Hashimoto, J.
Osaka Math. J.
15 (1963), 71–86.

CONGRUENCE RELATIONS AND CONGRUENCE CLASSES IN LATTICES

Dedicated to Professor K. Shoda on his sixtieth birthday

By

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1. Introduction

The general theory of abstract algebraic systems (algebras) introduced by Professor Shoda [7] has been successful not only to unify earlier results about many algebraic systems (groups, rings, lattices, etc.) but to develop further investigations into each individual system. The present paper is an additional work in those lines. We shall first consider the relation between congruence relations and congruence classes on universal algebras and next inquire precisely into the same problem on lattices.

In the present paper by an *algebra* A we shall mean, following Birkhoff [1], [2], a system with a number of operations f_λ: $(x_1, \cdots, x_n) \in A\times \cdots \times A \rightarrow f_\lambda(x_1, \cdots, x_n) \in A$. A homomorphism θ of A onto an algebra $B=\theta(A)$ yields a congruence relation on A, which shall be written $x\equiv y(\theta)$ or $x\theta y$; so

$$x\theta y \rightleftarrows \theta(x) = \theta(y)\,.$$

Conversely a congruence relation θ on A yields a homomorphism of A onto the algebra $\theta(A)$ of classes $S(a, \theta)=\{x\,;\, x\theta a\}$, which we shall denote also by the same notation θ.

For the investigation into the structure of algebras such as groups, rings, etc., the following properties on the congruence relations work effectively:

(α) Every congruence relation is determined by the congruence class containing a fixed element a; namely

$$S(a, \theta) = S(a, \varphi) \quad \text{implies} \quad \theta = \varphi\,,$$

(β) Congruence relations on A are permutable.
Some algebras however do not necessarily possess those properties. In this respect Birkhoff [2] has proposed the following problems.

Problem 33. Let A be an algebra with a one-element subalgebra a and permutable congruence relations. Can A have distinct congruence relations $\theta \neq \varphi$ such that $S(a, \theta)=S(a, \varphi)$?

Problem 73. Find necessary and sufficient conditions, in order that the correspondence between the congruence relations and ideals of a lattice be one-one.

About the latter problem we have got an answer in a previous paper [5], but as stated there, many related matters remain unexplored. It is rather easy to give examples of such algebras as mentioned in the former problem, but the connection between the two properties (α) and (β) is not so easily clarified. So we intend in the present paper to deal with those subjects.

First we shall state in § 2 what effect the property (α) or (β) of an algebra A and that of its subalgebras (Theorem 2. 1), homomorphic images (Theorem 2. 2) or direct unions (Theorem 2. 3) will have on each other. In § 3 we shall give for lattices L some necessary or sufficient conditions in order that (α) or (β) hold, such as Theorems 3. 2, 3. 4 and 3. 5. Especially we shall inquire into the connection between complementedness and the above properties (Cor. of Theorem 3. 2, Theorems 3. 3 and 3. 6). Finally in § 4 we shall deal with some stronger properties on a lattice L, in reference to which we give a condition that L be directly decomposed (Theorem 4. 4) and clarify the structure of some lattices (Cor. of Theorem 4. 4).

2. Properties (α) and (β) on universal algebras

We shall write $\alpha(a)$ to mean that an element a of an algebra A satisfies the condition

$$(\alpha) \qquad S(a, \theta) = S(a, \varphi) \quad \text{implies} \quad \theta = \varphi\,,$$

and $\alpha(A)$ to mean that every element a of A possesses the property $\alpha(a)$.

Now if we define $\theta \leqq \varphi$ to mean that $x\theta y$ implies $x\varphi y$, then all congruence relations on A form a complete, upper continuous lattice $\Theta(A)$, which we shall call the *structure lattice* of A. Let P be a set of pairs (a, b) of elements of A. We define the congruence relation $\theta(P)$ generated by P as the least of elements θ of $\Theta(A)$ satisfying $a\theta b$ for every pair $(a, b) \in P$. It follows that $\theta(P)=\bigvee_{(a,b)\in P}\theta(a, b)$, where $\theta(a, b)$ is the congruence relation generated by one pair (a, b), which shall be called a *monomial* congruence relation. Then the properties (α) and (β) can be restated as conditions about monomial congruence relations.

Put $\varphi=\bigvee_{b\in S(a,\theta)}\theta(a, b)$. Then it is easily seen that $\varphi \leqq \theta$ and $S(a, \varphi) =S(a, \theta)$. Hence if a satisfies $\alpha(a)$, then $\theta=\bigvee_{b\in S(a,\theta)}\theta(a, b)$. Further if

θ is monomial, we can find a finite number of $b_i \in S(a, \theta)$ such that $\theta = \bigvee_i \theta(a, b_i)$, since θ is inaccessible in $\Theta(A)$.[1] Conversely if every monomial congruence relation $\theta(x, y)$ is written $\theta(x, y) = \bigvee_i \theta(a, b_i)$, then for any congruence relation θ we get $\theta = \bigvee_{x\theta y} \theta(x, y) = \bigvee \theta(a, b_i) \leqq \bigvee_{b \in S(a,\theta)} \theta(a, b) \leqq \theta$, and hence $S(a, \theta) = S(a, \varphi)$ implies $\theta = \varphi$. Thus we have

Lemma 2.1. *An element a of an algebra A possesses the property $\alpha(a)$ if and only if, given $x, y \in A$, a finite number of elements b_i exist such that $\theta(x, y) = \bigvee_i \theta(a, b_i)$.*

If $\beta(A)$ holds, from $x \equiv y(\theta(x, z) \cup \theta(y, z))$ it follows that there exists $u \in A$ such that $x \equiv u(\theta(y, z))$ and $u \equiv y(\theta(x, z))$. Conversely if such u exists, then $x\varphi z\psi y$ implies $x\psi u\varphi y$, since $\theta(x, z) \leqq \varphi$ and $\theta(y, z) \leqq \psi$.

