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On the Geometry of Hopf Manifolds

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§1. Introduction

The purpose of the present note is to compute the cohomology groups $H^q(X, \Omega^p(F))$, $(0 \leq q \leq n)$ of an n-dimensional Hopf manifold X, where $\Omega^p(F)$ denotes the analytic sheaf of germs of holomorphic p-forms with values in a complex line bundle F over X. Throughout the arguments we make use of the fact that the Hopf manifold is a homogeneous compact complex manifold. Recently, R. Bott [3] and the author [7] have made some researches concerning the complex line bundles over a class of homogeneous compact complex manifolds ($=C$-manifolds in the sense of Wang). The essential difference between Hopf manifolds and Wang's C-manifolds lies in the non-triviality of the fundamental group of the former. But a Hopf manifold admits the so-called *Hopf fibering*, which plays the quite analogous role to the fundamental fiberings of non-kählerian C-manifolds (cf. [7]), and which allows us to have the similar results for them. Our principal tools are Leray's spectral sequences and the knowledge of the cohomology groups $H^q(P^n, \Omega^p(F))$ of the n-dimensional complex projective space P^n, which have been computed by Bott [3] and Matsumura [10].

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§2. Hopf manifolds

We recall here the definition of Hopf manifolds (cf. [6]). Let C^n denote a complex n-dimensional euclidean space and W^n the complement of the origin $o=(0, \cdots, 0) \in C^n$, and take a non-zero complex number d with the absolute value $|d| \neq 1$. Let Δ_d be the cyclic group generated by d in the multiplicative group C^* of all non-zero complex numbers. We denote also by Δ_d, for brevity, the subgroup of $GL(n, C)$ generated by the scaler matrix $d \cdot I$ (I is the unit matrix of $GL(n, C)$):

$$\Delta_d = \{d^m \cdot I \mid m \in Z\}$$

(Z is the ring of all integers)

The group Δ_d being a properly discontinuous group of W^n without fixed points, the quotient manifold $W^n/\Delta_d = X$ has a natural complex analytic structure. The complex manifold X thus obtained is, by definition, an *n-dimensional Hopf manifold* (corresponding to the number d)[1]. As is well known, X is diffeomorphic to the product manifold $S^1 \times S^{2n-1}$ of two odd dimensional spheres, since the group Δ_d is isomorphic to Z. The complex general linear group $GL(n, C)$ operates on W^n effectively and transitively; as Δ_d is contained in the centre of $GL(n, C)$, $GL(n, C)$ operates on X also transitively and holomorphically. Now we take the basic point $w_0 = (1, 0, \cdots, 0) \in W^n$ and the corresponding point $x_0 \in X$ modulo Δ_d. Then the isotropy subgroup U_d of $GL(n, C)$ at x_0 consists of the matrices of the form:

$$u = \left(\begin{array}{c|c} d^m & * \cdots\cdots * \\ \hline 0 & \\ \vdots & * \\ 0 & \end{array}\right), \qquad d^m \in \Delta_d .$$

So we can identify X with the complex coset space:

$$X = GL(n, C)/U_d .$$

Note that the action of $GL(n, C)$ is not effective; if we take the quotient groups $\tilde{G} = GL(n, C)/\Delta_d$ and $\tilde{U}_d = U_d/\Delta_d$, then $\tilde{G}$ acts on X effectively and we can put

$$X = \tilde{G}/\tilde{U}_d .$$

The connected complex reductive Lie group $\tilde{G}$ is considered as a Lie subgroup of the connected analytic automorphism group $A(X)$ of X. As a matter of fact, we show the following

Theorem 1. $A(X) = \tilde{G}$.

The proof is quite similar to the case of non-kählerian C-manifolds ([7], Proposition 6), but we state it here for the completeness sake and for the convenience in the later discussions. First we need some definitions:

DEFINITION 1. Let $G(1, n-1; C)$ denote the subgroup of $GL(n, C)$ consisting of the matrices:

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1) The 1-dimensional Hopf manifold in this definition is nothing but an elliptic curve T^1. We shall exclude this trivial case in the sequel; therefore assume $n \geqq 2$.

$$\left(\begin{array}{c|c} * & 0 \cdots\cdots 0 \\ \hline * & \\ \vdots & * \\ * & \end{array}\right).$$

Then we can identify $GL(n, C)/GL(1, n-1; C)$ with the $(n-1)$-dimensional complex projective space P^{n-1} and $GL(1, n-1; C)/U_d$ with an elliptic curve T^1 respectively. Therefore we have a natural (holomorphic) principal fibering of X with P^{n-1} as base, T^1 as group and the natural mapping ϕ of X onto P^{n-1} as projection; this fibering $X(P^{n-1}, T^1, \phi)$ is called the *Hopf fibering* of X.

DEFINITION 2. Let $X=G/U$ be a complex homogeneous space with a connected complex Lie group G and a (not nec. connected) closed complex Lie subgroup U, and let ρ be a holomorphic homomorphism of U into another complex Lie group B. Then the coset bundle $G(X, U, \pi)$ (π is the canonical projection of G onto X) and ρ induce a new holomorphic principal bundle $P(X, B, \varpi)$ over X, which is called a *homogeneous B-bundle over X with respect to the Klein form G/U.* In particular, when $B=GL(m, C)$, we have the associated m-dimensional vector bundle $E(X, C^m, GL(m, C), \varpi)$ which is simply written as $E_X(\rho, C^m)$ and which is called a *homogeneous vector bundle over X* (*with respect to the Klein form* G/U). This is the quotient space of $G\times C^m$ by the equivalence relation:

$$(g, \xi) \sim (gu, \rho(u^{-1})\xi)$$

for $g\in G$, $u\in U$ and $\xi\in C^m$.

Now we denote the Lie algebras of $GL(n, C)$, $GL(1, n-1; C)$ and U_d by $\mathfrak{g}$, $\hat{\mathfrak{u}}$ and $\mathfrak{u}$ respectively. Then $\mathfrak{u}$ is an ideal of $\hat{\mathfrak{u}}$ and the exact sequence of modules:

$$(1) \qquad 0 \to \hat{\mathfrak{u}}/\mathfrak{u} \to \mathfrak{g}/\mathfrak{u} \to \mathfrak{g}/\hat{\mathfrak{u}} \to 0$$

is considered as an exact sequence of U_d-modules under the adjoint actions. Therefore we can construct the exact sequence of homogeneous vector bundles over X with respect to the Klein form $GL(n, C)/U_d$:

$$0 \to E_X(Ad, \hat{\mathfrak{u}}/\mathfrak{u}) \to E_X(Ad, \mathfrak{g}/\mathfrak{u}) \to E_X(Ad, \mathfrak{g}/\hat{\mathfrak{u}}) \to 0\,.$$

If we denote here by Θ and $\hat{\Theta}$ the tangential vector bundles over X and $\hat{X}=P^{n-1}$ respectively, then, as is easily verified,

$$E_X(Ad, \mathfrak{g}/\mathfrak{u}) = \Theta\,, \quad E_X(Ad, \mathfrak{g}/\hat{\mathfrak{u}}) = \phi^*\hat{\Theta}$$

and $E_X(Ad, \hat{\mathfrak{u}}/\mathfrak{u})$ is the trivial line bundle (which is denoted by I).

