$, the naturality of g and Lemma 2.2,

$$p^a m[L^{a(p-1)}(p), i]+\frac{p^a m}{2}x_1[L^{a(p-1)-1}(p), i]+\cdots+\frac{p^a m x_{a(p-1)}}{a(p-1)+1}[L^0(p), i] = 0 .$$

Denote by $o([L^t(p), i])$ the order of $[L^t(p), i]$. Suppose that

$$t = a(p-1)-(p^k n-1),\ n\not\equiv 0 \bmod p ,$$

By Proposition 1.4,

$$o([L^t(p), i]) = p^a \quad \text{if } k=1 \text{ and } n=1 ,$$
$$o([L^t(p), i]) = p^v,\ v<a-k+1 \text{ otherwise} .$$

Therefore,

$$p^a m[L^{a(p-1)}p), i]+p^{a-1}mx_{p-1}[L^{(a-1)(p-1)}(p), i] = 0 .$$

Since $m\not\equiv 0 \bmod p$, using the induction hypothesis, the proposition follows. q. e. d.

Let $\Gamma(p)$ be the polynomial subring of U_* generated by all $[Y_{2k}]\in U_{2k}$ with $k\neq p-1$. We note that $\Gamma(p)[CP^{p-1}]=U_*$.

Proposition 2.5. *Suppose we are given a relation*

$$\sum_{k=0}^{n}[L^k(p), i][M^{2(l-k)}] = 0 ,$$

with $[M^{2(l-k)}]\in\Gamma(p)$. *Then* $[M^{2(l-k)}]\in p^{[k/(p-1)]+1}\Gamma(p)$.

Proof. The proof is by induction on n. Lemma 2.1 implies that the case $n=0$ is true. Suppose our assertion is true for $m<n$. We consider

$$\sum_{k=0}^{n}[L^k(p), i][M^{2(l-k)}] = 0 \tag{1}$$

Applying Smith homomorphism to this equation, we have

$$\sum_{k=0}^{n}[L^{k-1}(p), i][M^{2(l-k)}] = 0 .$$

By the induction hypothesis $[M^{2(l-k)}]=p^{[(k-1)/(p-1)]+1}[N^{2(l-k)}]$. Since $\left[\dfrac{k-1}{p-1}\right]$

$=\left[\frac{k}{p-1}\right]$ for $k\neq a(p-1)$ and the order of $[L^k(p), i]$ is $p^{[k/(p-1)]+1}$ by Proposition 1.4, the equation (1) becomes

$$\sum_a p^a[L^{a(p-1)}(p), i][N^{2l-2a(p-1)}] = 0 .$$

From Proposition 2.4,

$$\sum_a \alpha_a[N^{2l-2a(p-1)}][CP^{p-1}]^a[L^0(p), i] = 0 .$$

Since $\alpha_a \not\equiv 0 \bmod p$, it follows from Lemma 2.1 that $[N^{2l-2a(p-1)}]\in pU_*$. This completes the proof.

Let $\Gamma_{2k}(p)$ consist of $2k$-dimensional homogenuous polynomial. Finally we have the following

Theorem 2.6. *The homomorphism*

$$\Theta\colon \sum_{k=0}^{n} \Gamma_{2(n-k)}(p)/p^{[k/(p-1)]+1}\Gamma_{2(n-k)}(p) \to U_{2n+1}(BZ_p)$$

given by $\Theta(\sum_{k=0}^{n} [M^{2(n-k)}]) = \sum_{k=0}^{n} [M^{2(n-k)}][L^k(p), i]$ *is isomorphism.*

Proof. The Proposition 2.5 is precisely the statement that Θ is monomorphism. To check that Θ is epimorphism, we compute the order of the group

$$\sum_{k=0}^{n} \Gamma_{2(n-k)}(p)/p^{[k/p-1]+1}\Gamma_{2(n-k)}(p) ,$$

and compare it with that of $U_{2n+1}(BZ_p)$. The former is p^τ, $\tau=\sum_{k=0}^{k} t_k\left\{\left[\frac{k}{p-1}\right]+1\right\}$, where t_k is the number of partitions of k, containing no $(p-1)$, the latter is p^σ, $\sigma=\sum_{k=0}^{n} s_k$, where s_k is the number of partitions of k. Now

$$\begin{aligned}\sigma = \sum_{k=0}^{n} s_k = \sum_k\sum_a t_{k-a(p-1)} &= \sum_j (\max\{a \mid j = k-a(p-1)\}+1)\,t_j \\ &= \sum_j \left(\left[\frac{j}{p-1}\right]+1\right)t_j = \tau .\end{aligned}$$

Thus Θ is an isomorphism. q. e. d.

OSAKA CITY UNIVERSITY

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