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ON THE CLASS-FIELDS OBTAINED BY COMPLEX MULTIPLICATION OF ABELIAN VARIETIES

Dedicated to Professor K. Shoda on his sixtieth birthday

By

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By complex multiplication of abelian varieties, we get certain class-fields over a totally imaginary quadratic extension F of a totally real algebraic number field F_0. The corresponding ideal-groups are explicitly given in Main Theorems of [3]. On this subject, one may ask how large class-fields over F can be constructed by such a means. An answer to the question is given in [4, 5], to a certain degree, in terms of local characters attached to Grössen-characters. However, this does not give any information, for example, about unramified class-fields over F so obtained. The purpose of the present paper is to give some results concerning this problem, which are almost directly derived from the defining-relation for the ideal-groups mentioned above.

In general the ideal-class group $\mathfrak{K}$ of F is approximately decomposed into the ideal-class group $\mathfrak{K}_0$ of F_0 and its complementary part $\mathfrak{K}_1$. Adjoining the absolute class-field over F_0 to F, we get the unramified class-field over F corresponding to $\mathfrak{K}/\mathfrak{K}_1$. Now, roughly speaking, the unramified class-field over F corresponding to $\mathfrak{K}/\mathfrak{K}_0$ is generated by the fields of moduli of certain polarized abelian varieties. The ramified class-fields over F are found in a similar situation, if we consider the points of finite order on the varieties. In § 2, we show these facts under a condition on F, which is satisfied whenever F is normal over the rational number field. We shall prove that the class-fields over F_0 and complex multiplication yield at least a subfield B of the maximal abelian extension A of F such that $A \subset B(\sqrt{x} \mid x \in B)$ (Theorem 1); B contains the absolute class-field over F (Theorem 2). If F is an imaginary cyclotomic field, the results are stated in a little preciser and simpler form, as we shall see in § 3. The object of the final § 4 is the investigation of a special kind of CM-types, by which we can prove, without any condition on F, similar results for the class-fields over F obtained from complex multiplication of an abelian variety whose endomorphism-algebra contains a quadratic extension of F (Theorem 4). In all these cases, if the class-number of F is odd, the absolute class-field over F is contained in the composite of the absolute class-field over F_0 and the fields of moduli of certain polarized abelian varieties which we can specify in each case.

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NOTATION AND CONVENTION. $\boldsymbol{Q}$ and $\boldsymbol{C}$ denote respectively the field of rational numbers and the field of complex numbers. For every $x \in \boldsymbol{C}$, we denote by x^ρ the complex conjugate of x. Any algebraic number field will be considered as a subfield of $\boldsymbol{C}$. If K is an algebraic number field of finite degree and $\mathfrak{b}$ is an integral ideal of K, $I_\mathfrak{b}(K)$ denotes the group of all ideals prime to $\mathfrak{b}$, and $P_\mathfrak{b}(K)$ the subgroup of $I_\mathfrak{b}(K)$ consisting of all principal ideals (a) such that $a \in K$, $a \equiv 1 \bmod \mathfrak{b}$. For every positive integer b, the ideal (b) generated by b (in some algebraic number field) will be often denoted simply by b. Further we denote by $C_\mathfrak{b}(K)$ the class-field over K corresponding to the ideal-group $P_\mathfrak{b}(K)$, namely, the ray-class-field modulo $\mathfrak{b}$ over K. In particular, $C_1(K)$ is the absolute class-field (Hilbert's class-field) over K.

§ **1. Preliminaries.** Let F_0 be a totally real algebraic number field of finite degree, and F a totally imaginary quadratic extension of F_0. Define, for every positive integer b, a subgroup $I_b(F/F_0)$ of $I_b(F)$ by

$$(1) \qquad I_b(F/F_0) = \{\mathfrak{a} \in I_b(F) \mid \mathfrak{a}/\mathfrak{a}^\rho = (a) \text{ for some } a \in F \text{ such that } aa^\rho = 1,\ a \equiv 1 \bmod (b)\}.$$

We see easily that

$$(2) \qquad I_b(F/F_0) \supset P_b(F) \cdot \{\mathfrak{a} \in I_b(F) \mid \mathfrak{a}^\rho = \mathfrak{a}\} \supset P_b(F) I_b(F_0).$$

Consider the case $b=1$. If $\mathfrak{a} \in I_1(F/F_0)$, we have $\mathfrak{a}/\mathfrak{a}^\rho = (a)$ for some $a \in F$ such that $aa^\rho = 1$. By Hilbert's lemma, there exists an element w of F such that $a = w^\rho/w$. Then $(w\mathfrak{a})^\rho = w\mathfrak{a}$. It follows that

$$(3) \qquad I_1(F/F_0) = P_1(F) \cdot \{\mathfrak{a} \in I_1(F) \mid \mathfrak{a}^\rho = \mathfrak{a}\}.$$

Let $(F; \{\sigma_1, \cdots, \sigma_n\})$ be a CM-type and $(F^*; \{\tau_j\})$ be its dual (cf. [3, §§ 5.2, 8.3]). Let $\mathfrak{b}$ be an integral ideal of F^*, and b the smallest positive integer divisible by $\mathfrak{b}$. We denote by $I_\mathfrak{b}(F; \{\sigma_i\})$ the subgroup of $I_b(F)$ consisting of all ideals $\mathfrak{a}$ such that there exists an element u of F^* for which we have

$$(4)\qquad \prod_{i=1}^{n} \mathfrak{a}^{\sigma_i}=(u)\,,\quad N(\mathfrak{a}) = uu^{\rho}\,,\quad u\equiv 1 \bmod \mathfrak{b}\,.$$

Further we denote by $C_b(F/F_0)$ and $C_{\mathfrak{b}}(F;\ \{\sigma_i\})$ the class-fields over F corresponding to the ideal-groups $I_b(F/F_0)$ and $I_{\mathfrak{b}}(F;\ \{\sigma_i\})$, respectively. If $\mathfrak{a}\in I_b(F;\ \{\sigma_i\})$, we have $N(\mathfrak{a})\equiv 1 \bmod (b)$. It follows that $C_b(F;\ \{\sigma_i\})$ *contains the cyclotomic field* $\boldsymbol{Q}(\zeta)$ *for a primitive b-th root of unity* ζ.

