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A NOTE ON AZUMAYA'S THEOREM

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We say that a left Λ-module M is a cogenerator of the category of left Λ-modules if for every submodule N_1 of a left Λ-module N there exists a Λ-homomorphism f from N to M such that $f(N_1)\neq 0$. Let $\{I_\alpha\}$ be the full set of non isomorphic irreducible Λ-modules, and $\{E_\alpha\}$ be the set of their injective hulls. Then a left Λ-module M is a cogenerator if and only if M contains every E_α. In this case the sum of all E_α's is a direct sum by Zorn's Lemma (see Lemma 1 [3]). A cogenerator is a faithful module (Lemma 2). The aim of this paper is to compare the ring for which every faithful module is a cogenerator with the ring for which every faithful module is generator (see G. Azumaya [1]). We assume every ring has units and every module is unitary.

Lemma 1. *Let M be a left Λ-module and A be an arbitrary set of index. Then the followings are equivalent.*

(1) *M is a cogenerator*

(2) *$\sum^{\oplus}_{\nu\in A} M_\nu$ $(M_\nu\cong M)$ is a cogenerator*

(3) *$\Pi_{\nu\in A} M_\nu$ $(M_\nu\cong M)$ is a cogenerator*

Proof. (1)$\Rightarrow$(2)$\Rightarrow$(3) is clear by Lemma 1 [3]. So we shall prove (3)$\Rightarrow$(1). Choose any E_α, then we have Λ-maps $E_\alpha \xrightarrow{\tau} \Pi M_\nu \xrightarrow{\pi_\nu} M_\nu$ where τ is a monomorphism and π_ν are the canonical maps. Let $f_\nu=\pi_\nu\cdot\tau$ then we see $\bigcap \ker f_\nu=0$. If $\ker f_\nu\neq 0$ for every $\nu\in A$, then $I_\alpha\subseteq \bigcap \ker f_\nu \neq 0$ since I_α is irreducible and E_α is an essential extension of I_α. Hence M_ν has an isomorphic image of E_α and M is a cogenerator.

Corollary 1. *A ring Λ is a self-cogenerator ring if and only if every E_α, i.e., $\sum^{\oplus} E_\alpha$ is projective.*

Proof. If $\sum^{\oplus} E_\alpha$ is projective, $\sum^{\oplus} E_\alpha < \oplus\sum^{\oplus}\Lambda$, and $\sum^{\oplus}\Lambda$ is a cogenerator. Hence Λ is a cogenerator by Lemma 1. Conversely if Λ is a cogenerator, $\Lambda\oplus > E_\alpha$, and E_α is projective.

Lemma 2. *If M is a cogenerator and $\mathfrak{l}$ is a left ideal of Λ, then $\mathfrak{l}=l_\Lambda(r_M(\mathfrak{l}))$. Hence every cogenerator is faithful. Conversely assume that Λ is a left self-cogenerator ring. Then every faithful module is a cogenerator.*

Proof. In general we have $l_\Lambda(r_M(\mathfrak{l}))\supseteq\mathfrak{l}$. If $l_\Lambda(r_M(\mathfrak{l}))\neq\mathfrak{l}$, we have a Λ-homomorphism f from $\Lambda/\mathfrak{l}$ to M such that $f(l_\Lambda(r_M(\mathfrak{l})/\mathfrak{l})\neq 0$. This means that there exists $m\in M$ such that $\mathfrak{l}m=0$ and $l_\Lambda(r_M(\mathfrak{l}))\,m\neq 0$, which is a contradiction since $r_M(l_\Lambda(r_M(\mathfrak{l})))=r_M(\mathfrak{l})$. Thus $\mathfrak{l}=l_\Lambda(r_M(\mathfrak{l}))$. Since $0=l_\Lambda(r_M(0))=l(M)$, M is faithful. If Λ is a left self cogenerator ring and M is a faithful left module, we have a monomorphism f from Λ to ΠM such that $f(\lambda)=(\lambda m)_{m\in M}$. Hence ΠM is a cogenerator, consequently M is a cogenerator by Lemma 1.

