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K_U-GROUPS OF DOLD MANIFOLDS

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Introduction. J. F. Adams [1] calculated the Grothendieck rings K_U of the projective spaces. The manifold $D(m, n)$, defined by A. Dold in his study of cobordism theory [6], is regarded as a generalization of the projective spaces.

The purpose of this paper is to calculate K_U of the Dold manifold $D(m, n)$; the result is stated in Theorem (3.14) of §3. For this purpose, we construct a real 2-plane bundle η_1 over $D(m, n)$ which is a generalization of the real restriction of the canonical complex line bundle over $CP(n)$ and also of the bundle sum of the canonical real line bundle over $RP(m)$ and the trivial line bundle over $RP(m)$. This bundle η_1 plays an important role in computations. On the way of computations, we make use of mod 2 K_U-theory which is introduced by S. Araki and H. Toda [2].

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1. Cohomology rings of Dold manifolds

Let S^m, $m \geqq 0$, denote the unit m-sphere in R^{m+1} with the coordinates $x_0, x_1, \cdots, x_m$, and let $CP(n)$, $n \geqq 0$, denote the complex projective n-space with the homogeneous coordinates $z_0, z_1, \cdots, z_n$. Consider the product space $S^m \times CP(n)$ and difine a homeomorphism $T: S^m \times CP(n) \to S^m \times CP(n)$ by

$$(1.1) \qquad T(x, z) = (-x, \bar{z}) \qquad (x \in S^m, z \in CP(n)),$$

where $-x$ is the antipodal point of x and $\bar{z}$ is the conjugate point of z. Then, by definition, the Dold manifold $D(m, n)$ is the quotient space obtained from $S^m \times CP(n)$ by identifying (x, z) with $T(x, z)$.

The projection $S^m \times CP(n) \to S^m$ induces naturally a map p of $D(m, n)$ onto the real projective m-space $RP(m)$, and $\{D(m, n), p, RP(m), CP(n),$

$Z_2\}$ is a fibre bundle whose fibre is $CP(n)$ and structure group is the group of order 2 generated by a homeomorphism sending z to $\bar{z}$ ($z \in CP(n)$).

Let $C_i^+(C_i^-)$ denote an open i-cell of S^m defined by $x_{i+1}=x_{i+2}=\cdots=x_m=0$, $x_i>0$ ($x_i<0$), and D_j denote an open $2j$-cell of $CP(n)$ defined by $z_j=1$, $z_{j+1}=z_{j+2}=\cdots=z_n=0$. Then $\{C_i^\pm \times D_j \mid i=0, 1, \cdots, m; j=0, 1, \cdots, n\}$ forms an oriented cellular decomposition of $S^m \times CP(n)$ whose boundary relations are given by

$$(1.2) \qquad \begin{cases} \partial(C_i^\pm \times D_j) = \pm(C_{i-1}^+ \times D_j + C_{i-1}^- \times D_j) \\ \partial(C_0^\pm \times D_j) = 0, \ i=1, 2, \cdots, m; \ j=0, 1, \cdots, n. \end{cases}$$

The homeomorphism T is cellular with respect to the above cellular decomposition and satisfies

$$(1.3) \qquad T(C_i^\pm \times D_j) = (-1)^{i+j+1}(C_i^\mp \times D_j).$$

Let $\Phi: S^m \times CP(n) \to D(m, n)$ denote the projection, and write $(C_i, D_j) = \Phi(C_i^+ \times D_j)$. Then $\{(C_i, D_j) \mid i=0, 1, \cdots, m; j=0, 1, \cdots, n\}$ is a cellular decomposition of $D(m, n)$ whose boundary relations are given by

$$(1.4) \qquad \begin{cases} \partial(C_i, D_j) = (1+(-1)^{i+j})(C_{i-1}, D_j) \\ \partial(C_0, D_j) = 0, \ i=1, 2, \cdots, m; \ j=0, 1, \cdots, n, \end{cases}$$

and Φ is a cellular map. Let (c^i, d^j) denote the cochain dual to (C_i, D_j), then for the coboundary operation δ we have

$$(1.5) \qquad \delta(c^i, d^i) = (1+(-1)^{i+j+1})(c^{i+1}, d^j).$$

From this we obtain

Proposition (1.6). *The integral cohomology group $H^*(D(m, n); Z)$ is a direct sum of the following groups:*

case m: even

free abelian group generated by (c^0, d^{2j}) and (c^m, d^{2j+1}), torsion group generated by (c^{2i}, d^{2j}) and (c^{2i-1}, d^{2j+1}) whose order are 2.

case m: odd

free abelian group generated by (c^0, d^{2j}) and (c^m, d^{2j}), torsion group generated by (c^{2i}, d^{2j}) and (c^{2i-1}, d^{2j+1}) whose order are 2,

where $i=1, 2, \cdots, [m/2]$; $j=0, 1, \cdots, [n/2]$ ([ ] is the Gauss notation).

For $m' \leqq m$ and $n' \leqq n$ we identify $S^{m'} \times CP(n')$ with the subset $x_{m'+1}=\cdots=x_m=0$, $z_{n'+1}=\cdots=z_n=0$ of $S^m \times CP(n)$ and $D(m', n')$ with the subset $\Phi(S^{m'} \times CP(n'))$ of $D(m, n)$. Under this circumstance $D(m', n')$ is identified with the closure of the cell $(C_{m'}, D_{n'})$ of $D(m, n)$.

Consider the space $D(m, n)/D(m-1, n)$ obtained from $D(m, n)$ by collapsing $D(m-1, n)$ to a point, and let π denote the projection $D(m, n)\to D(m, n)/D(m-1, n)$. Then $\pi(C_m, D_j)$ $(j=0, 1, \cdots, n)$ together with a zero cell forms a cellular decomposition of $D(m, n)/D(m-1, n)$. Since obviously all $\pi(C_m, D_j)$ are cycles, their duals $(\bar{c}^m, \bar{d}^j)$ form a basis for $\tilde{H}^*(D(m, n)/D(m-1, n); Z)$.