Now Main Theorems 1 and 2 of [3] assert that if $(K^*;\ \{\psi_\alpha\})$ is a primitive CM-type, we get the class-fields $C_{\mathfrak{b}}(K^*;\ \{\psi_\alpha\})$ by means of complex multiplication of an abelian variety belonging to the dual of $(K^*;\ \{\psi_\alpha\})$. This result holds in a little more general form:

Proposition 1. *The assertions of Main Theorems* 1 *and* 2 *of* [3] *are true even in case where* $(K^*;\ \{\psi_\alpha\})$ *is not primitive.*

Proof. Let $(K^*;\ \{\psi_\alpha\})$ be a CM-type which is not necessarily primitive. Let $(K;\ \{\varphi_\lambda\})$ be the dual of $(K^*;\ \{\psi_\alpha\})$, and $(K_1^*;\ \{\chi_\nu\})$ be the dual of $(K;\ \{\varphi_\lambda\})$. Then $(K;\ \{\varphi_\lambda\})$ and $(K_1^*;\ \{\chi_\nu\})$ are primitive; and $(K;\ \{\varphi_\lambda\})$ is the dual of $(K_1^*;\ \{\chi_\nu\})$ (cf. [3, § 8. 3]). Let L be a Galois extension of $\boldsymbol{Q}$ containing K^*. Then K and K_1^* are subfields of L. Let G be the Galois group of L over $\boldsymbol{Q}$, and H^*, H_1^* be respectively the subgroups of G corresponding to K^*, K_1^* by Galois theory. We have $K^*\supset K_1^*$, $H^*\subset H_1^*$, in view of the result of [3, § 8. 3]. Extend ψ_α and χ_ν to elements of G and denote them again by the same letters. We have then

$$(5)\qquad \bigcup_{\alpha} H^*\psi_\alpha = \bigcup_{\nu} H_1^*\chi_\nu\,.$$

Let $\mathfrak{b}$ be an integral ideal of K and b the smallest positive integer divisible by $\mathfrak{b}$. Considering an abelian variety belonging to $(K;\ \{\varphi_\lambda\})$, we get the class-field $C_{\mathfrak{b}}(K_1^*;\ \{\chi_\nu\})$ over K_1^*. The composite of K^* and $C_{\mathfrak{b}}(K_1^*;\ \{\chi_\nu\})$ is a class-field over K^*; and by the "theorem of translation" of class-field theory, the corresponding ideal-group is the group of ideals $\mathfrak{a}\in I_b(K^*)$ such that $N_{K^*/K_1^*}(\mathfrak{a})\in I_{\mathfrak{b}}(K_1^*;\ \{\chi_\nu\})$. By the relation (5), this ideal-group is just $I_{\mathfrak{b}}(K^*;\ \{\psi_\alpha\})$; so we get our proposition.

For convenience, we state here a part of [3, § 8. 3, Prop. 28] as

Proposition 2. *Let* $(F;\ \{\sigma_i\})$ *be a CM-type and* $(F^*;\ \{\tau_j\})$ *its dual. Then* F^* *is generated over* $\boldsymbol{Q}$ *by the elements* $\sum_{i=1}^{n} x^{\sigma_i}$ *for* $x\in F$.

§ 2. **Class-fields obtained from two CM-types.** F and F_0 being as in § 1, let $(F;\ \{\sigma_1, \cdots, \sigma_n\})$ be a CM-type such that σ_1 is the identity mapping of F. Consider the condition:

(A) *If* $(F^*;\ \{\tau_j\})$ *is the dual of* $(F;\ \{\sigma_i\})$, *then* $F\supset F^*$.

This is satisfied whenever F is normal over $\boldsymbol{Q}$. Now we observe that $(F;\{\rho, \sigma_2, \cdots, \sigma_n\})$ is a CM-type. Let $(F_1^*;\{\varphi_\lambda\})$ be the dual of this CM-type. If $(F;\{\sigma_i\})$ satisfies the condition (A), we have $F_1^* \subset F$. In fact, by Proposition 2, for every $x \in F$, we see that $\sum_{i=1}^{n} x^{\sigma_i} \in F^* \subset F$, so that $x^\rho + \sum_{i=2}^{n} x^{\sigma_i} = x^\rho - x + \sum_{i=1}^{n} x^{\sigma_i} \in F$; this implies, again by Proposition 2, $F_1^* \subset F$.