Lemma 2.2. *Congruence relations on an algebra A are permutable if and only if, given $x, y, z \in A$, an element u exists such that $u \equiv x(\theta(y, z))$, $u \equiv y(\theta(x, z))$.*

Let S be a subalgebra of an algebra A. A congruence relation θ of A can be regarded as a congruence relation θ^* on S, provided the range of elements is restricted in S. When P is a set of pairs (a, b) in S, by $\theta^*(P)$ we denote the congruence relation on S generated by P. If θ^* is the congruence relation on S induced by $\theta = \theta(P) = \bigvee \theta(a, b)$, then we have $a\theta^* b$, $\theta^* \geqq \theta^*(a, b)$ and $\theta^* \geqq \bigvee \theta^*(a, b) = \theta^*(P)$. Hence $x \equiv y(\theta^*(P))$ implies $x\theta^* y$ and $x \equiv y(\theta(P))$. Next assume $\theta^*(P) = \theta^*(Q)$ in $\Theta(S)$ and $(a, b) \in P$. As shown above, from $a \equiv b(\theta^*(Q))$ we can deduce $a \equiv b(\theta(Q))$, that is $\theta(a, b) \leqq \theta(Q)$ and hence $\theta(P) \leqq \theta(Q)$. In summary

Lemma 2.3. *If $x \equiv y(\theta^*(P))$ holds in a subalgebra S of an algebra A, then $x \equiv y(\theta(P))$ holds in A. $\theta^*(P) = \theta^*(Q)$ in S implies $\theta(P) = \theta(Q)$ in A.*

Using the above lemmas, we can infer

Theorem 2.1.α. *An element a possesses the property $\alpha(a)$ in an algebra A if every triple $\{a, x, y\}$ is contained in a subalgebra S satisfying $\alpha(a)$.*

Proof. If a, x, y are contained in S satisfying $\alpha(a)$, we can choose by Lemma 2.1 $b_i \in S$ so that $\theta^*(x, y) = \bigvee_i \theta^*(a, b_i)$, whence $\theta(x, y) = \bigvee_i \theta(a, b_i)$ by Lemma 2.3.

Similarly from Lemmas 2.2 and 2, 3 we can deduce

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1) θ is called inaccessible if a set $\{\theta_\lambda\}$ satisfying $\bigvee \theta_\lambda \geqq \theta$ contains always a finite subset $\{\theta_i\} \subseteqq \{\theta_\lambda\}$ satisfying $\bigvee \theta_i \geqq \theta$.

cf. J. Hashimoto [6, Lemma 2.3].

Theorem 2.1.β. *Congruence relations on an algebra A are permutable if every triple $\{x, y, z\}$ is contained in a subalgebra S on which congruence relations are permutable.*

Now it is naturally guessed that the property of an algebra A may yield the same property of its homomorphic image $\theta(A)$. Let us affirm it.

Let φ^* and ψ^* be congruence relations on $\theta(A)$. Then $\varphi^*\theta(x)$ and $\psi^*\theta(x)$ are homomorphisms of A and generate congruence relations φ and ψ on A such that $\varphi \geqq \theta$, $\psi \geqq \theta$, and it is obvious that $\theta(x) \equiv \theta(y)$ (φ^*) is equivalent to $x \equiv y$ (φ). So $S(\theta(a), \varphi^*) = S(\theta(a), \psi^*)$ implies $S(a, \varphi) = S(a, \psi)$; hence if a possesses the property $\alpha(a)$ in A, then $S(\theta(a), \varphi^*) = S(\theta(a), \psi^*)$ implies $\varphi = \psi$ and $\varphi^* = \psi^*$. If φ and ψ are permutable, then $x\varphi z\psi y$ implies $x\psi u\varphi y$; namely $\theta(x)\varphi^*\theta(z)\psi^*\theta(y)$ implies $\theta(x)\psi^*\theta(u)\varphi^*\theta(y)$ and hence φ^* and ψ^* are permutable. Thus we have

Theorem 2.2.α. *If an element a possesses the property $\alpha(a)$ in an algebra A, then its homomorphic image $\theta(a)$ possesses the same property in $\theta(A)$.*

Theorem 2.2.β. *If congruence relations on an algebra A are permutable, then congruence relations on its homomorphic image $\theta(A)$ are also permutable.*

Next we shall consider the case that A is decomposed into a (finitely) *restricted direct union* of $\{A_\omega ; \omega \in \Omega\}$; namely A is a subsystem of the complete direct union ΠA_ω satisfying that $\{x_\omega\} \in A$ implies $\{y_\omega\} \in A$ if and only if the set of indices $\{\omega ; x_\omega \neq y_\omega\}$ is finite.[2] If A possesses one of the above properties, then each component A_ω satisfies the same property, since A_ω is a homomorphic image of A. But the converse does not necessarily hold. Indeed, let S be the (simple) semilattice of two elements $\{0, 1\}$ with an operation $\cup$. Then it is easy to see that no element of $S \times S$ satisfies (α) and congruence relations on $S \times S$ are not permutable. Yet if the structure lattice $\Theta(A)$ is distributive, then we can prove the converse.

Theorem 2.3. *Let A be an algebra with a distributive structure lattice $\Theta(A)$ and decomposed into a restricted direct union of algebras $\{A_\omega ; \omega \in \Omega\}$. Then*

α: *An element $a = \{a_\omega ; \omega \in \Omega\}$ satisfies $\alpha(a)$ in A if and only if every component a_ω satisfies $\alpha(a_\omega)$ in A_ω,*

β: *Congruence relations on A are permutable if and only if congruence relations on each A_ω are permutable.*

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2) cf. J. Hashimoto [6, p. 92].

Proof. Let θ_ω be the homomorphism from $x \in A$ to its ω-component x_ω: $\theta_\omega(x)=x_\omega$. Given a congruence relation φ on A, we put $\varphi_\omega=\theta_\omega \cup \varphi$. Since $x_\omega=y_\omega$ implies $\varphi_\omega(x)=\varphi_\omega(y)$, $\varphi^*_\omega(x_\omega)=\varphi_\omega(x)$ becomes a homomorphism of A_ω and $x_\omega \varphi^*_\omega y_\omega$ is equivalent to $x\varphi_\omega y$. We first show $\bigwedge_\omega \varphi_\omega=\varphi$. Assume $x \equiv y(\bigwedge_\omega \varphi_\omega)$ and $\Delta=\{\omega\,;\, x_\omega \neq y_\omega\}$. Then $x \equiv y(\bigwedge_{\omega \in \Omega-\Delta}\theta_\omega)$. Since Δ is finite and $\Theta(A)$ is distributive, we get $\bigwedge_{\omega \in \Delta}\varphi_\omega=\varphi \cup \bigwedge_{\omega\in\Delta}\theta_\omega$. It follows from $\bigwedge_{\omega \in \Omega}\theta_\omega=0$ that $\bigwedge_{\omega\in\Omega-\Delta}\theta_\omega \cap (\varphi \cup \bigwedge_{\omega\in\Delta}\theta_\omega) \leqq \varphi$. Hence we have $x\equiv y(\varphi)$, proving $\bigwedge_\omega \varphi_\omega=\varphi$.