Let E_+^m is the upper hemisphere of S^m, then we may regard $D(m, n)$ as the quotient space of the product space $E_+^m\times CP(n)$ under the identification $(x, z)=(-x, \bar{z})$, where $x\in\dot{E}_+^m$, $z\in CP(n)$ and $\dot{E}_+^m$ is the boundary of E_+^m. Let $CP(n)^+$ denote the disjoint union of $CP(n)$ and a point, and let $S^m\wedge CP(n)^+$ denote the reduced join of S^m and $CP(n)^+$. Then it is easily seen that a homeomorphism

$$h: D(m, n)/D(m-1, n) \approx S^m\wedge CP(n)^+ \tag{1.7}$$

can be defined by the following commutative diagram

$$\begin{array}{ccc}
E_+^m\times CP(n) & \xrightarrow{h_1} & D(m, n) \\
\cap & & \downarrow \pi \\
E_+^m\times CP(n)^+ & & D(m, n)/D(m-1, n) \\
\downarrow h_2 & & \downarrow h \\
S^m\times CP(n)^+ & \xrightarrow{h_3} & S^m\wedge CP(n)^+,
\end{array}$$

where h_1, h_3 are the identification maps and h_2 is the map collapsing $\dot{E}_+^m$ to a point. From this we obtain immediately the following

Proposition (1.8). *Let s_m and y be the generators of $\tilde{H}^m(S^m)$ and $H^2(CP(n))$ respectively, then isomorphism*

$$h^*: \tilde{H}^*(S^m\wedge CP(n)^+) \to \tilde{H}^*(D(m, n)/D(m-1, n))$$

sends $s_m\wedge y^i$ to $(\bar{c}^m, \bar{d}^j)$.

In the following, we denote by f the composition $h\circ\pi: D(m, n)\to S^m\wedge CP(n)^+$.

The following theorem is proved in A. Dold [6].

Theorem (1.9). *The mod 2 cohomology ring $H^*(D(m, n); Z_2)$ is a truncated polynomial ring $Z_2[c, d]/(c^{m+1}, d^{n+1})$, where $c=(c^1, d^0)$ and $d=(c^0, d^1)$.*

As for the structure of cohomology ring with coefficients in the field Q of rational numbers, we have also

Theorem (1.10).

i) $H^*(D(2t, 2r); Q) = Q[a, b]/(a^{r+1}, b^2, ba^r)$,

ii) $H^*(D(2t, 2r+1); Q) = Q[a, b]/(a^{r+1}, b^2)$,

iii) $H^*(D(2t+1, 2r); Q) = Q[a, b']/(a^{r+1}, b'^2)$,

iv) $H^*(D(2t+1, 2r+1); Q) = Q[a, b']/(a^{r+1}, b'^2)$,

where $a=(c^0, d^2)$, $b=(c^{2t}, d)$ *and* $b'=(c^{2t+1}, d^0)$.

Proof. Consider the spectral sequence associated with the covering $(S^m\times CP(n), \Phi, D(m, n))$. We then have an isomorphism

$$E_2^{p,q} \simeq H^p(Z_2; H^q(S^m\times CP(n); Q))$$

with the action of Z_2 to $H^q(S^m\times CP(n); Q)$ given by

$$T(1\times y^j) = (-1)^j 1\times y^j, \qquad T(s_m\times y^j) = (-1)^{m+j+1} s_m\times y^j .$$

Therefore we have

$$\begin{cases} E_2^{0,4k} \simeq H^0(S^m; Q)\otimes H^{4k}(CP(n); Q), \\ E_2^{0,m+2(2k+1)} \simeq H^m(S^m; Q)\otimes H^{2(2k+1)}(CP(n); Q), \end{cases} \quad \text{if } m = 4t,$$

$$E_2^{0,4k} \simeq H^0(S^m; Q)\otimes H^{4k}(CP(n); Q) + H^m(S^m; Q)\otimes H^{4k-m}(CP(n); Q), \quad \text{if } m = 4t+2,$$

$$\begin{cases} E_2^{0,4k} \simeq H^0(S^m; Q)\otimes H^{4k}(CP(n); Q), \\ E_2^{0,m+4k} \simeq H^m(S^m; Q)\otimes H^{4k}(CP(n); Q), \end{cases} \quad \text{if } m = 2t+1$$

and all other $E_2^{p,q}$ are zero. This proves that $d_r=0$ $(r\geqq 2)$ and $E_\infty^{p,q}=0$ for $p\neq 0$. Consequently we have

$$H^q(D(m, n); Q) \simeq E_2^{p,q} .$$

In case of $m=4t$ $(m=2t+1)$, obviously we may assume that $1\otimes y^2$ and $s_m\otimes y$ $(s_m\otimes 1)$ are the elements corresponding to $a=(c^0, d^2)$ and $b=(c^m, d)$ $(b'=(c^m, d^0))$ respectively.

In case of $m=4t+2$, since $a=(c^0, d^2)$ is induced from $a=(c^0, d^2)$ for $D(4(t+1), n)$ by the inclusion map $D(4t+2, n)\subset D(4(t+1), n)$, again we may assume in virtue of the naturality that $1\otimes y^2$ is the element corresponding to a. Furtheremore we may assume that $s_m\otimes y$ is the element corresponding to $b=(c^m, d)$ by the following reason. Let the element corresponding to b be $s_m\otimes y+k(1\otimes y^{2t+2})$ with $k\in Q$. Since $b=f^*(s_m\wedge y)$, we have $b^2=0$. Therefore $k=0$ for $n\geqq 2t+3$. For $n<2t+3$, since b is induced from b for $D(m, n')$ with $n'\geqq 2t+3$ by the inclusion map $D(m, n)\subset D(m, n')$, we have also $k=0$.