Proposition 3. *Notation being as above, suppose that the condition (A) is satisfied. Then, for every positive integer b, we have*

$$I_b(F;\{\sigma_i\}) \cap I_b(F;\{\rho, \sigma_2, \cdots, \sigma_n\}) \subset I_b(F/F_0)\,.$$

Proof. If $\mathfrak{a} \in I_b(F;\{\sigma_i\}) \cap I_b(F;\{\rho, \sigma_2, \cdots, \sigma_n\})$, we have $\mathfrak{a}\mathfrak{a}^{\sigma_2} \cdots \mathfrak{a}^{\sigma_n} = (u)$, $\mathfrak{a}^\rho \mathfrak{a}^{\sigma_1} \cdots \mathfrak{a}^{\sigma_n} = (v)$, $N(\mathfrak{a}) = uu^\rho = vv^\rho$ for an element u of F^* and an element v of F_1^* such that $u \equiv 1 \bmod (b)$, $v \equiv 1 \bmod (b)$. Put $a = u/v$. By our assumption and by the above consideration, a is an element of F; and we have $\mathfrak{a}/\mathfrak{a}^\rho = (a)$, $aa^\rho = 1$, $a \equiv 1 \bmod (b)$. This proves our proposition.

Theorem 1. *Let F_0 be a totally real algebraic number field of degree $n > 1$, and F a totally imaginary quadratic extension of F_0. Then, the composite D_b of $C_b(F/F_0)$ and $C_b(F_0)$ contains the class-field over F corresponding to the ideal-group $\{\mathfrak{a} \in I_b(F) \mid \mathfrak{a}^2 \in P_b(F)\}$. Let further $(F;\{\sigma_1, \cdots, \sigma_n\})$ be a CM-type such that σ_1 is the identity mapping of F. Suppose that the condition* (A) *is satisfied. Then, for every positive integer b, the composite of $C_b(F;\{\sigma_i\})$ and $C_b(F;\{\rho, \sigma_2, \cdots, \sigma_n\})$ contains $C_b(F/F_0)$.*

In other words, *if there exists a CM-type satisfying the condition* (A), *then, adjoining the ray-class-field modulo* (b) *over F_0, we get, by complex multiplication of abelian varieties, at least a subfield D_b of the ray-class-field $C_b(F)$ modulo* (b) *over F such that the Galois group of $C_b(F)/D_b$ is of exponent* 1 *or* 2.

Proof. The composite of $C_b(F_0)$ and F is the class-field over F corresponding to the ideal-group $\{\mathfrak{a} \in I_b(F) \mid \mathfrak{a}\mathfrak{a}^\rho \in P_b(F_0)\}$. If $\mathfrak{a}\mathfrak{a}^\rho \in P_b(F_0)$ and $\mathfrak{a}/\mathfrak{a}^\rho \in P_b(F)$, we have $\mathfrak{a}^2 \in P_b(F)$. This proves the first assertion. The second assertion is an immediate consequence of Proposition 3.

REMARK 1. If F is a non-abelian imaginary extension of $\boldsymbol{Q}$ of degree 4, the condition (A) is never satisfied by any CM-type $(F;\{\sigma_i\})$. In § 4, we shall give an example of a primitive CM-type $(F;\{\sigma_i\})$ satisfying (A) with an F which is not normal over $\boldsymbol{Q}$.

The author is ignorant of the difference between the maximal abelian extension $A=\bigcup_{b=1}^{\infty} C_b(F)$ and $B=\bigcup_{b=1}^{\infty} C_b(F_0)C_b(F; \{\sigma_i\})C_b(F; \{\rho, \sigma_2, \cdots, \sigma_n\})$[1]. If we put $D=\bigcup_{b=1}^{\infty} D_b$, we have $A\supset B\supset D$, and $A\subset D(\sqrt{x}\mid x\in D)$. We can at least prove:

Theorem 2. *F, F_0 and D_b being as in Theorem 1, the absolute class-field over F is contained in D_b for a suitable b.*

Proof. Let $E_1, \cdots, E_r$ be cyclic unramified extentions of F such that the composite of them is the maximal one among the unramified abelian extensions of F whose degrees are powers of 2. By [2, Satz 1b], we can find, for each i, a cyclic extension E_i' of F containing E_i such that $[E_i':E_i]=2$. Let b be a positive integer such that the ideal-groups corresponding to the E_i' are all defined modulo (b). Now let E_0 be the maximal one among the unramified abelian extensions of F of odd degree. Let $\mathfrak{H}, \mathfrak{K}, \mathfrak{L}$ denote respectively the subgroups of $I_b(F)$ corresponding to $E_0, E_0E_1\cdots E_r, E_0E_1'\cdots E_r'$. We have clearly $I_b(F)\supset\mathfrak{H}\supset\mathfrak{K}\supset\mathfrak{L}\supset P_b(F)$. If $\mathfrak{a}\in I_b(F)$ and $\mathfrak{a}^2\in P_b(F)$, then $\mathfrak{a}^2\in\mathfrak{L}$. As $\mathfrak{H}/\mathfrak{L}$ is the 2-Sylow subgroup of $I_b(F)/\mathfrak{L}$, we obtain $\mathfrak{a}\in\mathfrak{H}$. By our construction of the E_i', we must have $\mathfrak{a}\in\mathfrak{K}$. This shows that $\mathfrak{K}$ contains the ideal-group $\{\mathfrak{a}\in I_b(F)\mid\mathfrak{a}^2\in P_b(F)\}$. It follows that D_b contains the field $E_0E_1\cdots E_r$, the absolute class-field over F.

If either one or both of the groups

$$\{\mathfrak{a}\in I_b(F)\mid\mathfrak{a}\mathfrak{a}^\rho\in P_b(F_0)\}/P_b(F), \qquad I_b(F/F_0)/P_b(F)$$

have odd orders, then $D_b=C_b(F)$.