Now suppose that every a_ω satisfies $\alpha(a_\omega)$ in A_ω and $S(a, \varphi)=S(a, \psi)$ in A. If $x_\omega \varphi^*_\omega a_\omega$, then $x \equiv a(\theta_\omega \cup \varphi)$ and we can find $y \in A$ such that $x\theta_\omega y\varphi a$, since the decomposition congruence relation θ_ω is permutable with φ in the distributive $\Theta(A)$.[3] Then we have $x\theta_\omega y \psi a$, that is $x\psi_\omega a$ and hence $x_\omega \psi^*_\omega a_\omega$. Thus we can infer $S(a_\omega, \varphi^*_\omega)=S(a_\omega, \psi^*_\omega)$, $\varphi^*_\omega=\psi^*_\omega$, $\varphi_\omega=\psi_\omega$ and hence $\varphi=\psi$, proving α.

Next suppose that congruence relations on each A_ω are permutable and $x\equiv y(\varphi \cup \psi)$ in A. Then we can find z_ω in each A_ω satisfying $x_\omega \varphi^*_\omega z_\omega \psi^*_\omega y_\omega$ and an element u in A such that $u_\omega=z_\omega$ for $\omega \in \Delta=\{\omega\,;\, x_\omega \neq y_\omega\}$ and $u_\omega=x_\omega$ for $\omega \in \Omega-\Delta$. It is easily seen that $x\varphi_\omega u \psi_\omega y$ for all $\omega \in \Omega$ and hence $x\varphi u \psi y$, which shows that φ and ψ are permutable.

The example $S \times S$ mentioned before shows that the distributivity of $\Theta(A)$ cannot be dispensed with even if A is completely reducible. In that case however we can show that the condition $\beta(A)$ implies the condition $\alpha(A)$.

Theorem 2.4. *Let A be an algebra with permutable congruence relations and decomposed into restricted direct union of simple factors $\{A_\omega\,;\, \omega \in \Omega\}$. Then every congruence relation is determined by any one class.*

Proof. Assume that $S(a, \varphi)=S(a, \psi)$ and $\varphi \leqq \psi$. Using the results in our previous paper [6], we can find their complements φ' and ψ' such that $\varphi'=\bigwedge_{\omega\in M}\theta_\omega$, $\psi'=\bigwedge_{\omega \in N}\theta_\omega$ with $M \subseteqq N \subseteqq \Omega$,[4] where θ_ω are the same as defined in the proof of Theorem 2.3. If $\eta \in N-M$, then we can choose an element $x=\{x_\omega\} \in A$ so that $x_\omega=a_\omega$ for $\omega \neq \eta$ and $x_\eta \neq a_\eta$; accordingly $x\equiv a(\varphi')$ and $x \not\equiv a(\psi')$. Since $\psi \cup \psi'=1$ and they are permutable, $y \in A$ exists such that $x\psi' y \psi a$ and hence $y\varphi a$ by the assumption. If follows from $y\psi' x \varphi' a$ and $\varphi' \geqq \psi'$ that $y \varphi' a$ and $y=a$, since $\varphi \cap \varphi'=0$. Then we have $x\psi' a$, that is a contradiction. Therefore we can infer $M=N$, $\varphi'=\psi'$ and $\varphi=\varphi \cup (\psi' \cap \psi)=(\varphi \cup \psi') \cap \psi=\psi$, for permutable congruence relations satisfy the modular law.

3) cf. J. Hashimoto [6, Theorem 6.2].
4) cf. J. Hashimoto [6, Theorem 5.2 and Lemma 4.5].

The main results about universal algebras that we have obtained are mentioned above, but it seems that those theorems may hold under some weaker conditions. For instance we propose

PROBLEM 1. Can the distributivity of $\Theta(A)$ in Theorem 2.3 be replaced by the modularity?

As to the property (α) the way in which congruence relations are determined shall be a matter of question. In a group with the identity e a congruence relation θ is determined by an operation xy from the class $S(e, \theta)$ so that $S(x, \theta)=\{xy\,;\, y\in S(e, \theta)\}$. In general if a congruence relation θ on an algebra A satisfies

$$(\gamma^*) \qquad S(x, \theta) = \{f(x, y)\,;\, y\in S(a, \theta)\}$$

for a fixed element a, then it is shown that the permutability follows. Indeed, $x\theta z\varphi y$ implies $x=f(z, u)$ with $u\in S(a, \theta)$ and $x=f(z, u)\ \varphi f(y,u)\ \theta f(y, a)\in S(y, \theta)$, since $a\in S(a, \theta)$.

Theorem 2.5. *If a congruence relation θ satisfies (γ^*) for a fixed element a and an operation f, then θ is permutable with any congruence relation.*

Let G be a quasi-group with operations xy, x/y, $y\backslash x$ and relations $xy/y=y\backslash yx=(x/y)y=y(y\backslash x)=x$. Then $z\theta x$ implies $y=x\backslash za\theta a$ and $z=xy/a$; hence θ satisfies (γ^*) for the operation $f(x, y)=xy/a$.

Corollary. *Let G be a quasi-group with operations xy, x/y and $y\backslash x$. Then all congruence relations on G are permutable.*

3. Properties (α) and (β) on lattices

In a lattice the substance of Lemma 2.1 can be expressed more simply. Indeed, by putting $b=\bigwedge b_i$ and $c=\bigvee b_i$ in that lemma, we infer

Theorem 3.1. *An element a of a lattice L possesses the property $\alpha(a)$ if and only if, given $x, y\in L$, there exist $b, c\in L$ such that $\theta(x, y)=\theta(a\cap b,\ a\cup c)$.*

It is obvious that $\theta(a\cap b, a)=\theta(b, a\cup b)$ and if x' is a relative complement of x in a closed interval $[a, y]$, then $\theta(x, y)=\theta(a, x')$. Hence the following well-known proposition is immediately deduced from the theorem.