The above shows that the multiplicative structure of $H^*(D(m, n); Q)$ is induced by that of the spectral sequence. Thus we have the desired results.

The following corollary is obtained from Proposition (1.8) and Theorem (1.10).

Corollary (1.11). *We have an exact triangle*

$$\begin{array}{ccc} H^*(S^m \wedge CP(n)^+ ; Q) & \xrightarrow{f^*} & H^*(D(m, n) ; Q) \\ \delta \nwarrow & & \swarrow i^* \\ & H^*(D(m-1, n) ; Q) & \end{array}$$

such that

$$i^* a^k = a^k$$

and

$$\begin{cases} \delta(b' a^k) = 2 s_{2t} \wedge y^{2k} \\ f^*(s_{2t} \wedge y^{2k+1}) = b a^k \end{cases} \quad \text{if } m = 2t,$$

$$\begin{cases} \delta(b a^k) = 2 s_{2t+1} \wedge y^{2k+1} \\ f^*(s_{2t+1} \wedge y^{2k}) = b' a^k \end{cases} \quad \text{if } m = 2t+1.$$

2. Canonical real 2-plane bundle over $D(m, n)$

We shall recall from [4] that one can define operations

$$\varepsilon : K_O(X) \to K_U(X), \quad \rho : K_U(X) \to K_O(X), \quad * : K_U(X) \to K_U(X)$$

such that

$$\begin{cases} \rho\varepsilon = 2 \quad : K_O(X) \to K_O(X), \\ \varepsilon\rho = 1 + * : K_U(X) \to K_U(X). \end{cases} \tag{2.1}$$

The operations are natural with respect to maps and ring homomorphisms, excepting ρ which is a homomorphism of groups. ε and ρ come from the standerd inclusions, and $*$ is the conjugation (i.e. $*\mu = \bar{\mu}$).

Let ξ be the canonical real line bundle over $RP(m)$, and let η be the canonical complex line bundle over $CP(n)$.

In this section we shall prove the following

Theorem (2.2). *There is a real 2-plane bundle η_1 over $D(m, n)$ satisfying the following conditions*:

i) *η_1 restricted to $CP(n)$ is the 2-plane bundle $\rho\eta$,*

ii) *η_1 for $n=0$ is the 2-plane bundle $1 \oplus p^! \xi$,*

iii) *$\eta_1 \otimes p^! \xi$ is equivalent to η_1.*

iv) *the Chern character of the complex 2-plane bundle $\varepsilon\eta_1$ is given as follows*:

$$\text{ch}\, \varepsilon\eta_1 = 2(1 + a/2! + \cdots + a^r/(2r)!), \tag{2.3}$$

where $r=[n/2]$.

Proof. Every point of $D(m, n)$ can be represented by $[x, z]$ under the identification $(x, z)=(-x, \overline{\lambda z})$ for $x\in S^m$, $z\in S^{2n+1}\subset C^{n+1}$ and all $\lambda\in C$, $|\lambda|=1$. Then the total space $E(\eta_1)$ of η_1 is defined as the set of all triples $[(x, z), t]$ under the identification $((x, z), t)=((-x, \overline{\lambda z}), \overline{\lambda t})$, where $t\in C$ and x, z and λ are as above. The projection is given by $p([(x, z), t])=[x, z]$.

Local triviality is checked as follows: Define $\phi_{i,r}: U_{i,r}\times R^2\to p^{-1}(U_{i,r})$ by

$$\phi_{i,r}([x, z], t) = \begin{cases} [(x, z), z_r t] & \text{if } x_i>0\,, \\ [(x, z), z_r \bar{t}] & \text{if } x_i<0\,, \end{cases}$$

where $U_{i,r}$ is the set of points $[x, z]$ of $D(m, n)$ such that x_i and z_r are non-zero, and $\{U_{i,r}\,|\,i=0, 1, \cdots, m\,;\, r=0, 1, \cdots, n\}$ is an open covering of $D(m, n)$; the transition functions are given as follows:

$$(2.4) \qquad g_{(j,s)(i,r)}[x, z] = \begin{cases} \begin{pmatrix} a & -b \\ b & a \end{pmatrix} & (x_i,\, x_j>0)\,, \\ \begin{pmatrix} a & -b \\ -b & -a \end{pmatrix} & (x_i>0,\, x_j<0)\,, \\ \begin{pmatrix} a & b \\ b & -a \end{pmatrix} & (x_i<0,\, x_j>0)\,, \\ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} & (x_i,\, x_j<0)\,, \end{cases}$$

where $z_r/z_s=a+bi$, $a, b\in R$.

This real 2-plane bundle η_1 is the complex line bundle η for $m=0$, therefore we have

$$(2.5) \qquad i^!_R\eta_1 = \rho\eta$$

for the inclusion map $i: CP(n)\subset D(m, n)$.

Also, it is easy to see from (2.4) that in case of $n=0$ the 2-plane bundle η_1 is $1\oplus p^!\xi$.

Since the transition functions $h_{(j,s)(i,r)}[x, z]$ of $p^!\xi$ are 1 for $x_ix_j>0$ and -1 for $x_ix_j<0$, (2.4) implies

$$P(g_{(j,s)(i,r)}[x, z]\otimes h_{(j,s)(i,r)}[x, z]) = g_{(j,s)(i,r)}[x, z]\,P\,,$$

where $P=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. This shows iii).