Therefore we have the exact sequence:

$$(2) \qquad 0 \to I \to \Theta \to \phi^*\hat{\Theta} \to 0 .$$

This induces the corresponding exact sequence of the complex vector spaces of cross sections:

$$0 \to \Gamma_X(I) \to \Gamma_X(\Theta) \to \Gamma_X(\phi^*\hat{\Theta}) .$$

Here, $\Gamma_X(\Theta)$ may be identified with the Lie algebra $\mathfrak{a}(X)$ of all holomorphic vector fields on X, and $\Gamma_X(I)$ is then identified with the 1-dimensional ideal C of $\mathfrak{a}(X)$. Moreover $\Gamma_X(\phi^*\hat{\Theta})$ is isomorphic with $\Gamma_{\hat{X}}(\hat{\Theta})$ or with the Lie algebra $\mathfrak{a}(P^{n-1})$ of all holomorphic vector fields on P^{n-1}, since the fibres of $\phi: X \to P^{n-1}$ are compact, connected. Hence, we obtain the exact sequence of Lie algebras:

$$0 \to C \to \mathfrak{a}(X) \xrightarrow{\dot{\phi}} \mathfrak{a}(P^{n-1}) ,$$

where the homomorphism $\dot{\phi}$ means that every holomorphic vector field on X is constant on each fibre and that it induces a vector field over P^{n-1} (This fact has been recognized by Blanchard [2] in general case; see [2], Proposition 1.1). Now we know that $\mathfrak{a}(P^{n-1})$ is isomorphic to the Lie algebra $\mathfrak{sl}(n, C)$ of $SL(n, C)$ and that $\mathfrak{a}(X)$ contains the Lie algebra $\tilde{\mathfrak{g}}$ of $\tilde{G}$ which is clearly isomorphic to that of $GL(n, C)$. Therefore $\dot{\phi}$ is surjective and $\mathfrak{a}(X) = \tilde{\mathfrak{g}}$. Hence $A(X) = \tilde{G}$. This proves Theorem 1.

Here we add the following theorem which can be proved in the same way as in [7], Proposition 7 (see, also Remark 2 in §6).

Theorem 2. $\dim H^1(X, \Theta)) = n^2$ *and* $H^q(X, \Theta) = \{0\}$ *for* $q \geqq 2$.

The first result is known by Kodaira-Spencer [8] in the case $n=2$. The discussions in [8], §15 about deformations of complex analytic structures of a 2-dimensional Hopf manifold might be immediately extended to the case $n > 2$.

§3. Complex line bundles

In this section we shall concern ourselves with the homogeneous line bundles over a Hopf manifold X. For this sake, it is more appropriate to take the Klein form $GL(n, C)/U_d$ rather than to take G/U_d. We call, from now on, a homogeneous bundle with respect to the Klein form $GL(n, C)/U_d$ simply a *homogeneous bundle*. Then we have

Theorem 3. *Every complex line bundle over a Hopf manifold X is* *homogeneous and has an integrable holomorphic connection. Moreover the group $H^1(X, \boldsymbol{C}^*)$ of all complex line bundles over X is isomorphic to C^*.*

Proof. First consider the sheaf exact sequence over X:

$$0 \to Z \to \boldsymbol{C} \xrightarrow{\mathcal{E}} \boldsymbol{C}^* \to 0\,, \tag{1}$$

where $\boldsymbol{C}$ (resp. $\boldsymbol{C}^*$)[2] denotes the sheaf of germs of holomorphic (resp. non-vanishing holomorphic) functions on X, Z the constant sheaf of integers and $\mathcal{E}$ the homomorphism induced from the homomorphism $\mathcal{E}$ of C onto C^* defined by $\mathcal{E}(\xi)=\exp 2\pi\sqrt{-1}\,\xi$ for any $\xi\in C$. Because $H^2(X, Z)=\{0\}$, we have from (1) the following exact sequence:

$$0 \to H^1(X, Z) \to H^1(X, \boldsymbol{C}) \xrightarrow{\mathcal{E}} H^1(X, \boldsymbol{C}^*) \to 0\,.$$

On the other hand, we consider the exact sequence of the abelian groups:

$$0 \to \mathrm{Hom}\,(U_d, Z) \to \mathrm{Hom}\,(U_d, C) \xrightarrow{\mathcal{E}} \mathrm{Hom}\,(U_d, C^*)\,, \tag{2}$$

where $\mathrm{Hom}\,(U_d, B)$ means the abelian group of all holomorphic homomorphisms of U_d into the complex abelian Lie group B (If B is discrete, then holomorphic homomorphims should be understood as abstract ones). We see readily that $\mathrm{Hom}\,(U_d, B)\cong\mathrm{Hom}\,(\Delta_d\times GL(n-1, C), B)\cong\mathrm{Hom}\,(\Delta_d\times C^*, B)\cong\mathrm{Hom}\,(\Delta_d, B)\times\mathrm{Hom}\,(C^*, B)$; hence we have

$$\begin{cases}\mathrm{Hom}\,(U_d, Z) \cong \mathrm{Hom}\,(\Delta_d, Z) \cong Z\,,\\ \mathrm{Hom}\,(U_d, C) \cong \mathrm{Hom}\,(\Delta_d, C) \cong C\,,\\ \mathrm{Hom}\,(U_d, C^*) \cong \mathrm{Hom}\,(\Delta_d, C^*)\times\mathrm{Hom}\,(C^*, C^*) \cong C^*\times Z\,.\end{cases} \tag{3}$$