Corollary. *If all intervals $[0, x]$ of a lattice with 0 are complemented, then $\alpha(0)$ holds.*

Now we shall introduce some terms about elements of lattices. An element a of a lattice L shall be called *modular* if

$$x \leqq y \quad \text{implies} \quad x \cup (a \cap y) = (x \cup a) \cap y,$$

and *distributive* if it satisfies

$$a \cap (x \cup y) = (a \cap x) \cup (a \cap y), \;\; a \cup (x \cap y) = (a \cup x) \cap (a \cup y)$$

for all $x, y \in L$. It is easy to show that a is neutral if and only if it is both modular and distributive. In connection with those elements we intend to deal with the properties (α) and (β).

Theorem 3. 2. *Let an element a possess the property $\alpha(a)$ in a lattice L. If a is distributive in L, then a satisfies $\alpha(a)$ in $(a]$ and $[a)$. Conversely let a possess the property $\alpha(a)$ in $(a]$ and $[a)$. If a is modular, then a satisfies $\alpha(a)$ in L.*

Proof. If a is distributive, then the mapping $x \rightarrow a \cap x$ is an endomorphism of L onto $(a]$. So it follows from Theorem 2. 2. α that $\alpha(a)$ in L implies $\alpha(a)$ in $(a]$ and dually in $[a)$. Conversely let a possess the property $\alpha(a)$ in $(a]$ and $[a)$, and $[x, y]$ be any interval in L. Then by Theorem 3.1 we can choose $b \in (a]$ so that $\theta^*(a \cap x,\ a \cap y) = \theta^*(a, b)$ in $(a]$, whence $\theta(a \cap x,\ a \cap y) = \theta(a, b)$ in L by Lemma 2. 3. Similarly we have $\theta(a \cup x,\ a \cup y) = \theta(a, c)$. Put $\varphi = \theta(a \cap x,\ a \cap y) \cup \theta(a \cup x,\ a \cup y)$. It is easy to see $\varphi \leqq \theta(x, y)$. If a is modular, then we $x = x \cup (a \cap x) \varphi x \cup (a \cap y) = (x \cup a) \cap y \varphi (y \cup a) \cap y = y$, showing $\varphi \geqq \theta(x, y)$. Hence $\theta(x, y) = \theta(a, b) \cup \theta(a, c)$ $= \theta(b, c)$ and by Theorem 3.1 a satisfies $\alpha(a)$ in L.

Referring Cor. of Theorem 3. 1, we infer

Corollary 1. *If a is a modular element in a lattice L and all intervals of types $[x, a]$, $[a, y]$ are complemented, then $S(a, \theta) = S(a, \varphi)$ implies $\theta = \varphi$.*

The condition that a is modular cannot dispensed with. In fact, in the five-element non-modular lattice $\{a, b, c, 0, 1\}$, where $b < c$, $a \cap b = a \cap c = 0$, $a \cup b = a \cup c = 1$, the element a, which is distributive but not modular, does not satisfy $\alpha(a)$; nevertheless all intervals containing a are complemented (see also Theorem 3. 3). Further all congruence relations on this lattice are permutable; hence this gives a simple example of such algebras as stated in Birkhoff's Problem 33.

As is shown in a previous paper [5], a distributive lattice L is relatively complemented if every congruence relation on L having an ideal as a congruence class is determined by that ideal. Hence

Corollary 2. *The following conditions concerning an element a of a distributive lattice are equivalent*;

(1) *Every congruence relation is determined by the class containing* a,

(2) *All intervals of types* $[x, a]$, $[a, y]$ *are complemented.*

Though we have mentioned in Cor. of Theorem 3.1 that $\alpha(0)$ holds when all intervals $[0, x]$ are complemented, as a matter of fact not only 0 but all elements satisfy (α) in that case. We first show

Lemma 3.1. *Let an element a satisfy* $\alpha(a)$ *in a lattice L. If all intervals containing a are complemented, then every element x in L satisfies* $\alpha(x)$.

Proof. Let θ be a monomial congruence relation. By Theorem 3.1 we can choose b, c so that $b \leqq a \leqq c$ and $\theta = \theta(b, c)$. Let y be a relative complement of c in the interval $[b, c \cup x]$ containing a, and z a relative complement of $c \cap (x \cup y)$ in $[a \cap x, c]$. Then using the identity $\theta(u \cap v, u) = \theta(v, u \cup v)$, wet get

$$\theta(b, c) = \theta(y, c \cup x) = \theta(y, x \cup y) \cup \theta(x \cup y, c \cup x), \ \theta(y, x \cup y) = \theta(x \cap y, x)$$

and

$$\theta(x \cup y, c \cup x) = \theta(c \cap (x \cup y), c) = \theta(a \cap x, z) = \theta(x, x \cup z),$$

since $c \cup x \cup y = c \cup x$ and $x \cap z = a \cap x$. Hence $\theta = \theta(x \cap y, x) \cup \theta(x, x \cup z) = \theta(x \cap y, x \cup z)$ and thus x satisfies $\alpha(x)$ by Theorem 3.1.

Now let m be a modular element in a lattice L and all intervals containing m complemented. Then by Cor. 1 of Theorem 3.2 m satisfies $\alpha(m)$ and by Lemma 3.1 L satisfies $\alpha(L)$. Moreover we can show that congruence relations on such a lattice L are permutable.

Theorem 3.3. *Let m be a modular element in a lattice L. If all intervals containing m are complemented, then* $\alpha(L)$ *and* $\beta(L)$ *hold*; *namely*

α: *every congruence relation is determined by any one class,*

β: *all congruence relations on L are permutable.*

Proof. It is sufficient to prove β. We shall first show for $a \leqq c \leqq b$ that $a\theta c\varphi b$ implies $a\varphi c'\theta b$ for some element c' with $a \leqq c' \leqq b$. Let x be a relative complement of $c \cup m$ in the interval $[a \cap m, b \cup m]$ and y that of $(c \cup x) \cap m$ in $[(a \cup x) \cap m, m]$. Then we get

$$x = x \cap (b \cup m) \varphi x \cap (c \cup m) = a \cap m, \quad y = y \cup ((a \cup x) \cap m) \theta y \cup ((c \cup x) \cap m) = m,$$

and

$$\begin{aligned} a &= a \cup (a \cap m) \varphi a \cup x = ((a \cup x) \cap m) \cup (a \cup x) = ((c \cup x) \cap m \cap y) \cup (a \cup x) \\ &\varphi ((b \cup x) \cap m \cap y) \cup (a \cup x) \theta ((b \cup x) \cap m) \cup (c \cup x) = (b \cup x) \cap (m \cup c \cup x) \\ &= (b \cup x) \cap (b \cup m) = b \cup x. \end{aligned}$$

Hence if we set $c'=b\frown(((b\smile x)\frown m\frown y)\smile a\smile x)$, then $a\varphi c'\theta b$.