Lemma 1. *F and F_0 being as in Theorem 1, let h and h_0 be respectively the class-numbers of F and F_0. Then h is a multiple of h_0, and h/h_0 is the order of the group* $\{\mathfrak{a}\in I_1(F)\mid\mathfrak{a}\mathfrak{a}^\rho\in P_1(F_0)\}/P_1(F)$.

Proof. Let K be the absolute class-field over F_0. As the infinite prime spots of F_0 ramify in F, F is not contained in K, so that $[FK:F]=[K:F_0]=h_0$. Our lemma follows easily from this and class-field theory.

We call h/h_0 *the relative class-number of F*. Then we can conclude that, *if the relative class-number of F is odd, D_1 is the absolute class-field over F.* Further we obtain

Proposition 4. *F and F_0 being as in Theorem 1, let h and h_0 be respectively the class-numbers of F and F_0. Suppose that every prime ideal of F ramified in F/F_0 is a principal ideal. Then we have*

1) It would be meaningful to take account of the infinite prime spots of F_0, though we have not used them in the present investigation.

$$I_1(F/F_0) = P_1(F)I_1(F_0), \quad [C_1(F/F_0):F] \geqq h/h_0.$$

Moreover, if h_0 is odd, the composite D_1 of $C_1(F/F_0)$ and $C_1(F_0)$ is the absolute class-field over F.

Proof. The equality $I_1(F/F_0)=P_1(F)I_1(F_0)$ follows easily from our assumption and the relation (3) of § 1. Now the injection of $I_1(F_0)$ into $I_1(F)$ gives a homomorphism of $I_1(F_0)/P_1(F_0)$ onto $I_1(F/F_0)/P_1(F)$; so we have $[I_1(F/F_0): P_1(F)] \leqq h_0$, and hence $[I_1(F): I_1(F/F_0)] \geqq h/h_0$, which implies $[C_1(F/F_0):F] \geqq h/h_0$. If h_0 is odd, the order of the group $I_1(F/F_0)/P_1(F)$ must be odd; as remarked above, this implies $D_1=C_1(F)$.

§ 3. **Class-fields over cyclotomic fields.** Let F be an imaginary cyclotomic field and F_0 the maximal real subfield of F. As F is normal over $\boldsymbol{Q}$, we can apply to F the result of § 2. In particular, we get the following assertion. *If the relative class-number of an imaginary cyclotomic field F is odd, then the absolute class-field over F is generated by the absolute class-field over the maximal real subfield of F and the unramified class-fields over F obtained from the fields of moduli of certain two polarized abelian varieties having subfields of F as endomorphism algebras.* Several criteria for the oddness of relative class-number of imaginary cyclotomic fields are given in [1, Satz 38, 42, 46].

F being still an imaginary cyclotomic field, if $(F; \{\sigma_i\})$ is primitive, the dual of $(F; \{\sigma_i\})$ is $(F; \{\sigma_i^{-1}\})$ in virtue of [3, § 8. 4, (1)]. By (1) and (4) of § 1, we see easily

$$(6) \qquad I_b(F/F_0) \cap I_b(F; \{\sigma_i\}) = I_b(F/F_0) \cap I_b(F; \{\tau_i\})$$

for any two primitive CM-types $(F; \{\sigma_i\})$ and $(F; \{\tau_i\})$. For every automorphism γ of F and for every $(F; \{\sigma_i\})$, we have

$$(7) \qquad I_b(F; \{\sigma_i\}) = I_b(F; \{\gamma\sigma_i\}).$$

Theorem 3. *Let F be an imaginary cyclic extension of $\boldsymbol{Q}$ of degree $2n$ and F_0 the maximal real subfield of F; let σ be a generator of the Galois group of F over $\boldsymbol{Q}$. Then we have*

$$C_b(F; \{1, \sigma, \cdots, \sigma^{n-1}\}) \supset C_b(F/F_0),$$
$$C_b(F; \{1, \sigma, \cdots, \sigma^{n-1}\}) \supset C_b(F; \{\tau_i\})$$

for every positive integer b and for every primitive CM-type $(F; \{\tau_i\})$. Moreover, if every prime ideal of F ramified in F/F_0 is a principal ideal, then, we have, for every CM-type $(F; \{\tau_i\})$,

$$C_1(F; \{1, \sigma, \cdots, \sigma^{n-1}\}) = C_1(F/F_0) \supset C_1(F; \{\tau_i\}).$$

Proof. It is easy to see that $(F; \{1, \sigma, \cdots, \sigma^{n-1}\})$ is a primitive CM-type. By (7), we have $I_b(F; \{1, \sigma, \cdots, \sigma^{n-1}\}) = I_b(F; \{\sigma^n, \sigma, \sigma^2, \cdots, \sigma^{n-1}\})$. Then by Proposition 3, we have $I_b(F; \{1, \sigma, \cdots, \sigma^{n-1}\}) \subset I_b(F/F_0)$. This proves the first inclusion. The second inclusion follows from this and (6). Now assume that every prime ideal of F ramified in F/F_0 is a principal ideal. By Proposition 4, $I_1(F/F_0) = P_1(F) I_1(F_0)$. We can easily verify that $I_1(F_0) \subset I_1(F; \{\tau_i\})$ for every CM-type $(F; \{\tau_i\})$, so that $I_1(F/F_0) \subset I_1(F; \{\tau_i\})$, which implies $C_1(F/F_0) \supset C_1(F; \{\tau_i\})$. Apply this to the case $\{\tau_i\} = \{1, \sigma, \cdots, \sigma^{n-1}\}$. As we have already seen the inverse inclusion, we must have $C_1(F; \{1, \sigma, \cdots, \sigma^{n-1}\}) = C_1(F/F_0)$.