Now there is a natural homomorphism η_B of the group $\mathrm{Hom}\,(U_d, B)$ into the group $H^1(X, \boldsymbol{B})$ of all holomorphic B-bundles over X by assigning to every $\rho\in\mathrm{Hom}\,(U_d, B)$ the corresponding homogeneous B-bundle over X defined by ρ. Then we have the commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Hom}\,(U_d, Z) \arrow[r] \arrow[d, "\eta_Z"] & \mathrm{Hom}\,(U_d, C) \arrow[r] \arrow[d, "\eta_C"] & \mathrm{Hom}\,(U_d, C^*) \arrow[d, "\eta_{C^*}"] & \\ 0 \arrow[r] & H^1(X, Z) \arrow[r] & H^1(X, \boldsymbol{C}) \arrow[r] & H^1(X, \boldsymbol{C}^*) \arrow[r] & 0\,. \end{tikzcd} \tag{4}$$

On the other hand, the universal covering manifold $\tilde{X}$ of X is given by

2) Hereafter, for a given complex Lie group B, we denote by $\boldsymbol{B}$ the sheaf (of group) of holomorphic mappings of a certain complex manifold X into B. Similary, for a given complex analytic vector bundle E over X, we denote by $\boldsymbol{E}$ the sheaf of germs of holomorphic sections of E.

$\tilde{X}=GL(n, C)/U_1$ where U_1 is the subgroup of $GL(n, C)$ consisting of matrices of the form:

$$\left(\begin{array}{c|c} 1 & * \cdots\cdots * \\ \hline 0 & \\ \vdots & * \\ 0 & \end{array}\right)$$

and $U_d/U_1 \simeq \Delta_d$ is the covering transformation group of the covering $\psi: \tilde{X} \to X$. The bundle $\tilde{X}(X, \Delta_d, \psi)$ is obtained from the coset bundle $GL(n, C)(X, U_d, \pi)$ by the natural bundle homomorphism $\tau: GL(n, C) \to GL(n, C)/U_1 = \tilde{X}$, and is therefore the homogeneous Δ_d-bundle defined by the natural homomorphism $\tau: U_d \to U_d/U_1 = \Delta_d$. Now we say a *holomorphic B-bundle over X is defined by an (abstract) representation of the fundamental group* if it is induced from the bundle $\tilde{X}(X, \Delta_d, \psi)$ by a group homomorphism of Δ_d into B, and the homomorphism of $\mathrm{Hom}(\Delta_d, B)$ into $H^1(X, \boldsymbol{B})$ which is obtained in this procedure is denoted by ζ_B.

We know that a holomorphic B-bundle has an integrable holomorphic connection if and only if it is defined by a representation of the fundamental group [1]. On the other hand, the homomorphism $U_d \to \Delta_d$ induces naturally a homomorphism τ_B of $\mathrm{Hom}(\Delta_d, B)$ into $\mathrm{Hom}(U_d, B)$ and the following diagram is commuative.

$$\begin{tikzcd} \mathrm{Hom}(U_d, B) \arrow[dr, "\eta_B"'] & & \mathrm{Hom}(\Delta_d, B) \arrow[ll, "\tau_B"'] \arrow[dl, "\zeta_B"] \\ & H^1(X, \boldsymbol{B}) & \end{tikzcd}$$

Returning to the diagram (4), if we show that η_C is bijective, the theorem will be proved. In fact, in this case, it is obvious that η_{C^*} is surjective and this means that every line bundle is homogeneous. Moreover, $H^1(X, \boldsymbol{C}^*) \simeq \mathrm{Hom}(U_d, C)/\mathrm{Hom}(U_d, Z) \simeq C/Z \simeq C^*$, and the first isomorphism implies that $\zeta_{C^*}: \mathrm{Hom}(\Delta_d, C^*) \to H^1(X, \boldsymbol{C}^*)$ is bijective. Therefore, every line bundle is defined by a representation of the fundamental group Δ_d of X. Now we show first that η_C is injective. In fact, if we take the element $\rho_0 \in \mathrm{Hom}(\Delta_d, Z)$ which is defined by $\rho_0(d^m)=m$ for every $m \in Z$, then the bundle $\zeta_Z(\rho_0)$ is isomorphic with the bundle $\tilde{X}(X, \Delta_d, \psi)$ and so is not trivial, which implies that η_Z is injective. Then η_C being a linear homomorphism, η_C must be also injective. Next we shall prove that $H^1(X, \boldsymbol{C}) \simeq C$. For this sake we employ the spectral sequence associated to the Hopf fibering $X(P^{n-1}, T^1, \phi)$ and the sheaf C. That is to say, there exists a spectral sequence $\{E_k\}$ with $E_2^{r,s}=H^r(P^{n-1}, \phi^s(\boldsymbol{C}))$ and with the final term E_∞^q associated to $H^q(X, \boldsymbol{C})$, where $\phi^s(\boldsymbol{C})$

is the sheaf defined by the presheaf $\phi^s(\boldsymbol{C})_N = H^s(\phi^{-1}(N), \boldsymbol{C})$ (for every open set $N \subset P^{n-1}$). In our discussion, it needs only the case $q=1$; so we are concerned only with $E_2^1 = E_2^{1,0} + E_2^{0,1}$ and with $\phi^s(\boldsymbol{C})$ $(s=0.1)$. If we choose N as a Stein open set on which the bundle $\phi^{-1}(N)$ is trivial, then $\phi^{-1}(N) = N \times T^1$ and so by the Künneth relation we have:

$$H^0(\phi^{-1}(N), \boldsymbol{C}) \cong H^0(N, \boldsymbol{C})$$
$$H^1(\phi^{-1}(N), \boldsymbol{C}) \cong H^1(N, \boldsymbol{C}) \otimes H^0(T^1, \boldsymbol{C}) + H^0(N, \boldsymbol{C}) \otimes H^1(T^1, \boldsymbol{C}).$$

Because $H^1(N, \boldsymbol{C}) = \{0\}$, $H^s(T^1, \boldsymbol{C}) \cong C(s=0, 1)$, it follows that $\phi^0(\boldsymbol{C}) = \boldsymbol{C}$, $\phi^1(\boldsymbol{C}) = \boldsymbol{C}$. Therefore $E_2^{1,0} = H^1(P^{n-1}, \boldsymbol{C}) = \{0\}$, $E_2^{0,1} = H^0(P^{n-1}, \boldsymbol{C}) \cong C$. While, the d_2-differential operator sends $E_2^{0,1}$ into $E_2^{2,0} = H^2(P^{n-1}, \boldsymbol{C}) = \{0\}$. This implies that $E_2^{0,1} = E_3^{0,1} = E_\infty^{0,1} = E_\infty^1$, and consequently that $H^1(X, \boldsymbol{C}) \cong C$. The proof is now completed.