Now suppose that a, b, c are any elements and $a\theta c\varphi b$. Then we can deduce $a\theta a\smile c\varphi a\smile b\smile c\theta b\smile c\varphi b$ and find by the above proof u, v such that $a\leqq u\leqq a\smile b\smile c\geqq v\geqq b$ and $a\varphi u\theta a\smile b\smile c\varphi v\theta b$. It follows that $u=u\frown(a\smile b\smile c)\varphi u\frown v\theta(a\smile b\smile c)\frown v=v$ and $a\varphi u\frown v\theta b$, completing the proof.

The modularity of m cannot dispensed with. On the lattice of Fig. 1 shown below congruence relations are not permutable; nevertheless all intervals containing m are complemented.

It follows from this theorem that a lattice L with 0 satisfies $\alpha(L)$ and $\beta(L)$ if all intervals $[0, x]$ are complemented. Such a lattice in which all intervals $[0, x]$ are complemented is called *section-complemented*. For a lattice L without 0 we shall define L to be section-complemented when every element of L is contained in a section-complemented principal dual ideal. If a lattice L is section-complemented, then any three elements x, y, z are contained in a section-complemented dual ideal $S=[a)$, in which $\alpha(S)$ and $\beta(S)$ hold; hence by Theorem 2.1 we can infer

Corollary. *In a section-complemented lattice every congruence relation is determined by any one class and all congruence relations are permutable.*

Again in a distributive lattice L with 0 we see that $\alpha(0)$ implies $\alpha(L)$. Then one may question in general how the property of some elements influences other elements. It may be conjectured that, if $\alpha(a)$ and $\alpha(b)$ hold and $a\leqq c\leqq b$, then so does $\alpha(c)$, but the conjecture is affirmative only for an (upper) distributive element c.

We shall write $[a, b]\rightarrow[x, y]$ if an interval $[x, y]$ is contained in a transpose of $[a, b]$, and call $[x, y]$ to be *weakly projective* into $[a, b]$ if there exist a finite number of intervals $[x_i, y_i]$ such that

$$[a, b] = [x_0, y_0]\rightarrow[x_1, y_1]\rightarrow\cdots\rightarrow[x_n, y_n] = [x, y].$$

Then Dilworth [3] has proved

Lemma 3.2. $x\equiv y(\theta(a, b))$ *holds if and only if there exist a finite number of elements z_i such that*

$$x\frown y = z_0\leqq z_1\leqq\cdots\leqq z_n = x\smile y$$

and each $[z_{i-1}, z_i]$ is weakly projective into $[a\frown b, a\smile b]$.

If an element d is upper distributive, that is $d\smile(x\frown y)=(d\smile x)\frown(d\smile y)$ for all $x, y\in L$, then it is obvious that $[a, b]\rightarrow[x, y]$ implies $[a\smile d, b\smile d]\rightarrow[x\smile d, y\smile d]$; hence we obtain

Lemma 3.3. *If $[x, y]$ is weakly projective into $[a, b]$ and d is upper distributive, then $[x\smile d, y\smile d]$ is weakly projective into $[a\smile d, b\smile d]$.*

Then we show

Theorem 3.4. *Let d be an upper distributive element contained in an interval $[a, b]$. If $\alpha(a)$ and $\alpha(b)$ hold, so does $\alpha(d)$.*

Proof. Let θ be a monomial congruence relation. Then by Theorem 3.1. we can choose s, t, u, v so that $s \leqq a \leqq t$, $u \leqq b \leqq v$ and $\theta = \theta(s, t) = \theta(u, v)$, and it suffices to show $\theta = \theta(d \frown u, d \smile t)$. Since $d \frown u \theta d \frown v = d = d \smile s \theta d \smile t$, $\theta \geqq \varphi = \theta(d \frown u, d \smile t)$. Since $u \equiv v(\theta(s, t))$, by Lemma 3.2 there exists a chain $u = u_0 \leqq u_1 \leqq \cdots \leqq u_n = v$ such that every $[u_{i-1}, u_i]$ is weakly projective into $[s, t]$ and hence $[u_{i-1} \smile d, u_i \smile d]$ is weakly projective into $[s \smile d, t \smile d] \subseteqq [d \frown u, d \smile t]$. Therefore we get $u \smile d \varphi v$. On the other hand $d \frown u \varphi d$ implies $u \varphi u \smile d$. Thus we have $u \varphi v$ and $\varphi \geqq \theta(u, v) = \theta$, completing the proof.

Further a distributive element d satisfying $\alpha(d)$ is neutral. We show more generally

Theorem 3.5. *If a distributive element a satisfies $\alpha(a)$ in a lattice L, then every distributive element d in L is neutral.*

Proof. $\varphi(x) = d \frown x$ and $\psi(x) = d \smile x$ are endomorphisms in L. If $x \equiv a(\varphi \frown \psi)$, then $d \frown x = d \frown a$, $d \smile x = d \smile a$. Hence we get

$$a = a \frown (d \smile a) = a \frown (d \smile x) = (a \frown d) \smile (a \frown x) = (d \frown x) \smile (a \frown x) \leqq x$$

and dually $a \geqq x$. So $S(a, \varphi \frown \psi) = S(a, 0)$ and $\varphi \frown \psi = 0$. Then the mapping $x \rightarrow (d \frown x, d \smile x)$ is a subdirect decomposition; accordingly d is neutral. If either $d \leqq a$ or $d \geqq a$, we can dispense with the distributivity of a.

If a, $a \frown b$ and $a \smile b$ are neutral, then it is easy to show that b is neutral. Using this fact, we can prove by induction that a lattice of a finite length possessing a maximal chain which consists of neutral elements is distributive. Hence from the above theorem we can deduce

Corollary. *Let a lattice L of a finite length possess a maximal chain which consists of distributive elements. If $\alpha(0)$ holds, then L is a Boolean algebra.*

In contrast with this the five-element non-modular lattice cited before possesses a maximal chain $\{0, a, 1\}$ whose elements are distributive.