In general, for every positive integer b, we see that

$$I_b(F; \{\sigma_i\}) \supset P_b(F) \cdot \{\mathfrak{a} \in I_b(F_0) \mid N(\mathfrak{a}) \equiv 1 \bmod (b)\} .$$

If $(F; \{\sigma_i\}) = (F; \{1, \sigma, \cdots, \sigma^{n-1}\})$, *the factor group*

$$I_b(F; \{\sigma_i\}) / [P_b(F) \cdot \{\mathfrak{a} \in I_b(F_0) \mid N(\mathfrak{a}) \equiv 1 \bmod (b)\}]$$

is of exponent 1 *or* 2. In fact, in this case, if $\mathfrak{a} \in I_b(F; \{\sigma_i\})$, we have $\mathfrak{a} \in I_b(F/F_0)$ by Theorem 3, so that $\mathfrak{a}/\mathfrak{a}^\rho \in P_b(F)$; on the other hand, it is clear that $N_{F_0/\boldsymbol{Q}}(\mathfrak{a}\mathfrak{a}^\rho) \equiv 1 \bmod (b)$; therefore, we have

$$\mathfrak{a}^2 = (\mathfrak{a}/\mathfrak{a}^\rho)(\mathfrak{a}\mathfrak{a}^\rho) \in P_b(F) \cdot \{\mathfrak{a} \in I_b(F_0) \mid N(\mathfrak{a}) \equiv 1 \bmod (b)\} .$$

Let l^ν be a power of an odd prime number l and ζ a primitive l^ν-th root of unity. Put $F = \boldsymbol{Q}(\zeta)$, $F_0 = \boldsymbol{Q}(\zeta + \zeta^{-1})$. Then F is cyclic over $\boldsymbol{Q}$ and every prime ideal of F ramified in F/F_0 is a principal ideal. Therefore, we can apply Proposition 4 and Theorem 3 to the present case. In particular, *if the class-number of F_0 is odd, then, the field of moduli of a certain polarized abelian variety having F as endomorphism-algebra, together with the absolute class-field over F_0, generates the absolute class-field over F.* By a theorem of Kummer, the class-number of $F = \boldsymbol{Q}(\zeta)$ is odd if and only if the relative class-number of F is odd (cf. [1, Satz 45]). Hence, the class-number of $F_0 = \boldsymbol{Q}(\zeta + \zeta^{-1})$ is odd whenever the relative class-number of F is odd; the table of [1] shows that the relative class-number of $\boldsymbol{Q}(\zeta)$ is odd for $l^\nu < 100$, $l^\nu \neq 29$.

Remark 2. In Theorem 3, it may happen that $C_1(F/F_0) \neq C_1(F; \{\tau_i\})$ for some $\{\tau_i\}$. In fact, let l be a prime number $\geqq 5$ and ζ a primitive l-th root of unity. Choose as τ_i the automorphism of F defined by $\zeta^{\tau_i} = \zeta^i$ for $1 \leqq i \leqq n = (l-1)/2$. As observed in [3, § 8. 4, (1)], $(F; \{\tau_i\})$ is primitive; further by [3, § 15. 4, Example 2)], we have $C_1(F; \{\tau_i\}) = F$, so that $I_1(F; \{\tau_i\}) = I_1(F)$. Therefore, $C_1(F/F_0) \neq C_1(F; \{\tau_i\})$ if the relative class-number of F is greater than 1; the latter is of course the case for many l.

Now if we put $\{\sigma_i\} = \{\tau_1\sigma, \tau_2, \cdots, \tau_n\}$, we must have $I_1(F; \{\sigma_i\}) \subset I_1(F/F_0)$ in view of Proposition 3. We can prove that this CM-type $(F; \{\sigma_i\})$ is primitive. In fact, if $l \neq 17$, the trick of [3, § 8. 4, (1)] is applicable; and if $l=17$, this is shown by means of [3, § 8. 2, Prop. 26]. Then, by Theorem 3 and by what we have just proved, we get $I_1(F; \{\sigma_i\}) = I_1(F/F_0)$, which implies $C_1(F; \{\sigma_i\}) = C_1(F/F_0)$. In general, it is not necessarily true that there exists an automorphism γ of F such that $\{\gamma\sigma_i\} = \{1, \sigma, \cdots, \sigma^{n-1}\}$.

§ 4. A CM-type obtained from two CM-types. The argument of § 2 is powerless when F has no CM-type satisfying (A). In order to treat such a case, we consider a special kind of CM-type. We begin with an easy

Lemma 2. *Let F be a totally imaginary quadratic extension of a totally real algebraic number field F_0. Let L be the smallest normal extension of $\boldsymbol{Q}$ containing F, and G the Galois group of L over $\boldsymbol{Q}$. Then ρ, considered as an element of G, belongs to the center of G; and L is a totally imaginary quadratic extension of a totally real subfield.*

Proof. We can find an element z of F such that $F=F_0(z)$ and z^2 is a totally negative element of F_0. For every $\gamma \in G$, $(z^\gamma)^2$ is a totally negative element of F_0, so that $z^{\gamma\rho} = -z^\gamma = (-z)^\gamma = z^{\rho\gamma}$. Further, for every $x \in F_0$, we have $x^{\gamma\rho} = x^\gamma = x^{\rho\gamma}$. Therefore, for every $\gamma, \delta \in G$ and for every $x \in F_0$, we have $(x^\delta)^{\gamma\rho} = (x^\delta)^{\rho\gamma}$, $(z^\delta)^{\gamma\rho} = (z^\rho)^{\delta\gamma} = (z^\delta)^{\rho\gamma}$. These relations imply $y^{\gamma\rho} = y^{\rho\gamma}$ for every $y \in L$, since L is generated by F_0^δ and z^δ; this proves the first assertion. If we denote by L_0 the set of elements y of L such that $y^\rho = y$, we have $(y^\gamma)^\rho = y^{\rho\gamma} = y^\gamma$ for every $y \in L_0$. It follows that L_0 is totally real; this proves the last assertion.