§4. The cohomology groups $H^q(X, \Omega^p(F))$

By Theorem 3 we can write every complex line bundle over a Hopf manifold X as F_λ; $\lambda \in C^* = \mathrm{Hom}(\Delta_d, C^*)$. Our next step is to compute the cohomology groups $H^q(X, \Omega^p(F_\lambda))$, $(0 \leqq q \leqq n)$ with coefficients in the analytic sheaf $\Omega^p(F_\lambda)$ $(0 \leqq p \leqq n)$ of germs of holomorphic p-forms with values in F_λ.

To state our results of computations, we remark first the following situation. As to the Hopf fibering $X(P^{n-1}, T^1, \phi)$, we have the following commutative diagram:

$$\begin{array}{ccc}
\mathrm{Hom}(U_d, C^*) & \xrightarrow{\eta_{C^*}} H^1(X, \boldsymbol{C}^*) \to 0 \\
\uparrow \sigma & \nwarrow \phi^* \\
\mathrm{Hom}(GL(1, n-1; C), C^*) & \xrightarrow{\eta} H^1(P^{n-1}, \boldsymbol{C}^*) \to 0,
\end{array}$$

where σ is the restriction mapping of homomorphisms and η is the assignment of the defining homogeneous line bundle to each homomorphism belonging to $\mathrm{Hom}(GL(1, n-1; C), C^*)$ (cf. [7], §4). Now by the proof of Theorem 3 and Theorem 1 in [7], the above diagram yields the following one:

$$(1) \qquad \begin{array}{ccccccc}
0 \to & \mathrm{Hom}(\Delta_d, C^*) & \xrightarrow{\eta_{C^*}} & H^1(X, \boldsymbol{C}^*) & \to 0 \\
& \uparrow \sigma & & \uparrow \phi^* & \\
0 \to & \mathrm{Hom}(C^*, C^*) & \xrightarrow{\eta} & H^1(P^{n-1}, \boldsymbol{C}^*) & \to 0,
\end{array}$$

where each row is an exact sequence. Furthermore we shall identify, in the sequel, $\operatorname{Hom}(\Delta_d, C^*)$ with C^* and $\operatorname{Hom}(C^*, C^*)=\{\mu\in\operatorname{Hom}(C,C)\mid\mu(Z)\subset Z\}$ with Z respectively by means of the correspondences:

$$\operatorname{Hom}(\Delta_d, C^*)\ni\lambda\leftrightarrow\lambda(d)\in C^*\,,$$

$$\operatorname{Hom}(C^*, C^*)\ni\mu\leftrightarrow\mu(1)\in Z\,.$$

Under these identifications, the mapping σ is given by $\sigma(m)=d^m$ for $m\in Z$, so that σ is an isomorphism of Z into C^* and its image is nothing but Δ_d. The mapping ϕ^* is, therefore, injective.

Thus we can state our main result.

Theorem 4. *Set* $h^{p,q}(\lambda)=\dim H^q(X,\Omega^p(F_\lambda))$. *Then* $h^{p,q}(\lambda)=0$ *for any* p *and* q *if* $\lambda\notin\Delta_d$, *and in the case* $\lambda\in\Delta_d$ *we have*:

(A) *if* $\lambda=d^m$, $m\neq 0$, $p>0$,

$(n>2)$

(i) $h^{p,q}(\lambda)=0$, *if* $2\leqq q\leqq n-2$

(ii) $h^{p,0}(\lambda)=h^{p,1}(\lambda)$

$$=\begin{cases}0\,, & \text{if } m<p\\ \binom{n}{p}\,, & \text{if } m=p\\ \binom{n+m-p-1}{n-p-1}\binom{m-1}{p}+\binom{n+m-p}{n-p}\binom{m-1}{p-1}^{3)}\,, & \text{if } m>p\,.\end{cases}$$

(iii) $h^{p,n-1}(\lambda)=h^{p,n}(\lambda)$

$$=\begin{cases}0\,, & \text{if } m>p-n\\ \binom{n}{p}\,, & \text{if } m=p-n\\ \binom{-m+p}{p}\binom{-m-1}{n-p-1}+\binom{-m+p-1}{p-1}\binom{-m-1}{n-p}\,, & \text{if } m<p-n.\end{cases}$$

$(n=2)$

In this case $h^{p,0}(\lambda)$ *and* $h^{p,2}(\lambda)$ *are given by the same formula as the case* $n>2$, *setting* $n=2$; *but* $h^{p,1}(\lambda)$ *is not, and* $h^{p,1}(\lambda)=h^{p,0}(\lambda)+h^{p,2}(\lambda)$.

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3) For any given two integers r and s, $\binom{r}{s}$ means the usual combination if $r,s>0$ and, otherwise we shall understand it as follow; $\binom{r}{s}=0$ if r or s is negative and $\binom{r}{s}=1$ if $r,s\geqq 0$, $rs=0$.

(B) *if* $\lambda=d^m$, $p=0$,

(i) $h^{0,q}(\lambda)=0$, *if* $2\leqq q\leqq n-2$

(ii) $(n>2)$, $h^{0,0}(\lambda)=h^{0,1}(\lambda)=\binom{n+m-1}{m}$

$$h^{0,n-1}(\lambda)=h^{0,n}(\lambda)=\binom{-m-1}{-m-n}$$

(iii) $(n=2)$, $h^{0,0}(\lambda)=\binom{m+1}{m}$

$$h^{0,1}(\lambda)=\binom{-m-1}{1}+\binom{m+1}{m}$$

$$h^{0,2}(\lambda)=\binom{-m-1}{1}$$

(C) *if* $\lambda=1$,

(i) $h^{n,q}(1)=\begin{cases}0, & \textit{if} \quad q\leqq n-2\\ 1, & \textit{if} \quad q=n-1, n\end{cases}$

(ii) $h^{p,q}(1)=0$, *if* $1\leqq p\leqq n-1$,

(iii) $h^{0,q}(1)=\begin{cases}0, & \textit{if} \quad q\geqq 2\\ 1, & \textit{if} \quad q=0, 1\end{cases}$

§ 5. Summary of some known results.