As to the converse of Theorem 3.3, we can assert the following theorem concerning distributive elements.

Theorem 3.6. *Let all congruence relations on a lattice L be permutable. Then a distributive element d has a relative complement in every* *interval* $[a, b]$ *containing it. If* d *possesses the property* $\alpha(d)$ *moreover, then* $L \cong (d] \times [d)$.

Proof. Put $\varphi(x) = d \cap x$ and $\psi(x) = d \cup x$. Then $a\psi d\varphi b$. Since φ and ψ are permutable, we can find c such that $a\varphi c\psi b$, whence $d \cap c = d \cap a = a$, $d \cup c = d \cup b = b$. The latter half is evident, since d is neutral by Theorem 3. 5.

Corollary. *On a distributive lattice* L *the following conditions are equivalent*:

(1) *L is relatively complemented,*

(2) *Every congruence relation is determined by any one class,*

(3) *Congruence relations on L are permutable.*

One of our objects is to inquire into the connection between the conditions (2) and (3) given above, but in arbitrary lattices we cannot find such a close connection as in distributive lattices. Only on locally finite modular lattices[5] L we can affirm that $\beta(L)$ implies $\alpha(L)$. For congruence relations on such a lattice L form a Boolean algebra; hence if congruence relations are permutable, L is decomposed into a restricted direct union of simple factors and accordingly all elements in L satisfy (α) by Theorem 2. 4. However not so does a modular lattice having an interval of infinite length. The lattice of Fig. 2 shown below is modular and congruence relations on it are permutable, but $\alpha(0)$ does not hold. Again the converse $\alpha(L) \rightarrow \beta(L)$ does not hold even in a finite modular lattice. Fig. 3 shows the simplest example of such lattices all of whose elements satisfy (α) but on which congruence relations are not permutable.

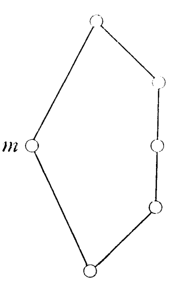

Fig. 1

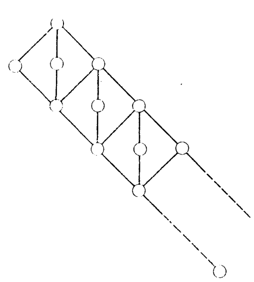
Fig. 2

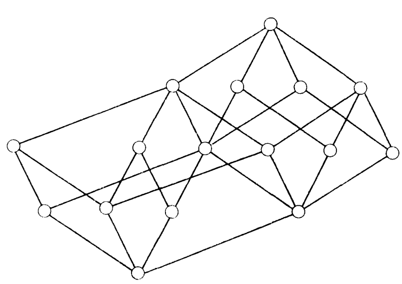
Fig. 3

4. A structure theorem for lattices

The property (γ^*) mentioned in § 2 is meaningless in lattices, for

5) A locally finite lattice is a lattice in which every closed interval has a finite length.

the set $\{f(x, y)\,;\, y \in S(a, \theta)\}$ is contained in the sublattice generated by $S(a, \theta)$ and x; so even the lattice $\mathbf{2}\times\mathbf{2}$ does not satisfy (γ^*). But in a section-complemented lattice a congruence relation θ is determined by some operations in the following way.

Let a be an element of a lattice L and θ a congruence relation on L. By $J(a, \theta)$ and $J'(a, \theta)$ we shall denote the ideal and the dual ideal respectively generated by the class $S(a, \theta)$. If $x \in J(a, \theta)$ we can find $y \geqq x$ with $y\theta a$, whence $x \cup a \theta x \cup y = y \theta a$; so we may write $J(a, \theta) = \{x\,;\, x \cup a\theta a\}$. Then a congruence relation θ on a section-complemented lattice possess the property

(γ) If $x\theta y$, there exists $s \in J(a, \theta)$ satisfying $x \cup y = (x \cap y) \cup s$.

Indeed, let a, x, y be contained in a section-complemented dual ideal $[b)$ and s a relative complement of $x \cap y$ in $[b, x \cup y]$. If $x\theta y$, we get $s\theta b$, $s \cup a\theta a$ and $s \in J(a, \theta)$. Even in the case that intervals $[a, x]$ are complemented, we see, by taking a relative complement s of $a \cup (x \cap y)$ in $[a, a \cup x \cup y]$, that the element a satisfies a somewhat weaker condition

(δ) If $x\theta y$, there exists $s \in S(a, \theta)$ satisfying $x \cup y \leqq (x \cap y) \cup s$.

We shall deal in the present section with those properties. By (γ') and (δ') we shall mean the dual of (γ) and (δ) respectively. Further we shall write $\gamma(a, \theta)$ to mean that (γ) holds for an element a and a congruence relation θ, $\gamma(a)$ that all congruence relations θ satisfy $\gamma(a, \theta)$ and $\gamma(\theta)$ that all elements a satisfy $\gamma(a, \theta)$.

Now we shall call a lattice L to satisfy the *restricted chain condition* if every closed interval of L satisfies either one of chain conditions.

Lemma 4. 1. *Let L be a section-complemented lattice satisfying the restricted chain condition. Then every congruence relation θ satisfies $\delta'(\theta)$.*

Proof. Suppose $x\theta y$ and $x \leqq y$. Let $[b)$ be section-complemented and contain a, x, y. If $[b, a]$ satisfies the ascending condition, we can find c such that $[b, a] \cap S(b, \theta) = [b, c]$. Let t be a relative complement of c in $[b, a]$ and u that of $x \cap t$ in $[b, y \cap t]$. Then we get $u\theta b$, $u \leqq c$ and $u \leqq c \cap t = b$. Hence $y \cap t = x \cap t \leqq x$ with $t\theta a$. If $[b, a]$ satisfies the descending condition, then $[b, a] \cap S(a, \theta) = [t, a]$ for some t. Let u be a relative complement of $x \cap t$ in $[b, y \cap t]$ and v that of u in $[b, t]$. Then $u\theta b$, $v\theta a$ and $v \geqq t$, whence $v = t$ and $u = b$. So we have $y \cap t = x \cap t$ with $t\theta a$.

PROBLEM 2. Can the chain condition in Lemma 4. 1 be dispensed with?