Let F_0 be a totally real algebraic number field of degree $n > 1$. Let F and M be totally imaginary quadratic extensions of F_0. We assume $F \neq M$. Let K be the composite of F and M. Obviously, K contains a totally real algebraic number field K_0 such that $[K_0 : F_0] = 2$. Let $(F; \{\sigma_i\})$ and $(M; \{\tau_i\})$ be CM-types. We assume $\sigma_i = \tau_i$ on F_0. This is not an essential restriction, since for any $\{\sigma_i\}$ and $\{\tau_i\}$, we can reorder them so that $\sigma_i = \tau_i$ on F_0.

Now fix an integer r such that $1 \leqq r \leqq n$, and define $2n$ isomorphisms $\alpha_1, \beta_1, \cdots, \alpha_n, \beta_n$ of K into $\boldsymbol{C}$ by

$$
(8) \qquad \begin{cases} \alpha_i = \sigma_i \text{ on } F, \ \alpha_i = \tau_i \text{ on } M \text{ for } 1 \leqq i \leqq n, \\ \beta_j = \sigma_j \text{ on } F, \ \beta_j = \tau_j\rho \text{ on } M \text{ for } 1 \leqq j \leqq r, \\ \beta_k = \sigma_k\rho \text{ on } F, \ \beta_k = \tau_k \text{ on } M \text{ for } r < k \leqq n. \end{cases}
$$

It can be easily seen that $(K; \{\alpha_1, \beta_1, \cdots, \alpha_n, \beta_n\})$ is a CM-type. We assume henceforth that σ_1 is the identity mapping of F and τ_1 is the identity mapping of M, and consider only the case $r=1$.

Let $(M^*; \{\chi_\mu\})$ be the dual of $(M; \{\tau_i\})$; let M^{**} be the field generated over $\boldsymbol{Q}$ by the elements $\sum_{\nu=2}^{n} x^{\tau_\nu}$ for $x \in M$. By Proposition 2, M^* is generated over $\boldsymbol{Q}$ by the elements $\sum_{i=1}^{n} x^{\tau_i}$ for $x \in M$. It follows that

$$M^*M = M^{**}M. \tag{9}$$

Proposition 5. *Let $(K^*; \{\varphi_\lambda\})$ be the dual of $(K; \{\alpha_i, \beta_i\})$. Then we have $K^*=FM^{**}$.*

Proof. Put $g(y)=\sum_{i=1}^{n}(y^{\alpha_i}+y^{\beta_i})$ for $y \in K$. By Proposition 2, K^* is generated over $\boldsymbol{Q}$ by the elements $g(y)$ for $y \in K$. For any $y \in K$, we see easily $y^{\alpha_1}+y^{\beta_1}=\mathrm{Tr}_{K/F}(y)$, $y^{\alpha_\nu}+y^{\beta_\nu}=\mathrm{Tr}_{K/M}(y)^{\tau_\nu}$ for $\nu>1$, so that

$$g(y) = \mathrm{Tr}_{K/F}(y)+\sum_{\nu=2}^{n}\mathrm{Tr}_{K/M}(y)^{\tau_\nu}.$$

This implies $K^* \subset FM^{**}$. Now take elements z and w so that $F=F_0(z)$, $M=F_0(w)$, $z^2 \in F_0$, $w^2 \in F_0$. If $x \in F_0$, we have

$$g(x) = 2\mathrm{Tr}_{F_0/\boldsymbol{Q}}(x),\ g(xz) = 2xz,\ g(xw) = 2\sum_{\nu=2}^{n}(xw)^{\tau_\nu}.$$

These relations show that K^* contains F and M^{**}; this completes the proof.

Proposition 6. *M^{**} is a totally imaginary quadratic extension of a totally real algebraic number field containing F_0. Moreover, for every $x \in M$, we have $x^{\tau_2} \cdots x^{\tau_n} \in M^{**}$; and for every ideal $\mathfrak{c}$ of M, $\mathfrak{c}^{\tau_2} \cdots \mathfrak{c}^{\tau_n}$ is an ideal of M^{**}.*

Proof. If $x \in F_0$, we have $x=\mathrm{Tr}_{F_0/\boldsymbol{Q}}(x)-\sum_{\nu=2}^{n} x^{\tau_\nu} \in M^{**}$, so that $F_0 \subset M^{**}$. Now let L be the smallest normal extension of $\boldsymbol{Q}$ containing K and G the Galois group of L over $\boldsymbol{Q}$. Denote by H the set of elements $\gamma \in G$ such that $\{\tau_2\gamma, \cdots, \tau_n\gamma\}$ coincides with $\{\tau_2, \cdots, \tau_n\}$ on M as a whole. Then, by the same argument as in the proof of [3, § 8. 3, Prop. 28], we can prove that $H=\{\gamma \in G \mid x^\gamma = x$ for every $x \in M^{**}\}$. Using this fact, the second and last assertions are proved in the same manner as in the proof of [3, § 8. 3, Prop. 28]. Now, by the definition of CM-type, $\tau_2\rho$ does not coincide with any τ_ν on M; so ρ is not contained in H. If we put $H_1=H \cup H\rho$, then H_1 is a subgroup of G on account of Lemma 2. Call M_1 the subfield of L corresponding to H_1 by Galois theory. Then we see easily that M_1 is totally real and M^{**} is a totally imaginary quadratic extension of M_1.