We next show iv). In virtue of Theorem (1.10), we see that the kernel of the homomorphism

$$i^*: H^*(D(m, n); Q) \to H^*(CP(n); Q)$$

consists of the elements divisible by b or b' and that

$$i^*a = y^2.$$

Also, by (2.1) and (2.5) we have

$$i_c^! \varepsilon\eta_1 = \eta \oplus \bar{\eta}.$$

Since

$$i^* \operatorname{ch} \varepsilon\eta_1 = \operatorname{ch}(\eta\oplus\bar{\eta}) = 2(1+y^2/2!+\cdots+y^{2r}/(2r)!),$$

we have

$$i^* \operatorname{ch} \varepsilon\eta_1 = i^* 2(1+a/2!+\cdots+a^r/(2r)!) .$$

Hence

(2. 6) $\qquad \operatorname{ch} \varepsilon\eta_1 - 2(1+a/2!+\cdots+a^r/(2r)!) \in \operatorname{Ker} i^*,$

that is

(2. 7) $\qquad \operatorname{ch} \varepsilon\eta_1 - 2(1+a/2!+\cdots+a^r/(2r)!)$ *is divisible by b or b'.*

On the other hand, the total Chern class $c(\varepsilon\eta_1)$ of the complex 2-plane bundle $\varepsilon\eta_1$ is a polynomial on a for $m \geqq 5$ and so is the Chern character $\operatorname{ch} \varepsilon\eta_1$ of $\varepsilon\eta_1$ for $m \geqq 5$.

Therefore the left hand side of (2.6) is a polynomial on a. Thus we obtain (2.3) from (2.7).

In case of $m<5$, since the bundle η_1 over $D(m, n)$ is induced from η_1 over $D(m', n)$ ($m' \geqq 5$) by the inclusion $D(m, n) \subset D(m', n)$, the naturality of the Chern character shows (2.3) for every $D(m, n)$. This completes the proof of Theorem (2. 2).

Finally we shall prove

Theorem (2.8). *On the real tangent bundle $\tau(D(m, n))$ of $D(m, n)$, we have the following relation:*

$$\tau(D(m, n)) \oplus 1 \oplus p^! \xi = p^! \tau(RP(m)) \oplus \overbrace{\eta_1 \oplus \cdots \oplus \eta_1}^{n+1}.$$

Proof. The total space $E(\tau(D(m, n)))$ of the real tangent vector bundle of $D(m, n)$ can be represented as the set of all pairs $[(x, z), (u, v)]$, wtih $x \in S^m \subset R^{m+1}$, $z \in S^{2n+1} \subset C^{n+1}$, $u \in R^{m+1}$, $v \in C^{n+1}$ and $\vec{x}\cdot\vec{u}=0$, $\vec{z}\cdot\vec{v}=0$ in the Hermitian metric, under the identification $((x, z), (u, v)) = ((-x, \overline{\lambda z}), (-u, \overline{\lambda v}))$ for all $\lambda \in C$, $|\lambda|=1$. Therefore we have the following decomposition:

$$\tau(D(m, n)) = p^!\tau(RP(m))\oplus\zeta,$$

where the total space $E(\zeta)$ of ζ is the set of all triples $[(x, z), v]$ under the identification $((x, z), v) = ((-x, \overline{\lambda z}), \overline{\lambda v})$ for x, z, v and λ are as above.

Consider the $(n+1)$-fold bundle sum $\eta_1\oplus\cdots\oplus\eta_1$. Then the total space $E(\eta_1\oplus\cdots\oplus\eta_1)$ can be represented as the set of all triples $[(x, z), v]$ with the identification $((x, z), v)=((-x, \overline{\lambda z}), \overline{\lambda v})$, where $x\in S^m$, $z\in S^{2n+1}\subset C^{n+1}$, $v\in C^{n+1}$ and λ is as above. Comparing this with $E(\zeta)$, we see $E(\eta_1\oplus\cdots\oplus\eta_1)\supset E(\zeta)$.

Let θ be the real 2-plane bundle over $D(m, n)$ with $E(\theta)=\{[(x, z), rz]\}$ modulo the identification $((x, z), rz)=((-x, \overline{\lambda z}), \overline{r\lambda z})$, where $x\in S^m$, $z\in S^{2n+1}\subset C^{n+1}$, $r\in R^2\equiv C$ and λ is as above. Clearly θ is equivalent to $1\oplus p^!\xi$.

As can readily be seen, we have

$$\tau(D(m, n))\oplus\theta = p^!\tau(RP(m))\oplus\overbrace{\eta_1\oplus\cdots\oplus\eta_1}^{n+1}.$$

3. Calculation *of* $\tilde{K}^i_U(D(m, n))$

In terms of the canonical line bundle and the canonical 2-plane bundle, we introduce the following elements λ, μ, ν, α_1, α.

$$\begin{aligned}
\lambda &= \xi-1\in\tilde{K}_O(RP(m)),\\
\mu &= \eta-1\in\tilde{K}_U(CP(n)),\\
\nu &= \varepsilon\lambda\in\tilde{K}_U(RP(m)),\\
\alpha_1 &= \eta_1-p^!\xi-1\in\tilde{K}_O(D(m, n)),\\
\alpha &= \varepsilon\alpha_1\in\tilde{K}_U(D(m, n)).
\end{aligned}$$

According to J. F. Adams [1] we have the following theorems.

Theorem (3.1). *$\tilde{K}^0_U(RP(m))=Z_{2^f}$, the cyclic group of order 2^f, where $f=[m/2]$. ν generates the group, and the multiplicative structure is given by $\nu^2=-2\nu$.*

Theorem (3.2). *$K^0_U(CP(n))$ is a truncated polynomial ring (over the integers) with one generator μ and one relation $\mu^{n+1}=0$.*

Also, we have the following theorem.