We shall now state the relation between the above conditions and the conditions (α), (β).

Lemma 4.2. *$x\cup y=(x\cap y)\cup s$ and $x\cap y\geqq(x\cup y)\cap t$ with $s\in J(a, \theta)$, $t\in S(a, \theta)$ imply $x\theta y$.*

Proof. It suffices to prove for $x\leqq y$. It follows that $y\geqq x\cup(y\cap(s\cup a))\geqq x\cup((x\cup s)\cap s)=x\cup s=y$ and $x\cup(y\cap(s\cup a))\theta x\cup(y\cap t)=x$, since $s\cup a\theta a\theta t$. Hence $x\theta y$.

Lemma 4.3. *Let m be a modular element. $x\cup y\leqq(x\cap y)\cup s$ and $x\cap y\geqq(x\cup y)\cap t$* with $s, t\in S(m, \theta)$ imply $x\theta y$.

Proof. If $x\leqq y$, we have $x=x\cup(t\cap y)\theta x\cup(m\cap y)=(x\cup m)\cap y\theta(x\cup s)\cap y=y$.

From those lemmas the following theorem is immediately deduced.

Theorem 4.1. *$\gamma(a)$ and $\delta'(a)$ imply $\alpha(a)$. If m is a modular element, then $\delta(m)$ and $\delta'(m)$ imply $\alpha(m)$.*

Theorem 4.2. *If a congruence relation θ on a lattice satisfies $\delta(\theta)$ and $\delta'(\theta)$, then θ is permutable with any congruence relation φ.*

Proof. Suppose $x\theta y\varphi z$. If $x\geqq y\geqq z$, we can find $s\in S(z, \theta)$ such that $x\leqq y\cup s$ and hence $x=x\cap(y\cup s)\varphi x\cap(z\cup s)\theta x\cap z=z$. Dually for the case $x\leqq y\leqq z$ we obtain $u\in[x, z]$ with $x\varphi u\theta z$. Then, whenever $x\theta y\varphi z$, we can derive $x\varphi u\theta z$ in the same way as the proof of Theorem 3.3.

Next we shall investigate direct decompositions of a lattice in connection with the property (γ). Let θ be a decomposition congruence relation and suppose $x\theta y$ with $x\leqq y$. If we choose s so that $a\theta s\theta' y$, where θ' is the complement of θ, then it is easy to see $x\cup(s\cap y)\theta y$, $x\cup(s\cap y)\theta' y$ and so $x\cup(s\cap y)=y$ with $s\cap y\ni J(a, \theta)$. Hence a decomposition congruence relation θ satisfies $\gamma(\theta)$ and $\gamma'(\theta)$. The converse of this fact is our main object in this section. Before stating it, we must deal with some related matters. Since $a\leqq b$ implies $J(a, \theta)\subseteqq J(b, \theta)$, we have first

Lemma 4.4. *If $a\leqq b$, $\gamma(a, \theta)$ implies $\gamma(b, \theta)$, and $\delta(a, \theta)$ implies $\delta(b, \theta)$.*

Lemma 4.5. *Let θ^* be the congruence relation induced by θ on an interval $[b, c]$ containing a. If θ satisfies $\gamma(a, \theta)$ or $\delta(a, \theta)$ in the whole lattice L, then θ^* satisfies the same condition in $[b, c]$.*

Proof. If $y=x\cup s$, $s\in J(a, \theta)$ for $x, y\in[b, c]$, then $y=x\cup b\cup s$, $b\cup s\in J(a, \theta)\cap[b, c]=J(a, \theta^*)$. If $y\leqq x\cup t$, $t\in S(a, \theta)$ for $x, y\in[b, c]$, then $y\leqq x\cup b\cup t$, $b\cup t\in S(a, \theta)\cap[b, c]=S(a, \theta^*)$.

Lemma 4.6. *Let X, Y be any ideals containing a and set $J=J(a, \theta)$. If θ satisfies $\delta(a, \theta)$, then $J\cup(X\cap Y)=(J\cup X)\cap(J\cup Y)$, and if θ satisfies $\gamma(a, \theta)$, then $X\cap(J\cup Y)=(X\cap J)\cup(X\cap Y)$.*

Proof. $x \in X$, $y \in Y$ and $t \in J$, we set $u = x \cup a$, $v = y \cup a$. Then it is easy to show $(t \cup u) \cap (t \cup v) \theta u \cap v$ and $u \cap (t \cup v) \theta u \cap v$. If θ satisfies $\delta(a, \theta)$, we can find $s \in S(a, \theta)$ such that $(t \cup u) \cap (t \cup v) \leqq (u \cap v) \cup s$ and hence $(t \cup x) \cap (t \cup y) \leqq s \cup (u \cap v) \in J \cup (X \cap Y)$. If θ satisfies $\gamma(a, \theta)$, we can find $s \in J(a, \theta)$ such that $u \cap (t \cup v) = (u \cap v) \cup s$. From $s \leqq u$, $s \in X \cap J$ follows. Then $x \cap (t \cup y) \leqq s \cup (u \cap v) \in (X \cap J) \cup (X \cap Y)$.

According to Grätzer and Schmidt [4], an ideal J satisfying $X \cap (J \cup Y) = (X \cap J) \cup (X \cap Y)$ for all ideals X, Y is called *standard*. Every standard ideal J is upper distributive, i.e. $J \cup (X \cap Y) = (J \cup X) \cap (J \cup Y)$ for all ideals X, Y. Now, given an ideal J, put $\theta(J) = \bigvee_{a,b \in J} \theta(a, b)$. If J is upper distributive, then $\theta(x) = J \cup (x]$ is a homomorphism of the lattice into its ideal lattice with the kernel J and it is easy to see $\theta = \theta(J)$. If $a \in J$, then $x \theta y$ and $x \leqq y$ imply $y \leqq x \cup s$ for some $s \in J = S(a, \theta)$; namely θ satisfies $\delta(a, \theta)$, and by Lemma 4.4 we can show that θ satisfies $\delta(\theta)$. Moreover if J is standard, then the above y satisfies $y \in (y] \cap (J \cup (x]) = ((y] \cap J) \cup (x]$, whence there exists $t \in J$ such that $y \leqq (y \cap t) \cup x$; namely $y = x \cup (y \cap t)$ with $y \cap t \in J$. Hence θ satisfies $\gamma(a, \theta)$ and so $\gamma(\theta)$. Referring Lemma 4.6 again, we get

Lemma 4.7. *An ideal J is upper distributive if and only if J is a class of a congruence relation θ satisfying $\delta(\theta)$, and standard if and only if J is a class of θ satisfying $\gamma(\theta)$.*

Then we show the first main theorem.