Lemma 3. *If Λ is a self cogenerator ring, $l(\mathfrak{r})$ is 0 or a minimal left ideal for every maximal right ideal of Λ and $r(\mathfrak{l})$ is an essential extension of a minimal right ideal for every maximal left ideal. Consequently every right ideal (resp. if Λ is right lower distinguished, every left ideal) constains a minimal right (resp. left) ideal of Λ.*

Proof. If $l(\mathfrak{r})\neq 0$, $\mathfrak{r}=r(l(\mathfrak{r}))=r(\mathfrak{l}')$, and $l(\mathfrak{r})=l(r(\mathfrak{l}'))=\mathfrak{l}'$, where $\mathfrak{l}'$ is an arbitrary left subideal of $l(\mathfrak{r})$. Thus $l(\mathfrak{r})$ is a minimal left ideal. Let $\mathfrak{l}$ be a maximal left ideal. Then there exists a minimal left ideal Λx such that $\Lambda/\mathfrak{l}\cong\Lambda x$ and $E(\Lambda x)\subseteq\Lambda$. Let $y\in r(\mathfrak{l})$, then $\Lambda y\cong\Lambda/\mathfrak{l}\cong\Lambda x$. Hence there exists $z\in E(\Lambda x)$ such that $yz=x$, since $E(\Lambda x)$ is injective. Therefore $y\Lambda\supseteq x\Lambda$ for any $y\in r(\mathfrak{l})$, and we see $x\Lambda$ is minimal and $r(\mathfrak{l})$ is an essential extension of $x\Lambda$. Let y be an arbitrary element of Λ and $\mathfrak{l}$ a maximal left ideal such that $\mathfrak{l}\supseteq l(y)$. Then there is a Λ-homomorphism $\Lambda y\cong\Lambda/l(y)\to\Lambda/\mathfrak{l}\cong\Lambda x$, where $E(\Lambda x)\subseteq\Lambda$. Then $yz=x$ for some $z\in E(\Lambda x)$, and $x\Lambda\subseteq y\Lambda$. Since $x\in r(\mathfrak{l})$, $x\Lambda$, consequently, $y\Lambda$ contains a minimal right ideal. Thus every right ideal contains a minimal right ideal. If Λ is right lower distinguished, $l(\mathfrak{r})\neq 0$ for every maximal right ideal. Let $\mathfrak{l}$ be an arbitrary left ideal. Then $\mathfrak{l}=l(r(\mathfrak{l}))\supseteq 1(\mathfrak{r})$, where $\mathfrak{r}$ is a maximal right ideal such that $\mathfrak{r}\supseteq r(\mathfrak{l})$. Therefore $\mathfrak{l}$ contains a minimal left ideal.

Proposition 1. *Let Λ be a left self cogenerator ring and $\{E_\alpha\}$, $\{I_\alpha\}$ be as before. If $E_\alpha=\Lambda e_\alpha$, then $e_\alpha\Lambda$ is an essential extension of a minimal right ideal $x_\alpha\Lambda$. $x_\alpha\Lambda$ and $x_\beta\Lambda$ are not isomorphic for any $e_\alpha\neq e_\beta$.*

Proof. $l(e_\alpha)=\Lambda(1-e_\alpha)$. By Lemma 3 [3] $\Lambda(1-e_\alpha)\oplus Ne_\alpha$ is a maximal ideal. Hence $\Lambda/\Lambda(l-e_\alpha)\oplus Ne_\alpha\cong\Lambda e_\alpha/Ne_\alpha\cong\Lambda x_\alpha$ where $E(\Lambda x_\alpha)\subseteq\Lambda$. Then the same argument as the proof of Lemma 3 shows that $x_\alpha\Lambda$ is a minimal right ideal and $e_\alpha\Lambda$ is an essential extension of $x_\alpha\Lambda$. If $x_\alpha\Lambda\cong x_\beta\Lambda$, then $r(x_\alpha)=r(x_\beta z)$ for some $z\in\Lambda$. Then $\Lambda x_\alpha=l(r(\Lambda x_\alpha))=l(r(\Lambda x_\beta z))=\Lambda x_\beta z\cong\Lambda x_\beta$, since $\Lambda x_\beta\cong\Lambda e_\beta/Ne_\beta$ is minimal. This is a contradiction since $\Lambda e_\alpha/Ne_\alpha\not\cong\Lambda e_\beta/Ne_\beta$. Hence $e_\alpha\Lambda\not\cong e_\beta\Lambda$.

The following lemma is well known.