Theorem (3.3). i) *$\tilde{K}^1_U(RP(2t))=0$ and $\tilde{K}^1_U(RP(2t+1))=Z$,*
ii) *$K^1_U(CP(n))=0$.*

Proof. i) Considering the spectral sequence of $\tilde{K}_U$-theory for

$RP(2t)$, we have

$$E_2^{p+1,-p}(RP(2t)) = \tilde{H}^{p+1}(RP(2t); K_U^{-p}(*)) = 0,$$

and hence $\tilde{K}_U^1(RP(2t))=0$. Next, considering the exact sequence

$$\tilde{K}_U^0(RP(2t+1)) \to \tilde{K}_U^0(RP(2t)) \to \tilde{K}_U^1(S^{2t+1}) \to \tilde{K}_U^1(RP(2t+1)) \to \tilde{K}_U^1(RP(2t)) = 0,$$

we have

$$\tilde{K}_U^1(RP(2t+1)) \cong \tilde{K}_U^1(S^{2t+1}) = Z.$$

ii) Since

$$E_2^{p+1,-p}(CP(n)) = H^{p+1}(CP(n); K_U^{-p}(*)) = 0,$$

we have ii).

The following three lemmas are useful for the computation of $\tilde{K}_U^i(D(m, n))$.

Lemma (3.4). *The homomorphism, induced by projection,*

$$p_\Lambda^! : \tilde{K}_\Lambda^i(RP(m)) \to \tilde{K}_\Lambda^i(D(m, n)) \qquad (\Lambda = O \text{ or } U)$$

is a monomorphism and $Im\, p_\Lambda^!$ *is a direct summand of* $\tilde{K}_\Lambda^i(D(m, n))$.

Proof. Since there is a cross section

$$r : RP(m) \to D(m, n)$$

defined by $r([x]) = [x_0, \cdots, x_m, 1, 0, \cdots, 0]$, we have immediately the lemma.

Lemma (3.5). *Both of the following systems of elements of the type* i) *and* ii) *form an integral basis of* $\tilde{K}_U^0(CP(n))$.

i) $\mu, \mu(\mu+\bar{\mu}), \cdots, \mu(\mu+\bar{\mu})^{r-1}, (\mu+\bar{\mu}), (\mu+\bar{\mu})^2, \cdots, (\mu+\bar{\mu})^r$, *and also, in case* n *is odd,* $\mu^{2r+1} (=\mu(\mu+\bar{\mu})^r)$;

ii) $\mu, \mu(\mu+\bar{\mu}), \cdots, \mu(\mu+\bar{\mu})^{r-1}, \mu-\bar{\mu}, (\mu-\bar{\mu})(\mu+\bar{\mu}), \cdots, (\mu-\bar{\mu})(\mu+\bar{\mu})^{r-1}$, *and also, in case* n *is odd,* μ^{2r+1}, *where* $r=[n/2]$.

Proof. First we consider the elements of type i). It is sufficient to ensure that $\mu, \mu^2, \cdots, \mu^n$ can be written as linear combinations of the elements of type i).

From Theorem (7.2) of [1] we have

$$\bar{\mu} = -\mu + \mu^2 - \mu^3 + \cdots + (-1)^n \mu^n.$$

Therefore

$$(\mu+\bar{\mu})^k = \{\mu^2 - \mu^3 + \cdots + (-1)^n \mu^n\}^k.$$

Since

$$(\mu+\bar{\mu})^j = \mu^{2j} + highter\ terms$$

and

$$\mu(\mu+\bar{\mu})^{j-1} = \mu^{2j-1} + higher\ terms,$$

an easy inductive argument on i shows that μ^{n-i} $(i=0, \cdots, n-1)$ are represented as linear combinations of the elements of type i).

As for ii), in virtue of the relation

$$(\mu+\bar{\mu})^j = 2\mu(\mu+\bar{\mu})^{j-1} - (\mu-\bar{\mu})(\mu+\bar{\mu})^{j-1},$$

the elements of type i) are rewritten as linear combinations of the elements of type ii), thus the elements of type ii) also form a basis of $\tilde{K}_U^0(CP(n))$.

Lemma (3.6). ch $\alpha = 2(a/2! + \cdots + a^r/(2r)!)$, *where* $r=[n/2]$.

Proof. Since $\alpha_1 = \eta_1 - 2 - (p^! \xi - 1)$, we have $\alpha = \varepsilon\eta_1 - 2 - p^! \nu$. On the other hand ch $\nu = 0$. Therefore Theorem (2.2) implies the lemma.

Considering the spectral sequence in $\tilde{K}_U$-theory for $D(m, n)$, we have

$$E_2^{p,q}(D(m, n)) = \begin{cases} \tilde{H}^p(D(m, n); Z) & \text{if } q = \text{even} \\ 0 & \text{if } q = \text{odd}. \end{cases}$$

By Proposition (1.6) we can enumerate $E_2^{p,q}$ with $p+q=0$ or 1, and we obtain the following result as for the rank of $E_2^{*,*} = \sum_{p+q=i} E_2^{p,q}$:

(3.7)

i \ (m, n)	$(2t, 2r)$	$(2t+1, 2r)$	$(2t, 2r+1)$	$(2t+1, 2r+1)$
0	$2r$	r	$2r+1$	r
1	0	$r+1$	0	$r+1$

Next, we shall show that the rank of $\tilde{K}_U^i(D(m, n))$ is no less than that of $E_2^{*,*}$. For this purpose, by (1.7) we identify $\tilde{K}_U^i(D(m, n)/D(m-1, n))$ with $K_U^{i-m}(CP(n))$. Then in virtue of Lemma (3.5) the basis of $\tilde{K}_U^{-1}(D(2t+1, n)/D(2t, n))$ can be represented by

$$g^{t+1}, g^{t+1}\mu, g^{t+1}\mu(\mu+\bar{\mu}), \cdots, g^{t+1}\mu(\mu+\bar{\mu})^{r-1},$$
$$g^{t+1}(\mu-\bar{\mu}), g^{t+1}(\mu-\bar{\mu})(\mu+\bar{\mu}), \cdots, g^{t+1}(\mu-\bar{\mu})(\mu+\bar{\mu})^{r-1},$$

and also, in case n is odd, $g^{t+1}\mu^{2r+1}$ with $r=[n/2]$, where g denotes the canonical generator of $\tilde{K}_U^0(S^2)$. Also, in virtue of Proposition (1.8) we may identify $\tilde{H}^*(D(m, n)/D(m-1, n); Q)$ with $\tilde{H}^*(S^m \wedge CP(n)^+; Q)$.