Let $X(P^{n-1}, T^1, \phi)$ be the Hopf fibering of X and let $E=E_X(\rho, C^m)$ be the homogeneous vector bundle over X defined by the representation (ρ, C^m) of U_d and let $\boldsymbol{E}$ be the sheaf of germs of holomorphic sections of E. The restriction E/T^1 of E on $T^1=GL(1, n-1, C)/U_d$ is also homogeneous with respect to the Klein form $GL(1, n-1; C)/U_d$. Therefore every element of $GL(1, n-1; C)$ induces a bundle automorphism of $E|T^1$, and so a linear isomorphism of the cohomology group $H^s(T^1, \boldsymbol{E}|T^1)$, $(s=0, 1)$. The holomorphic representation of $GL(1, n-1; C)$ thus obtained will be denoted by ρ^s $(s=0, 1)$[4], and the corresponding homogeneous vector bundle $E_{P^{n-1}}(\rho^s, H^s(T^1, \boldsymbol{E}|T^1))$ over P^{n-1} will be denoted simply by $\phi^s(E)$.

Now we take a spectral sequence $\{E_k\}$ whose final term E_∞ is associated to $H^*(X, \boldsymbol{E})$ and the second term E_2 is given by $E_2^{r,s}=H^r(P^{n-1}, \phi^s(\boldsymbol{E}))$, where $\phi^s(\boldsymbol{E})$ is the so-called *s-dimensional direct image sheaf of* $\boldsymbol{E}$ *by* ϕ. While, in our case, it is known the following result of Bott (cf. [3], Theorem VI).

4) This representation ρ^s is called, according to Bott, the s-dimensional induced representation of ρ.

Lemma 1. *$\phi^s(\boldsymbol{E})$ coincides with the sheaf of germs of holomorphic sections of $\phi^s(E)$; therefore $\phi^s(\boldsymbol{E})$ are zero sheaves for $s\geqq 2$.*

In particular, let E be a (homogeneous) line bundle. Then as to the restriction $E|T^1$ we know the following lemma (cf. [9], Proposition 3. 6).

Lemma 2. *The 0-dimensional cohomology group $H^0(T^1, \boldsymbol{E}|T^1)$ does not vanish if and only if $E|T^1$ is the trivial line bundle; therefore $\phi^0(E)$ is the zero sheaf unless $E|T^1$ is trivial.*

Now, for the computations in the next section, we need to know the cohomology groups $H^q(P^{n-1}, \Omega^p(\hat{F}))$, where $\Omega^p(\hat{F})$ is the sheaf of germs of holomorphic p-forms with values in the line bundle $\hat{F}$ over P^{n-1}; the dimensions of these cohomology groups have been computed by Bott [3] and by Matsumura [10] independently. That is,

Lemma 3. *Let $\hat{F}_m$ $(m\in Z)$ be the line bundle over P^{n-1} corresponding to $m\in \mathrm{Hom}(C^*, C^*)$ (cf. §4, (1)), and set $\hat{h}^{p,q}(m)=\dim H^q(P^{n-1}, \Omega^p(\hat{F}_m))$. Then we have,*

(i) $\hat{h}^{p,p}(0) = 1 \quad (o\leqq p\leqq n-1)$

(ii) $\hat{h}^{p,0}(m) = \binom{n+m-p-1}{n-p-1}\binom{m-1}{p} \quad (m>p)$

(iii) $\hat{h}^{p,n-1}(m) = \binom{-m+p}{p}\binom{-m-1}{n-p-1} \quad (p-n+1>m)$

(iv) $\hat{h}^{p,q}(m) = 0$ *for other cases.*

§6. The proof of the main theorem.

For the proof of our Theorem 4, we need the following extension of vector bundles over P^{n-1}:

$$(1) \qquad 0\to I\to Q(X)\to \hat{\Theta}\to 0\,,$$

which are the homogeneous vector bundles over P^{n-1} induced by the exact sequence (1) in §2 of $GL(1, n-1; C)$-modules. For instance $Q(X)=E_{P^{n-1}}(Ad, \mathfrak{g}/\mathfrak{u})$; and we note that $\Theta=\phi^*Q(X)$. By Ω and $\hat{\Omega}$ are meant the analytic sheaves of germs of holomorphic sections of Θ^* and $\hat{\Theta}^*$ respectively ($*$ means the dual vector bundle). Moreover we denote by Ξ the sheaf of germs of holomorphic sections of $Q(X)^*$. From (1) we have the exact sequence of analytic sheaves on P^{n-1}:

$$0\to\hat{\Omega}\to\Xi\to\boldsymbol{C}\to 0\,.$$

From this we can construct the following exact sequences:

$$(2)\qquad 0 \to \hat{\Omega}^p \to \Xi^p \to \hat{\Omega}^{p-1} \to 0 \quad (1 \leqq p \leqq n),$$

where Ξ^p is the sheaf of germs of holomorphic sections of the vector bundle $Q(X)^{*p}$ which is the p-exterior product of $Q(X)^*$ (see, for detail, [5], Satz 4.1.3*) and $\hat{\Omega}^p$ denotes the sheaf of germs of holomorphic p-forms on P^{n-1}.

Now we consider the spectral sequence $\{E_k\}$ associated to the Hopf fibering and the sheaf $\Omega^p(F_\lambda)$ over X. Then, the sheaf $\phi^s(\Omega^p(F_\lambda))=0$ except for $s=0,1$ by Lemma 1 and $\phi^s(\Omega^p(F_\lambda))=\phi^s(\boldsymbol{F}_\lambda)\otimes \Xi^p$, as is known by an easy argument on the induced representation, since $\Theta^{*p}=\phi^*(\Xi^p)$. On the other hand, the theorem of Riemann-Roch concerning the elliptic curve T^1 and the line bundle $F_\lambda | T^1$ (=the restriction of F_λ on T^1) implies that

$$\dim \phi^0(F_\lambda)-\dim \phi^1(F_\lambda) = 0,$$

because $F_\lambda | T^1$ has a holomorphic connection by a theorem of Matsushima [9] and so has the vanishing Chern class (cf. Atiyah [1]). Moreover, by Lemma 2, $\dim \phi^0(F_\lambda)>0$ if and only if $F_\lambda | T^1$ is trivial. The latter condition means that F_λ is induced from a line bundle $\hat{F}_m$ over P^{n-1} by ϕ; therefore in this case $\lambda=d^m$ (cf. (1) in § 4). Hence, if $\lambda \notin \Delta_d$ then $\phi^s(\Omega^p(F_\lambda))=0$ for all s (and p), which implies $E_2=E_\infty=H^*(X, \Omega^p(F_\lambda))=\{0\}$.