Consider a particular case where F_0 is normal over $\boldsymbol{Q}$. Take an element w of M so that $M=F_0(w)$. Choose $n-1$ elements $x_2, \cdots, x_n$ of F_0 in such a way that $\det(x_\mu^{\tau_\nu})_{\mu,\nu=2,\cdots,n} \neq 0$. We have $\sum_{\nu=2}^{n} x_\mu^{\tau_\nu} w^{\tau_\nu} \in M^{**}$ for every μ, so that $w^{\tau_2}, \cdots, w^{\tau_n}$ are contained in M^{**} since $F_0 \subset M^{**}$. This shows that M^{**} is the composite of $M^{\tau_2}, \cdots, M^{\tau_n}$. We can similarly prove that F_0M^* is the composite of $M^{\tau_1}, \cdots, M^{\tau_n}$. Now assume that the composite of $M^{\tau_1}, M^{\tau_2}, \cdots, M^{\tau_n}$ is of degree 2^n over F_0. This is the case for example, if there exists a prime ideal $\mathfrak{p}$ of F_0 of absolute degree 1 such that $\mathfrak{p}$ is inertial in M while the conjugates of $\mathfrak{p}$, other than $\mathfrak{p}$ itself, decompose in M. Then, we have $[F_0M^* : \boldsymbol{Q}] = 2^n \cdot n$, and hence $[M^* : \boldsymbol{Q}] \geqq 2^n$.[2] This gives an example of CM-type $(M : \{\tau_i\})$ such that $[M^* : \boldsymbol{Q}] > [M : \boldsymbol{Q}]$ for the dual $(M^* ; \{\chi_\mu\})$ of $(M ; \{\tau_i\})$. This shows also that the case $[K^* : F] > 2$ may happen.

Coming back to the general case, we get

Proposition 7. *Three CM-types $(F ; \{\sigma_i\})$, $(M ; \{\tau_i\})$, $(K ; \{\alpha_i, \beta_i\})$ being as above, let $(K^* ; \{\varphi_\lambda\})$ be the dual of $(K ; \{\alpha_i, \beta_i\})$. Then, for every positive integer b, the composite of $C_b(M)$ and $C_b(K ; \{\alpha_i, \beta_i\})$ contains the class-field H_b over F corresponding to the ideal group $I_b(F) \cap P_b(K^*)$.*

Proof. Let $\mathfrak{a}$ be an ideal of K. In the same way as in the proof of Proposition 5, we see that

$$\mathfrak{a}^{\alpha_1}\mathfrak{a}^{\beta_1} \cdots \mathfrak{a}^{\alpha_n}\mathfrak{a}^{\beta_n} = N_{K/F}(\mathfrak{a}) \prod_{\nu=2}^{n} N_{K/M}(\mathfrak{a})^{\tau_\nu} . \tag{10}$$

The composite of $C_b(M)$ and $C_b(K ; \{\alpha_i, \beta_i\})$ is a class-field over K; denote by $\mathfrak{H}$ the corresponding ideal-group. If $\mathfrak{a} \in \mathfrak{H}$, we have $\mathfrak{a}^{\alpha_1}\mathfrak{a}^{\beta_1} \cdots \mathfrak{a}^{\alpha_n}\mathfrak{a}^{\beta_n} \in P_b(K^*)$ and $N_{K/M}(\mathfrak{a}) \in P_b(M)$; so we see that $\prod_{\nu=2}^{n} N_{K/M}(\mathfrak{a})^{\tau_\nu} \in P_b(M^{**})$ in view of Proposition 6. By (10) and by Proposition 5, we have $N_{K/F}(\mathfrak{a}) \in P_b(K^*)$. This shows that $\mathfrak{H}$ is contained in the ideal-group corresponding to the composite of K and H_b; our proposition is thereby proved.

If we put $m=[K^* : F]$, we see easily that

$$P_b(F) \subset I_b(F) \cap P_b(K^*) \subset \{\mathfrak{a} \in I_b(F) \mid \mathfrak{a}^m \in P_b(F)\} .$$

Therefore, the exponent of the Galois group of $C_b(F)/H_b$ is a divisor of $m=[K^* : F]$.

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2) In reality we can show that $[M^* : \boldsymbol{Q}] = 2^n$.

If we fix M (and hence F_0) and consider M and $C_b(M)$ auxiliary, Proposition 7 may be regarded as a statement concerning the class-fields over the variant field F, which can be obtained by complex multiplication of abelian varieties having a certain overfield K^* of F as endomorphism-algebra.[3] In order to get a more transparent result, we consider a restrictive case.

Proposition 8. *$(M: \{\tau_i\})$ satisfies the condition (A) if and only if $M \supset M^{**}$; and if this is satisfied, we have $M = M^{**}$, $K = K^*$.*

Proof. The first assertion is a direct consequence of (9). If $M \supset M^{**}$, we must have $M = M^{**}$ on account of Proposition 6, so that $K^* = FM = K$ by Proposition 5.