Consider the following commutative diagram

$$\begin{array}{ccc}\tilde{K}_U^{-1}(D(2t+1,n)/D(2t,n)) & \xrightarrow{(sf)^!} & \tilde{K}_U^{-1}(D(2t+1,n)) \\ \downarrow \mathrm{ch} & & \downarrow \mathrm{ch} \\ \tilde{H}^*(D(2t+1,n)/D(2t,n);Q) & \xrightarrow{f^*} & \tilde{H}^*(D(2t+1,n);Q),\end{array}$$

where f is the map defined after Proposition (1.8) and sf is its suspension. Since we have

$$\begin{aligned}\mathrm{ch}(sf)^!g^{t+1} &= f^*\,\mathrm{ch}\,g^{t+1} = b', \\ \mathrm{ch}(sf)^!g^{t+1}\mu(\mu+\bar{\mu})^{k-1} &= f^*\,\mathrm{ch}\,g^{t+1}\mu(\mu+\bar{\mu})^{k-1} \\ &= 2^{k-1}b'(a/2!+\cdots+a^r/(2r)!)^k,\end{aligned}$$

there are $r+1$ independent elements $(sf)^!g^{t+1}$, $(sf)^!g^{t+1}\mu$, $(sf)^!g^{t+1}\mu(\mu+\bar{\mu}),\cdots,(sf)^!g^{t+1}\mu(\mu+\bar{\mu})^{r-1}$ in $\tilde{K}_U^{-1}(D(2t+1,n))$ with $r=[n/2]$. We put

$$(3.8)\qquad \begin{cases}(sf)^!g^{t+1}=g', \\ (sf)^!g^{t+1}\mu(\mu+\bar{\mu})^{k-1}=\beta_{k-1} \quad (k=1,2,\cdots,r).\end{cases}$$

Next, in virture of Lemma (3.5) the basis of $\tilde{K}_U^0(D(2t,n)/D(2t-1,n))$ can be represented by

$$g^t,\ g^t(\mu+\bar{\mu}),\cdots,g^t(\mu+\bar{\mu})^r,\ g^t\mu,\ g^t\mu(\mu+\bar{\mu}),\cdots,g^t\mu(\mu+\bar{\mu})^{r-1},$$

and also, in case n is odd, $g^t\mu^{2r+1}$ with $r=[n/2]$.

Consider the following commutative diagram

$$\begin{array}{ccc}\tilde{K}_U^0(D(2t,n)/D(2t-1,n)) & \xrightarrow{f^!} & \tilde{K}_U^0(D(2t,n)) \\ \downarrow \mathrm{ch} & & \downarrow \mathrm{ch} \\ \tilde{H}^*(D(2t,n)/D(2t-1,n);Q) & \xrightarrow{f^*} & \tilde{H}^*(D(2t,n);Q).\end{array}$$

Since we have

$$\begin{aligned}&\mathrm{ch}\,f^!g^t\mu(\mu+\bar{\mu})^{k-1} = f^*\,\mathrm{ch}\,g^t\mu(\mu+\bar{\mu})^{k-1} \\ &=\begin{cases}2^{k-1}b(1+a/3!+\cdots+a^{r-1}/(2r-1)!)(a/2!+\cdots+a^r/(2r)!)^{k-1} & \text{if } n=2r, \\ 2^{k-1}b(1+a/3!+\cdots+a^r/(2r+1)!)(a/2!+\cdots+a^r/(2r)!)^{k-1} & \text{if } n=2r+1,\end{cases}\end{aligned}$$

there are independent elements $f^!g^t\mu$, $f^!g^t\mu(\mu+\bar{\mu}),\cdots,f^!g^t\mu(\mu+\bar{\mu})^{r-1}$, and also, in case n is odd, $f^!g^t\mu^{2r+1}$ in $\tilde{K}_U^0(D(2t,n))$ with $r=[n/2]$. We put

$$(3.9)\qquad \begin{cases}f^!g^t\mu(\mu+\bar{\mu})^{k-1}=\gamma_{k-1} \qquad (k=1,2,\cdots,r) \\ f^!g^t\mu^{2r+1}=\gamma_{r+1}.\end{cases}$$

Moreover, by Lemma (3.6) there are r independent elements $\alpha, \alpha^2, \cdots, \alpha^r$ in $\tilde{K}_U^0(D(m, n))$ with $r=[n/2]$.

From the above mentioned facts, we have the following results as for the rank of $\tilde{K}_U^i(D(m, n))$:

(3.10)

i \ (m, n)	$(2t, 2r)$	$(2t+1, 2r)$	$(2t, 2r+1)$	$(2t+1, 2r+1)$
0	$2r$	r	$2r+1$	r
1	0	$r+1$	0	$r+1$

Now, in virtue of Proposition (1.6) $\tilde{K}_U^i(D(m, n))$ must be a direct sum of Z's and Z_{2^*}'s, and it remains to settle the question of how many Z_{2^*}'s occur in $\tilde{K}_U^i(D(m, n))$. For this purpose we consider the spectral sequence of mod 2 $\tilde{K}_U$-theory. Let M_2 be $RP(2)$ and let (X, A) be a pair of finite CW-complex and its subcomplex. The mod 2 K_U-theory [2], $K_U(\ ;Z_2)$ and $\tilde{K}_U(\ ;Z_2)$, is defined by

$$K_U^i(X, A; Z_2) = K_U^{i+2}(X\times M_2, X\times * \cup A\times M_2),$$
$$\tilde{K}_U^i(X; Z_2) = \tilde{K}_U^{i+2}(X\wedge M_2) \qquad \text{for all } i.$$

Let X be a finite simplicial complex and X^n be the n-skeleton of X. When we filter $K_U^i(X; Z_2)$ by defining

$$K_p^i(X; Z_2) = \text{Kernel}[K_U^i(X; Z_2)\to K_U^i(X^{p-1}; Z_2)],$$

we have the following theorem.