We assume hereafter that $\lambda=d^m \in \Delta_d$, and that $F_\lambda=\phi^*\hat{F}_m$. Then $F_\lambda | T^1$ is trivial and $\phi^s(\Theta^{*p}\otimes F_\lambda)\cong Q(X)^{*p}\otimes \hat{F}_m$ for $s=0,1$ by an easy argument on the induced representations; hence we have $E_2^{r,s}=H^r(P^{n-1}, \Xi^p\otimes \hat{\boldsymbol{F}}_m)$ $(s=0,1)$ and $E_2^{r,s}=\{0\}$ $(s\geqq 2)$, which implies that

$$(3)\qquad E_2^q = E_2^{q,0}+E_2^{q-1,1} = H^q(P^{n-1}, \Xi^p\otimes \hat{\boldsymbol{F}}_m)+H^{q-1}(P^{n-1}, \Xi^p\otimes \hat{\boldsymbol{F}}_m)$$

for $0\leqq q\leqq n$. Now we shall devide the subsequent discussions into three cases.

(A) The case $m\neq 0$, $p>0$.

The sequence (2) implies the following sheaf exact sequences:

$$(4)\qquad 0 \to \hat{\Omega}^p(\hat{F}_m) \to \Xi^p\otimes \hat{\boldsymbol{F}}_m \to \hat{\Omega}^{p-1}(\hat{F}_m) \to 0 \qquad (1\leqq p\leqq n).$$

The corresponding cohomology exact sequence is

$$\begin{aligned}&\to H^{q-1}(P^{n-1}, \hat{\Omega}^p(\hat{F}_m)) \to H^{q-1}(P^{n-1}, \Xi^p\otimes \hat{\boldsymbol{F}}_m) \to H^{q-1}(P^{n-1}\hat{\Omega}^{p-1}(\hat{F}_m))\\ &\to H^q(P^{n-1}, \hat{\Omega}^p(\hat{F}_m)) \to H^q(P^{n-1}, \Xi^p\otimes \hat{\boldsymbol{F}}_m) \to H^q(P^{n-1}, \hat{\Omega}^{p-1}(\hat{F}_m)) \to\end{aligned}$$

Therefore, if $1\leqq q\leqq n-2$, then $H^q(P^{n-1}, \hat{\Omega}^p(\hat{F}_m))=\{0\}$ for any p by Lemma 3, so that $E_2^{q,0}=\{0\}$, and $E_2^{q-1,1}=\{0\}$ also for the case $q>1$. Let $n>2$. If $q=0$, $E_2^0=E_\infty^0$ and $E_2^0=E_2^{0,1}=H^0(P^{n-1}, \Xi^p\otimes \hat{\boldsymbol{F}}_m)$ is given by

$$0 \to H^0(P^{n-1}, \hat{\Omega}^p(\hat{F}_m)) \to E_2^0 \to H^0(P^{n-1}, \hat{\Omega}^{p-1}(\hat{F}_m)) \to 0\,;$$

hence $\dim E_2^0 = \hat{h}^{p,0}(m) + \hat{h}^{p-1,0}(m)$. If $q = n-1$, $E_2^{n-2,1} = E_2^{n-2,0} = \{0\}$ and $E_2^{n-1,0} = E_2^{n-1}$ is given by

$$0 \to H^{n-1}(P^{n-1}, \hat{\Omega}^p(\hat{F}_m)) \to E_2^{n-1} \to H^{n-1}(P^{n-1}, \hat{\Omega}^{p-1}(\hat{F}_m)) \to 0\,;$$

hence $\dim E_2^{n-1} = \hat{h}^{p,n-1}(m) + \hat{h}^{p-1,n-1}(m)$. If $q = n$, $E_2^{n,0} = H^n(P^{n-1}, \Xi^p \otimes \hat{\boldsymbol{F}}_m) = \{0\}$ and $E_2^{n-1,1} = E_2^{n-1,0}$. Thus the spectral sequence is trivial and we obtain

$$\text{(i)}\quad h^{p,q}(\lambda) = 0 \quad \text{for} \quad 2 \leqq q \leqq n-2$$
$$\text{(ii)}\quad h^{p,0}(\lambda) = h^{p,1}(\lambda) = \hat{h}^{p,0}(m) + \hat{h}^{p-1,0}(m)$$
$$\text{(iii)}\quad h^{p,n-1}(\lambda) = h^{p,n}(\lambda) = \hat{h}^{p,n-1}(m) + \hat{h}^{p-1,n-1}(m)\,.$$

In case $n=2$, from (3), (4) and Lemma 3, we can deduce readily the following results:

$$\text{(i)}\quad h^{p,0}(\lambda) = \begin{cases} \hat{h}^{p,0}(m) + \hat{h}^{p-1,0}(m), & \text{if} \quad m \geqq p-1 \\ \hat{h}^{p,0}(m), & \text{if} \quad m \leqq p-1\,. \end{cases}$$
$$\text{(ii)}\quad h^{p,2}(\lambda) = \begin{cases} \hat{h}^{p-1,1}(m), & \text{if} \quad m \geqq p-1 \\ \hat{h}^{p,1}(m) + \hat{h}^{p-1,1}(m), & \text{if} \quad m \leqq p-1\,. \end{cases}$$
$$\text{(iii)}\quad h^{p,1}(\lambda) = h^{p,0}(\lambda) + h^{p,2}(\lambda)\,.$$

(B) The case $p=0$.

By Lemma 3 and (3), we have $E_2^q = \{0\}$ for $2 \leqq q \leqq n-2$. Furthermore, if $n > 2$, we have $E_2^0 = E_2^1 = H^0(P^{n-1}, \hat{\boldsymbol{F}}_m)$, $E_2^{n-1} = E_2^n = H^{n-1}(P^{n-1}, \hat{\boldsymbol{F}}_m)$, and if $n=2$, we have $E_2^0 = H^0(P^1, \hat{\boldsymbol{F}}_m)$, $E_2^1 = H^0(P^1, \hat{\boldsymbol{F}}_m) + H^1(P^1, \hat{\boldsymbol{F}}_m)$ and $E_2^2 = H^1(P^1, \hat{\boldsymbol{F}}_m)$. Moreover the spectral sequence is trivial, and we obtain:

$$\text{(i)}\quad h^{0,q}(\lambda) = 0\,, \quad \text{for } 2 \leqq q \leqq n-2$$
$$\text{(ii)}\quad (n > 2) \begin{cases} h^{0,0}(\lambda) = h^{0,1}(\lambda) = \hat{h}^{0,0}(m) \\ h^{0,n-1}(\lambda) = h^{0,n}(\lambda) = \hat{h}^{0,n-1}(m) \end{cases}$$
$$\text{(iii)}\quad (n = 2) \begin{cases} h^{0,0}(\lambda) = \hat{h}^{0,0}(m), \\ h^{0,1}(\lambda) = \hat{h}^{0,0}(m) + \hat{h}^{0,1}(m) \\ h^{0,2}(\lambda) = \hat{h}^{0,1}(m)\,. \end{cases}$$

(C) The case $m=0$, $p > 0$.