In particular, if M is normal over $\boldsymbol{Q}$, then $(K; \{\alpha_i, \beta_i\})$ satisfies (A); in this case, K is normal over $\boldsymbol{Q}$ if and only if F is normal over $\boldsymbol{Q}$; and if we take as F a non-abelian extension of $\boldsymbol{Q}$ of degree 4, we see easily that $(K; \{\alpha_i, \beta_i\})$ is primitive.

For any totally real algebraic number field F_0, we can find a CM-type $(M; \{\tau_i\})$ such that $[M: F_0] = 2$ and $M \supset M^*$. In fact, for any positive integer s, put $M = F_0(\sqrt{-s})$ and define $\tau_1, \cdots, \tau_n$ so that $(\sqrt{-s})^{\tau_\nu} = \sqrt{-s}$. Then it is easy to see $M^* = \boldsymbol{Q}(\sqrt{-s})$.

Theorem 4. *Let F_0 be a totally real algebraic number field of degree $n > 1$. Let F and M be distinct totally imaginary quadratic extensions of F_0, and K the composite of F and M. Let $(F; \{\sigma_i\})$ and $(M; \{\tau_i\})$ be CM-types such that σ_1 is the identity on F, τ_1 is the identity on M, and $\sigma_i = \tau_i$ on F_0. Define a CM-type $(K; \{\alpha_i, \beta_i\})$ by the relation (8) with $r = 1$. Suppose that $(M; \{\tau_i\})$ satisfies the condition (A) of § 2. Then, for every positive integer b, $C_b(K; \{\alpha_i, \beta_i\})$ contains the class-field $C_b(F/F_0)$ over F.*

Proof. For every ideal $\mathfrak{a}$ of K, the equality (10) is also written in the form

$$\text{(11)} \qquad \mathfrak{a}^{\alpha_1}\mathfrak{a}^{\beta_1} \cdots \mathfrak{a}^{\alpha_n}\mathfrak{a}^{\beta_n} = N_{K/F}(\mathfrak{a}) N_{K/M}(\mathfrak{a})^{-1} \prod_{i=1}^{n} N_{K/M}(\mathfrak{a})^{\tau_i}.$$

By our assumption and Proposition 8, we have $K = K^*$. Hence, if $\mathfrak{a} \in I_b(K; \{\alpha_i, \beta_i\})$, there exists an element u of K such that

$$\mathfrak{a}^{\alpha_1}\mathfrak{a}^{\beta_1} \cdots \mathfrak{a}^{\alpha_n}\mathfrak{a}^{\beta_n} = (u), \quad N(\mathfrak{a}) = uu^{\rho}, \quad u \equiv 1 \bmod (b).$$

3) In fact, the abelian varieties belonging to $(K; \{\alpha_i, \beta_i\})$ are special members of an analytic family of polarized abelian varieties whose moduli are given by certain automorphic functions of one variable.

Put $v=N_{K/F}(u)$, $\mathfrak{c}=N_{K/F}(\mathfrak{a})$. Now take $N_{K/F}$ of the both sides of (11). We note that for any ideal $\mathfrak{e}$ of M, $N_{K/F}(\mathfrak{e})=N_{M/F_0}(\mathfrak{e})$; especially,

$$N_{K/F}(N_{K/M}(\mathfrak{a})) = N_{M/F_0}(N_{K/M}(\mathfrak{a})) = N_{K/F_0}(\mathfrak{a}) = N_{F/F_0}(\mathfrak{c}) = \mathfrak{c}\mathfrak{c}^{\rho},$$

and

$$N_{K/F}\left(\prod_{i=1}^{n} N_{K/M}(\mathfrak{a})^{\tau_i}\right) = N_{M/F_0}\left(\prod_{i=1}^{n} N_{K/M}(\mathfrak{a})^{\tau_i}\right) = \prod_{i=1}^{n} N_{K/M}(\mathfrak{a})^{\tau_i} N_{K/M}(\mathfrak{a})^{\tau_i\rho}$$
$$= N_{M/Q}(N_{K/M}(\mathfrak{a})) = N_{K/Q}(\mathfrak{a}) = (uu^{\rho}).$$

Therefore, we obtain from (11), $(v)=\mathfrak{c}^2(\mathfrak{c}\mathfrak{c}^{\rho})^{-1}(uu^{\rho})$. Put $w=v(uu^{\rho})^{-1}$. Then w is an element of F, and $\mathfrak{c}/\mathfrak{c}^{\rho}=(w)$, $ww^{\rho}=1$, $w\equiv 1 \bmod (b)$. Thus we have shown $N_{K/F}[I_b(K; \{\alpha_i, \beta_i\})]\subset I_b(F/F_0)$. This proves our theorem.

By means of Theorem 4, we obtain several assertions concerning the class-fields over F similar to those given in § 2. In particular, the absolute class-field $C_1(F)$ is contained in the composite $C_b(K; \{\alpha_i, \beta_i\})$ and $C_b(F_0)$ for a suitable positive integer b. As another example of specializations (or degenerations) of Theorem 4, we get the following conclusion: *F and F_0 being as in Theorem 4, let s be a positive integer such that $\sqrt{-s}\notin F$. Then the field of moduli of a certain polarized abelian variety having $F(\sqrt{-s})$ as endomorphism-algebra, together with the absolute class-field over F_0, generates a class-field over $F(\sqrt{-s})$ containing the absolute class-field over F, if the class-number of F is odd.*

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