Theorem 4.3. *Let θ be a congruence relation on a lattice L with $0, 1$ satisfying the conditions $\gamma(0, \theta)$ and $\gamma'(1, \theta)$. If $S(0, \theta)$ has a maximal element c, then c is in the center of L; accordingly θ is a decomposition congruence relation.*

Proof. It follows from Lemma 4.7 that $x \cap (c \cup y) = (x \cap c) \cup (x \cap y)$ for all $x, y \in L$, since $(c] = S(0, \theta)$. Accordingly it suffices to show that $x \cup (c \cap y) = (x \cup c) \cap (x \cup y)$ and c has a complement c'. By $\gamma'(1, \theta)$ we can find c' such that $0 = c \cap c'$ with $c' \theta 1$. If $a \leqq c'$ and $a \theta 1$, then by $\gamma(0, \theta)$ we get $1 = a \cup b$ for some $b \in S(0, \theta) = (c]$ whence $a \cup c = 1$ and especially $c \cup c' = 1$. Moreover we see $c' = c' \cap (c \cup a) = (c' \cap c) \cup (c' \cap a) = a$; hence we obtain $S(1, \theta) = [c')$ and $x \cup (c' \cap y) = (x \cup c') \cap (x \cup y)$ by the dual of Lemma 4.7. Put $u = x \cup (c \cap y)$ and $v = (x \cup c) \cap (x \cup y)$. Since $y = y \cap (c \cup c') = (y \cap c) \cup (y \cap c')$, we have $y \cup c' = (y \cap c) \cup c'$ and $u \cup c' = x \cup y \cup c' \geqq v$. Then

$$v = (u \cup c') \cap (u \cup v) = u \cup (c' \cap v) = u \cup (c' \cap (c \cup x) \cap (x \cup y)) = u \cup (c' \cap x) = u .$$

Further the theorem is true for a lattice L without $0, 1$. If $(c]$ is a class of θ satisfying $\gamma(\theta)$ and $\gamma'(\theta)$, then by Lemma 4.5 $\gamma(a, \theta^*)$ and

$\gamma'(b, \theta^*)$ hold in every interval $[a, b]$ containing c and hence c is in the center of $[a, b]$. Then corresponding (x_1, x_2), where $x_1 \in (c]$, $x_2 \in [c)$, to the relative complement x of c in $[x_1, x_2]$, we have $L \cong (c] \times [c)$.

Corollary. *Let θ be a congruence relation satisfying $\gamma(\theta)$ and $r'(\theta)$ in a lattice L. If a principal ideal $(c]$ is a class of θ, then $L \cong (c] \times [c)$.*

If the ideal $J = S(0, \theta)$ is not principal, then J cannot become a direct component of the lattice with 0, 1; however we doubt if it should be neutral.

PROBLEM 3. Let θ be a congruence relation satisfying $\gamma(\theta)$ and $\gamma'(\theta)$, and possessing an ideal J as its class. Is J necessarily neutral?

Theorem 4.4. *Let L be a lattice satisfying the restricted chain condition. Then a congruence relation θ on L is a decomposition congruence relation if and only if it satisfies $\gamma(\theta)$ and $\gamma'(\theta)$.*

Proof. Assume $\gamma(\theta)$ and $\gamma'(\theta)$. As θ is permutable with every congruence relation, we need only show that θ has a complement θ'. Let $\{\varphi_\lambda\}$ be the set of all congruence relations φ_λ satisfying $\theta \cap \varphi_\lambda = 0$ and put $\theta' = \bigvee \varphi_\lambda$. Then $\theta \cap \theta' = 0$, since $\Theta(L)$ is distributive and upper continuous. We shall show $\theta \cup \theta' = 1$. Let $[a, b]$ be any interval in L. If $[a, b]$ satisfies the ascending condition, then we can find c such that $S(a, \theta) \cap [a, b] = [a, c]$ and, as is shown above, c' such that $S(b, \theta) \cap [a, b] = [c', b]$. Put $\varphi = \theta(c, b)$ and assume $x \equiv y(\theta \cap \varphi)$ with $x \leqq y$. We can choose a chain $x = x_0 \leqq x_1 \leqq \cdots \leqq x_n = y$ so that each subinterval $[x_{i-1}, x_i]$ be weakly projective into $[c, b]$. Further, taking a sufficiently large interval $[f, e]$ containing a, b, c, x, y, we can make every $[x_{i-1}, x_i]$ be weakly projective into $[c, b]$ in that interval $[f, e]$. There exists d such that $S(f, \theta) \cap [f, e] = [f, d]$. Then, since d is neutral in $[f, e]$, $[d \cap x_{i-1}, d \cap x_i]$ is weakly projective into $[d \cap c, d \cap b]$ by Lemma 3.3. On the other hand we get $(a \cup d) \cap b \theta a$, $(a \cup d) \cap b \leqq c \leqq b$ and hence $d \cap c = d \cap b$. So we see $d \cap x_{i-1} = d \cap x_i$ and $d \cap x = d \cap y$. Morever since $x \theta y$ in $[f, e]$, $y \leqq x \cup d$ and $d \cup x = d \cup y$. From those equalities and the neutrality of d, we infer $x = y$, showing $\theta \cap \varphi = 0$ and $\varphi \leqq \theta'$. Then $c \equiv b(\theta')$ and $a \equiv b(\theta \cup \theta')$; hence $\theta \cup \theta' = 1$.

And we can immediately deduce

Corollary 1. *Let L be a lattice satisfying the restricted chain condition. Then L is decomposed into a (restricted) direct union of simple lattices if and only if all congruence relations on L satisfy the conditions (γ) and (γ').*

It has been proved that a congruence relation θ on a section-complemented lattice satisfies $\gamma(\theta)$; hence we infer

Corollary 2. *Let L be a section-complemented lattice satisfying the restricted chain condition. If its dual is also section-complemented, then L is a (restricted) direct union of simple lattices.*

This is a generalization of the results about relatively complemented lattices in Dilworth's [3] and the author's [6] previous papers.

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(Received March 28, 1963)

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