From (2) and (3) we have

$$\to H^{q-1}(P^{n-1}, \hat{\Omega}^p) \to E_2^{q-1,0} = E_2^{q-1,1} \to H^{q-1}(P^{n-1}, \hat{\Omega}^{p-1})$$
$$\to H^q(P^{n-1}, \hat{\Omega}^p) \to E_2^{q,0} = E_2^{q,1} \to H^q(P^{n-1}, \hat{\Omega}^{p-1})$$

If $p=n$, then $H^q(P^{n-1}, \hat{\Omega}^n)=\{0\}$ and so $E_2^q=E_2^{q,0}+E_2^{q-1,1}=H^q(P^{n-1}, \hat{\Omega}^{n-1}) + H^{q-1}(P^{n-1}, \hat{\Omega}^{n-1})$; hence by Lemma 3 we have

$$h^{n,q}(1) = \begin{cases} 0, & \text{if } q \leqq n-2 \\ 1, & \text{if } q = n-1, n. \end{cases}$$

We assume hereafter that $1 \leqq p \leqq n-1$. If $q \neq p \pm 1$, p, then we have $E_2^{q,0}=\{0\}$, $E_2^{q-1,1}=\{0\}$; hence $h^{p,q}(1)=0$. If $q=p-1$, then $E_2^{p-2,1}=\{0\}$, $E_2^{p-1,0}=E_\infty^{p-1}$. If $q=p+1$, then $E_2^{p+1,0}=\{0\}$, $E_2^{p,1}=E_\infty^{p+1}$. If $q=p$, then we have

$$(5) \qquad 0 \to E_2^{p-1,0} \to H^{p-1}(P^{n-1}, \hat{\Omega}^{p-1}) \xrightarrow{\delta^*} H^p(P^{n-1}, \hat{\Omega}^p) \to E_2^{p,0} \to 0\,.$$

We remark here that $\dim H^{p-1}(P^{n-1}, \hat{\Omega}^{p-1}) = \dim H^p(P^{n-1}, \hat{\Omega}^p)=1$ and that $E_2^{p-1,0}=E_2^{p,0}=\{0\}$ if and only if δ^* is not the zero homomorphism (i.e. bijective). While by the following Lemma 4, we have in reality $E_2^{p-1,0}= E_2^{p-1,1}=E_2^{p,0}=E_2^{p,1}=\{0\}$ for $1 \leqq p \leqq n-2$; hence we have then $h^{p,q}(1)=0$ also for $q=p\pm 1$, p. It remains only the case $p=n-1$; in this case we have $h^{n-1,q}(1)=h^{1,n-q}(1)=0$ by Serre's duality for $n>2$ (The case $n=2$ is contained in the proof of Lemma 4).

Lemma 4. *In the above exact sequence* (5), *if* $1 \leqq p \leqq n-2$, δ^* *is bijective*; *hence* $E_2^{p-1,0}=E_2^{p,0}=\{0\}$.

Proof. First we shall consider the case $p=1$ (in (2) we set $\hat{\Omega}^0=\boldsymbol{C}$). In this case, $\delta^*: H^0(P^{n-1}, \boldsymbol{C}) \to H^1(P^{n-1}, \hat{\Omega})$ is not the zero homomorphism; in fact, if otherwise, the extension: $0 \to \hat{\Omega} \to \Xi \to \boldsymbol{C} \to 0$ is splittable by a lemma of Atiyah (Proc. London Math. Soc., 7 (1957), p. 429, Lemma 13), and then, by the same argument as in our previous paper [7] (cf. the foot-note 7)), the Hopf fibering must be trivial; however $S^1 \times S^{2n-1}$ and $T^1 \times P^{n-1}$ are clearly not homeomorphic. This proves the lemma in our case.

In general case we prove the lemma by induction on n. In case $n=2$, it must be $p=1$; therefore the lemma has been proved by the above discussions. We assume $n>2$ and consider the exact sequence (2) over the base space P^{n-2}, which will be written as:

$$(2_*) \qquad 0 \to \hat{\Omega}_*^p \to \Xi_*^p \to \hat{\Omega}_*^{p-1} \to 0 \quad \text{over} \quad P^{n-2}\,.$$

On the other hand, the imbedding of $GL(n-1, C)$ into $GL(n, C)$, defined by $g \to \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}$ for $g \in GL(n-1, C)$, induces an imbedding of $P^{n-?}$ into P^{n-1} as a hyperplane, which we shall fix once for all. The sheaves in (2_*) are naturally extendable to the sheaves over P^{n-1} by assuming that the fibres on the complement of P^{n-1} vanish, and they shall be denoted with the same letters as in (2_*). Now we shall show that there are natural sheaf homomorphisms $\alpha_p : \Xi^p \to \Xi^p_*$ and $\beta_p : \hat{\Omega}^p \to \hat{\Omega}^p_*$ which yield the following commutative diagram:

$$
(6) \qquad \begin{array}{ccccccccc}
0 \to & \hat{\Omega}^p & \to & \Xi^p & \to & \hat{\Omega}^{p-1} & \to 0 \\
& \downarrow \beta_p & & \downarrow \alpha_p & & \downarrow \beta_{p-1} & \\
0 \to & \hat{\Omega}^p_* & \to & \Xi^p_* & \to & \hat{\Omega}^{p-1}_* & \to 0 .
\end{array}
$$