Lemma 4. *Let P_1, P_2 be finitely generated projective module. Then $P_1\cong P_2$ if and only if $P_1/NP_1\cong P_2/NP_2$.*

Proposition 2. *If Λ/N is a semi-simple ring where N is the radical of Λ,* *and Λ is a left self cogenerator ring, then Λ is a finite direct sum of injective hulls of irreducible left ideals. Every commutative self cogenerator ring is self-injective and a direct sum of a finite number of injective hulls of irreducible ideals such that any two of which are not isomorphic.*

Proof. If Λ/N is semisimple, Λ has only a finite number of non isomorphic irreducible modules, say $I_1, I_2, \cdots, I_n$. Let E_i $(i=1, \cdots, n)$ be their injective hulls. Then E_i/NE_i's are all the non isomorphic irreducible modules by Lemma 3 [3] and Lemma 4. Since $\Lambda/N \cong I_1^{\alpha_1}+I_2^{\alpha_2}+\cdots+I_n^{\alpha_n} \cong (E_1/NE_1)^{\alpha_1}+(E_2/NE_2)^{\alpha_2}+\cdots+(E_n/NE_n)^{\alpha_n}$, $\Lambda \cong E_1^{\alpha_1}+E_2^{\alpha_2}+\cdots+E_n^{\alpha_n}$ by Lemma 4. Let R be a commutative self cogenerator ring. Then $R \supseteq \sum^{\oplus} E_\alpha$. Since $\sum^{\oplus} E_\alpha$ is faithful $\sum^{\oplus} E_\alpha = l(r(\sum^{\oplus} E_\alpha)) = l(0) = R$. Then the number of E_α's is finite and any two of which are not isomorphic.

Now we consider following two properties of a ring Λ.

(*) Every faithful left module is a generator

(**) Every faithful left module is a cogenerator.

Then we have

Theorem 1. *Λ has Property (*) if and only if Λ/N is semisimple and Λ has Property (**). In case Λ is commutative Property (*) and Property (**) are equivalent.*

Proof. By Theorem 6 G. Azumaya [1] we see that a ring Λ has Property (*) if and only if Λ is left self injective and a finite direct sum of indecomposable left ideals each of which is an essential extension of a minimal left ideal. In this case Λ/N is a semisimple ring (see Theorem 7 [7].) Hence the proof is straightforward by Proposition 2 and Lemma 2.

REMARK 1. Proposition 1 means that a left self cogenerator ring contains at least the same number of the isomorphism types of irreducible right ideals as that of irreducible left ideals. Hence if in addition Λ/N is a semisimple ring Λ is right lower distinguished, since Λ/N, consequently Λ has the same number of isomorphism types of irreducible right ideals as that of irreducible left ideals (see Theorem 7 [1]).

REMARK 2. Lemma 3 [3] can be stated more generally as follows.

Lemma 5. *If M is a projective, injective and indecomposable Λ-module, then M/NM is an irreducible Λ-module, and M is isomorphic to a principal left ideal of Λ.*

Proof. We need only to prove that $M \cong \Lambda e$, $e^2=e$, and $\Omega = \mathrm{Hom}(\Lambda e, \Lambda e)$ is a local ring. Since M is Λ-projective, M is contained in a free Λ-module, $F=\sum \Lambda u_\alpha$. Let m be a non zero element of M and $m=\sum_{i=1}^{m} \lambda_i u_{\alpha_i}$, $\lambda_i \in \Lambda$.

If $\lambda\lambda_i=0$ implies $\lambda m=0$ for any $\lambda\in\Lambda$ and some i, then $\Lambda m\cong\Lambda\lambda_i\subseteq\Lambda$. If we consider the projection π_{α_i} of F to Λ, then $M\cong\pi_{\alpha_i}(M)$ since M is an essential extension of Λm. If $\lambda m\neq 0$ and $\lambda\lambda_1=0$ for some $\lambda\in\Lambda$, then replacing m by λm we may assume $m=\sum_{i=2}^{n}\lambda_i u_{\alpha_i}$. Repeating this argument we can show as above that $M\cong\Lambda e$, $e^2=e$. Since Λe is an essential extension of every subideal, we can easily show that $\mathrm{Hom}(\Lambda e, \Lambda e)\cong e\Lambda e$ is a local ring. Then the same proof as Lemma 3 [3] shows that M/NM is irreducible.

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References

[1] G. Azumaya: *Completely faithful modules and self-injective rings,* Nagoya Math. J. **27** (1966), 697–708.

[2] E. Matlis: *Injective modules over Noetherian rings,* Pacific J. Math. **8** (1958), 511–528.

[3] B.L. Osofsky: *A generalization of quasi-Frobenius rings,* J. Algebra **4** (1966), 373–387.