Theorem (3.11). *Let X be a finite simplicial complex. Let M_2 be $RP(2)$, so that $\tilde{K}_U^q(M_2)\cong Z_2$ if q is even and $\tilde{K}_U^q(M_2)=0$ if q is odd. Then there is a spectral sequence $E_r^{p,q}(X; Z_2)$ $(r\geqq 1, -\infty<p, q<\infty)$ with*

$$(1) \qquad E_1^{p,q}(X; Z_2) \cong C^p(X; \tilde{K}_U^q(M_2)),$$

d_1 being the ordinary coboundary operator,

$$(2) \qquad E_2^{p,q}(X; Z_2) \cong H^p(X; \tilde{K}_U^q(M_2)),$$
$$(3) \qquad E_\infty^{p,q}(X; Z_2) \cong K_p^{p+q}(X; Z_2)/K_{p+1}^{p+q}(X; Z_2).$$

The differential $d_r: E_r^{p,q}(X; Z_2)\to E_r^{p+r,q-r+1}(X; Z_2)$ vanishes for even r since $E_r^{p,q}(X; Z_2)=0$ for all odd values of q. Also $d_3=Sq^3+Sq^2Sq^1$ is known.

The $E_r^p(X; Z_2)$ together with the differentials d_r are homotopy type invariants of X for $r\geqq 2$. Also $K_U(X)$ is a homotopy type invariant. By a theorem of J.H.C. Whitehead [8, p. 239, Theorem 13], any finite CW-complex is of the homotopy type of a finite simplicial complex.

Hence the spectral sequence $\{E_r^{p,q}(X;Z_2), r\geqq 2\}$ is well defined for any finite CW-complex.

We now apply the spectral sequence of mod 2 $\tilde{K}_U$-theory to $D(m, n)$. We have $Sq^1 d=cd$ from (1.5). Since the operator d_3 is a derivation, we obtain

$$d_3(c^i d^j) = (i+j)c^{i+3}d^j + jc^{i+1}d^{j+1}, \tag{3.12}$$

We can enumerate easily the additive basis in E_4-term which is the d_3-cohomology of $H^*(D(m, n); Z_2)$:

$$\begin{cases} c^2, d^2, d^4, \cdots, d^{2r}, c^{2t}d, c^{2t}d^3, \cdots, c^{2t}d^{2r-1}, \\ c^{2t-1}, & \text{if } (m, n)=(2t, 2r), \end{cases}$$

$$\begin{cases} c^2, d^2, d^4, \cdots, d^{2r}, \\ c^{2t-1}, c^{2t+1}, c^{2t+1}d, \cdots, c^{2t+1}d^{2r-1}, & \text{if } (m, n)=(2t+1, 2r), \end{cases}$$

$$\begin{cases} c^2, d^2, \cdots, d^{2r}, c^{2t}d, \cdots, c^{2t}d^{2r+1}, c^{2t-2}d^{2r+1}, \\ c^{2t-1}, cd^{2r+1}, & \text{if } (m, n)=(2t, 2r+1), \end{cases}$$

$$\begin{cases} c^2, d^2, \cdots, d^{2r}, c^{2t}d^{2r+1}, \\ c^{2t-1}, c^{2t+1}, c^{2t+1}d, \cdots, c^{2t+1}d^{2r-1}, cd^{2r+1}, & \text{if } (m, n)=(2t+1, 2r+1), \end{cases}$$

where elements in the first rows are the basis of E_4-term of total degree 0 and the second are those of total degree 1.

Now, note that $\tilde{K}_U^0(D(m, n))$ has a 2-primary compoent Z_{2^f} by Lemma (3.4). By Künneth relation of $\tilde{K}_U$-theory [2, Cor. 2.8]

$$\tilde{K}_U^0(D(m, n); Z_2) \cong \tilde{K}_U^0(D(m, n)) \otimes Z_2 + \mathrm{Tor}(\tilde{K}_U^1(D(m, n)), Z_2),$$
$$\tilde{K}_U^1(D(m, n); Z_2) \cong \tilde{K}_U^1(D(m, n)) \otimes Z_2 + \mathrm{Tor}(\tilde{K}_U^0(D(m, n)), Z_2).$$

Comparing the number of copies of Z_2 of both sides, as for the 2-torsion part of $\tilde{K}_U^i(D(m, n))$, we obtain the following results:

(3.13) *If n is even, the torsion of $\tilde{K}_U^0(D(m, n))$ is $p^!\tilde{K}_U^0(RP(m))$, and $\tilde{K}_U^1(D(m, n))$ has no torsion.*
If n is odd, the torsion of $\tilde{K}_U^1(D(m, n))$ is Z_{2^*} or 0.

Now we obtain the following

Theorem (3.14).

i) $\tilde{K}_U^0(D(2t, 2r)) = \overbrace{Z+\cdots+Z}^{2r}+Z_{2^t}$,
$\tilde{K}_U^1(D(2t, 2r)) = 0$,

$$\text{ii)}\quad \tilde{K}_U^0(D(2t+1, 2r)) = \overbrace{Z+\cdots+Z}^{r}+Z_{2^t},$$

$$\tilde{K}_U^1(D(2t+1, 2r)) = \overbrace{Z+\cdots+Z}^{r+1},$$

$$\text{iii)}\quad \tilde{K}_U^0(D(2t, 2r+1)) = \overbrace{Z+\cdots+Z}^{2r+1}+Z_{2^t},$$

$$\tilde{K}_U^1(D(2t, 2r+1)) = Z_{2^t}$$

$$\text{iv)}\quad \tilde{K}_U^0(D(2t+1, 2r+1)) = \overbrace{Z+\cdots+Z}^{r}+Z_{2^t},$$