For this sake, we identity the exact sequence of $\hat{U}$-modules (1) in §2 with the one:

$$0 \to C^1 \to C^n \to C^{n-1} \to 0 ,$$

where $\hat{U}$ acts on each module as the identity representation on C^1, as $\frac{1}{b}\hat{u} = \begin{pmatrix} 1 & \frac{1}{b}* \\ 0 & \frac{1}{b}B \end{pmatrix}$ on C^n and as $\frac{1}{b}B$ on C^{n-1} respectively, for every element $\hat{u} = \begin{pmatrix} b & * \\ 0 & B \end{pmatrix} \in \hat{U}$. Then the restrictions of $\hat{\Theta}^{*p}$ and $Q(X)^{*p}$ on P^{n-2} are given by $GL(n-1, C) \times_{\hat{U}_*} (C^{n-1})^{*p}$ [5] and $GL(n-1, C) \times_{\hat{U}_*} (C^n)^{*p}$ respectively where $\hat{U}_* = GL(1, n-2\,; C)$ acts on $(C^n)^{*p}$ and $(C^{n-1})^{*p}$ as defined above. Then we have the commutative diagram of modules:

$$
\begin{array}{ccccccc}
0 \to (C^{n-1})^{*p} & \to & (C^n)^{*p} & \to & (C^{n-1})^{*p-1} \to 0 \\
\downarrow \tilde{\beta}_p & & \downarrow \tilde{\alpha}_p & & \downarrow \tilde{\beta}_{p-1} \\
0 \to (C^{n-2})^{*p} & \to & (C^{n-1})^{*p} & \to & (C^{n-2})^{*p-1} \to 0 ,
\end{array}
$$

where $\tilde{\alpha}_p$ and $\tilde{\beta}_p$ denote the restriction mappings of alternating p-forms. This diagram, considered as the one of $\hat{U}_*$-modules, is commutative as is easily seen. Therefore it induces the commutative diagram of homogeneous vector bundles over P^{n-2}; this implies that there are corresponding sheaf homomorphisms α_p, β_p and β_{p-1} as in (6). Thus we have the following commutative diagram:

$$
\begin{array}{ccc}
H^{p-1}(P^{n-1}, \hat{\Omega}^{p-1}) & \xrightarrow{\delta^*} & H^p(P^{n-1}, \hat{\Omega}^p) \\
\downarrow \beta_{p-1} & & \downarrow \beta_p \\
H^{p-1}(P^{n-2}, \hat{\Omega}^{p-1}_*) & \xrightarrow{\delta^*} & H^p(P^{n-2}, \hat{\Omega}^p_*) ,
\end{array}
$$

where the mappings β_{p-1} and β_p are bijective for $1 \leqq p \leqq n-2$ since they coincide with the restriction mappings of harmonic forms *via* the

5) $(C^{n-1})^{*p}$ denotes the vector space of all alternating p-forms on C^{n-1}.

Dolbealt isomorphisms. While, δ^* in the under column is bijective by induction assumption, so our δ^* must be bijective.

REMARK 1. Theorem 4 tells us that both Riemann-Roch's theorem with respect to any line bundle and Hodge's index theorem are valid for Hopf manifolds (cf. [8]). In fact, for any line bundle F_λ over a Hopf manifold X, we can readily check that

$$\chi(X, \boldsymbol{F}_\lambda) = \sum_{q=0}^{n} (-1)^q \dim H^q(X, \boldsymbol{F}_\lambda) = \sum_{q=0}^{n} (-1)^q h^{0,q}(\lambda) = 0\,;$$

while the Todd genus $T(X, \boldsymbol{F}_\lambda)=0$ since $H^2(X, Z)=\{0\}$. Furthermore, the index $\tau(X)$ of X is clearly 0, since X is homeomorphic to $S^1\times S^{2n-1}$; while we see immediately, from (C) in Theorem 4, that

$$\sum_{p,q} (-1)^q h^{p,q}(1) = 0\,.$$

REMARK 2. Theorem 1 and Theorem 2 can be readily derived from Theorem 4. In fact, by Serre's duality theorem, we have $H^q(X, \Theta)\simeq H^{n-q}(X, \Omega^1(K))$ where K denotes the canonical line bundle of X. While, from the exact sequence (2) in § 2, we get immediately $K=\phi^*\hat{K}$, where $\hat{K}$ is the canonical bundle of P^{n-1} and coincides with $\hat{F}_{-n}$. Therefore, Theorem 4, (A) yields that $\dim H^0(X, \Theta)=\dim H^1(X, \Theta)=n^2$, $H^q(X, \Theta)=\{0\}$ for $q\geqq 2$.

REMARK 3. The proof of Theorem 4 suggests us the possibilities of computing the cohomology groups $H^p(X, \Omega^p(F))$ for other class of C-manifolds with the *fundamental fibering* $X(\hat{X}, T^1, \phi)$ (cf. [7]) provided that the cohomology groups $H^q(\hat{X}, \Omega(\hat{F}))$ are known. For instance, Calabi-Eckmann's example (cf. [4], [7]) or $SU(3)$ with a left invariant complex structure is such a manifold. However, for them $\hat{X}=P^p\times P^q$ or $\hat{X}=F(3)$ (=the 3-dimensional flag manifold) respectively and the corresponding cohomology groups $H^q(\hat{X}, \Omega^p(\hat{F}))$ are rather complicated; consequently the computations of $H^q(X, \Omega^p(F))$ might be more difficult than for Hopf manifolds.

But we shall exhibit here the number $h^{p,q}=h^{p,q}(1)$ for $SU(3)$, since Bott's computations in [3] for then are incorrect.

$$h^{0.0} = h^{0.1} = h^{1.1} = h^{1.2} = 1\,,$$
$$h^{4.4} = h^{4.3} = h^{3.3} = h^{3.2} = 1\,,$$
$$h^{p,q} = 0 \quad \text{otherwise.}$$

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References

[1] M. F. Atiyah: Complex analytic connections in fibre bundles, Trans. Amer. Math. Soc. **85** (1957), 181–207.

[2] A. Blanchard: Sur les variétés analytiques complexes, Ann. Ecol. Norm. Sup. **73** (1956), 157–202.

[3] R. Bott: Homogeneous vector bundles, Ann. of Math. **66** (1957), 203–248.

[4] E. Calabi and B. Eckmann: A class of compact complex manifolds which are not algebraic, Ann. of Math. **98** (1953), 494–500.

[5] F. Hirzebruch: Neue topologische Methoden in der algebraischen Geometrie, Erg. de Math. **9**, 1956.

[6] H. Hopf: Zur Topologie der komplexen Mannigfaltigkeiten, in Studies and Essays presented to R. Courant, New York, 1948.

[7] M. Ise: Some properties of complex analytic vector bundles over compact complex homogeneous spaces, Osaka Math. J. **12** (1960), 217–252.

[8] K. Kodaira and D. C. Spencer: On deformations of complex analytic structures, I–II, Ann. of Math. **67** (1958), 328–466.

[9] Y. Matsushima: Fibrés holomorphes sur un tore complexe, Nagoya Math. J. **14** (1958), 1–24.

[10] H. Matsumura: Geometric structure of the cohomology rings in abstract algebraic geometry, Mem. Coll. Sci. Univ. Kyoto, Ser. A Math. **32** (1959), 33–84.