$$\tilde{K}_U^1(D(2t+1, 2r+1)) = \overbrace{Z+\cdots+Z}^{r+1}+Z_{2^{t+1}};$$

the basis of the free part of $\tilde{K}_U^0(D(m, n))$ *are* $\alpha, \alpha^2, \cdots, \alpha^r, \gamma, \gamma\alpha, \cdots, \gamma\alpha^{r-1}$, *and also, in case n is odd,* $\gamma\alpha^r$, *and the basis of the free part of* $\tilde{K}_U^1(D(2t+1, n))$ *are* $g', \beta, \beta\alpha, \cdots, \beta\alpha^{r-1}$, *where* $\gamma=f^!g^t\mu$ *and* $\beta=(sf)^!g^{t+1}\mu$; *the generator of 2-torsion part of* $\tilde{K}_U^0(D(m, n))$ *is* $\nu_1=p^!\nu$. *Also we have* $\alpha\cdot\nu_1=0$.

Proof. Proof of i) and ii). Since we have $D(0, 2r)=CP(2r)$, our assertions are trivial for $m=0$, and the basis of the free part are given by $\mu+\bar{\mu}, (\mu+\bar{\mu})^2, \cdots, (\mu+\bar{\mu})^r, \mu, \mu(\mu+\bar{\mu}), \cdots, \mu(\mu+\bar{\mu})^{r-1}$.

Suppose that i) is true for $m=2t$, $n=2r$ and that the basis of the free part of $\tilde{K}_U^0(D(2t, 2r))$ are $\alpha, \alpha^2, \cdots, \alpha^r, \gamma, \gamma\alpha, \cdots, \gamma\alpha^{r-1}$. And consider the exact sequence

$$0 \longrightarrow \tilde{K}_U^{-2}(D(2t+1, 2r)) \xrightarrow{i^!} \tilde{K}_U^{-2}(D(2t, 2r)) \xrightarrow{\delta^!}$$
$$\tilde{K}_U^{-1}(D(2t+1, 2r)/D(2t, 2r)) \xrightarrow{(sf)^!} \tilde{K}_U^{-1}(D(2t+1, 2r)) \longrightarrow 0.$$

It is easy to see that the basis of the free part of $\tilde{K}_U^0(D(2t+1, 2r))$ are given by $\alpha, \alpha^2, \cdots, \alpha^r$ and the basis of $\tilde{K}_U^{-1}(D(2t+1, 2r))$ are given by $g', \beta, \beta\alpha, \cdots, \beta\alpha^{r-1}$, because of $\mathrm{ch}\,\beta_{k-1}=\mathrm{ch}\,\beta\alpha^{k-1}$ $(k=1, 2, \cdots, r)$.

Now, if we use the exact sequence

$$0 \longrightarrow \tilde{K}_U^{-1}(D(2t+1, 2r)) \xrightarrow{\delta^!} \tilde{K}_U^0(D(2t+2, 2r)/D(2t+1, 2r)) \xrightarrow{f^!}$$
$$\tilde{K}_U^0(D(2t+2, 2r)) \xrightarrow{i^!} \tilde{K}_U^0(D(2t+1, 2r)) \longrightarrow 0,$$

the induction on m shows i) and ii).

Proof of iii) and iv). Consider the exact sequence

$$0 \longrightarrow \tilde{K}_U^{-2}(D(2t+1, 2r+1)) \xrightarrow{i^!} \tilde{K}_U^{-2}(D(2t, 2r+1)) \xrightarrow{\delta^!}$$
$$\tilde{K}_U^{-1}(D(2t+1, 2r+1)/D(2t, 2r+1)) \xrightarrow{(sf)^!} \tilde{K}_U^{-1}(D(2t+1, 2r+1))$$
$$\xrightarrow{i^!} \tilde{K}_U^{-1}(D(2t, 2r+1)) \longrightarrow 0 .$$

Assume inductively that the basis of the free part of $\tilde{K}_U^{-2}(D(2t, 2r+1))$ are given by $g\alpha, g\alpha^2, \cdots, g\alpha^r, g\gamma, g\gamma\alpha, \cdots, g\gamma\alpha^r$ and that $\tilde{K}_U^1(D(2t, 2r+1)) = Z_{2^t}$. Then we have

$$\delta^! g\gamma\alpha^r = g^{t+1}(\mu-\bar{\mu})(\mu+\bar{\mu})^r = 2g^{t+1}\mu^{2r+1},$$

and hence $\tilde{K}_U^{-1}(D(2t+1, 2r+1))$ has 2-torsion part $Z_{2^{t+1}}$.

Consider the exact sequence

$$0 \longrightarrow \tilde{K}_U^{-1}(D(2t+2, 2r+1)) \xrightarrow{i^!} \tilde{K}_U^{-1}(D(2t+1, 2r+1)) \xrightarrow{\delta^!}$$
$$\tilde{K}_U^0(D(2t+2, 2r+1)/D(2t+1, 2r+1)) \xrightarrow{f^!} \tilde{K}_U^0(D(2t+2, 2r+1))$$
$$\xrightarrow{i^!} \tilde{K}_U^0(D(2t+1, 2r+1)) \longrightarrow 0 .$$

Since $\tilde{K}_U^0(D(2t+2, 2r+1)/D(2t+1, 2r+1))$ is free, we have

$$\tilde{K}_U^{-1}(D(2t+2, 2r+1)) = Z_{2^{t+1}} .$$

The rest of the proof of iii) and iv) can be treated in the similar way as in the case i) and ii).

Since $\alpha\nu_1 \in p^! \tilde{K}_U^0(RP(m))$ and $r^! \alpha = 0$ (cf. Theorem (2.2) and Lemma (3.4)), we have $\alpha\nu_1 = 0$. The proof is complete.

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