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# *On Diffeomorphic Approximations of Polyhedral Surfaces in 4-Space*

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Let $f$ be a semilinear homeomorphism of a connected closed surface $F$, with or without boundary, in 4-space $E^4$. Hence $f(F)$ is polyhedral. Then $f$ is said to be *diffeomorphically approximable* if for each positive real number $\varepsilon$ there exists a diffeomorphism $g$ of $F$ into $E^4$ such that $|f(x)-g(x)|<\varepsilon$ for each $x \in F$[1].

First suppose that $F$ is a connected closed surface without boundary. Let $p_1, \cdots, p_n$ be all the *singularities*[2] of $f(F)$ in $E^4$ and $\tilde{k}_1, \cdots, \tilde{k}_n$ the knot types of these singularities, respectively. Further let $\tilde{k}$ be the *knot product* of $\tilde{k}_1, \cdots, \tilde{k}_n$. Then the purpose of this note is to prove the following

**Theorem 1.** *$f$ is diffeomorphically approximable if $\tilde{k}$ is null-equivalent*[2].

It is easy to see that if $F$ is the 2-sphere, then $\tilde{k}$ is null-equivalent. Therefore, as a special case of Theorem 1, we have

**Theorem 2.** *If $F$ is the 2-sphere, then $f$ has a diffeomorphic approximation.*

Now let $F$ be a connected closed surface with boundary. In this case, by the same method of proof, we have the following

**Theorem 3.** *$f$ can be diffeomorphically approximated.*

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The proof of Theorem 1 is divided into two steps.

(1) *To construct a semilinear homeomorphism $h$ of $F$ in 4-space $E^4$, which is an approximation of $f$ such that $h(F)$ is locally flat*[4] *at every point of $h(F)$.*

1) For the case of polyheral $(n-1)$-manifold in $n$-space, see [3] [5].

2) See [1] [2].

3) See [1] [6].

4) See [1] [2].

(2) *To construct a diffeomorphism $g$ of $F$ in 4-space $E^4$, which is an approximation of $h$.*

In the second step there occurs no difficulty, since $h(F)$ is a locally flat closed surface in 4-space. Hence we shall omit the proof of it.

Now we shall prove the first step. As far as this step is concerned we may take the semilinear point of view. By definition, for each of the singularities $p_1, \cdots, p_n$ we can take sufficiently small 4-cells $C_1, \cdots, C_n$ such that $C_i \cap f(F)$ is a cone whose base is the knot $k_i$, where the knot type of $k_i$ is $\tilde{k}_i$, in the bdry $C_i$ and that the interior of the 2-cell $f(F) \cap C_i$ is contained in Int $C_i$, respectively.

Further let $D_i (i=2, \cdots, n)$ be a sufficiently narrow 4-cell satisfying the following conditions:

(i) $D_i \cap D_j = 0$ $(i \neq j)$,

(ii) $D_i \cap C_i$ is a 3-cell,

(iii) $D_i \cap C_i \cap f(F)$ is an arc which represents a trivial knot in $D_i \cap C_i$[5],

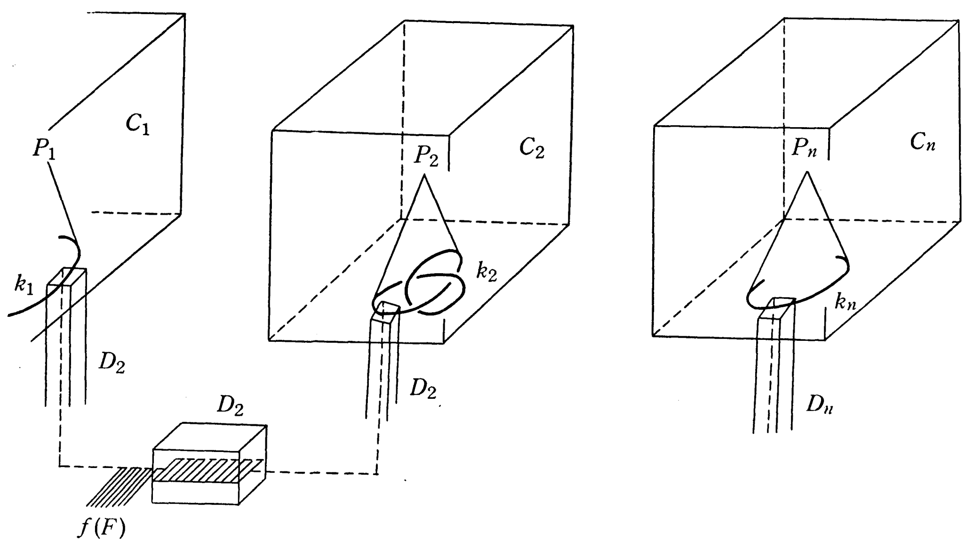
(iv) $D_i \cap f(F)$ is a narrow 2-cell which is imbedded trivially in $D_i$,

(v) $D_i \cap C_1$ is a 3-cell, and

(vi) $D_i \cap C_1 \cap f(F)$ is an arc which represents a trivial knot in $D_i \cap C_1$.

The existence of such $D_i$ is obvious, since $f(F)$ is locally flat at every point of $F - \bigcup_{i=1}^{n} p_i$.

Now we shall construct the semilinear homeomorphism $h$ in question. We shall only construct the image $h(F)$, for from this it will be easy to define the required $h$.




5) See [4].

First, for each $i$ $(i=2, \cdots, n)$ we construct a knot $k'_i$ in bdry $C_i$ such that

(i) $k'_i = k_i \times \tilde{k}_i^{-1}$ and

(ii) $k'_i \cap (\text{bdry } C_i - D_i) = f(F) \cap (\text{bdry } C_i - D_i)$.

Since the knot type of the singularity $p_i$ is $\tilde{k}_i$, this is possible. Then the arc $k'_i \cap D_i \cap C_i$ represents the knot type $\tilde{k}_i^{-1}$ in the 3-cell $D_i \cap C_i$. From (i) follows the existence of a locally flat 2-cell $d_i$ such that $d_i \subset C_i$, the interior of the 2-cell $d_i$ is contained in Int $C_i$ and the boundary of the 2-cell $d_i$ is $k'_i$.

Next we construct a knot $k = k_1 \times \cdots \times k_n$ in bdry $C_1$ such that

(i) $k \cap (\text{bdry } C_1 - \bigcup_{i=2}^{n} D_i) = f(F) \cap (\text{bdry } C_1 - \bigcup_{i=2}^{n} D_i)$ and

(ii) the arc $k'_i \cap D_i \cap C_i$ represents the knot type $\tilde{k}_i$ in the 3-cell $D_i \cap C_i$ $(i=2, \cdots, n)$.

Since the knot type of the singularity $p_1$ is $\tilde{k}_1$, this is possible. As remarked before, the arc $k'_i \cap D_i \cap C_i$ $(i=2, \cdots, n)$ represents the knot type $\tilde{k}_i^{-1}$ in the 3-cell $D_i \cap C_i$. Since $D_i \cap C_i$ and $D_i \cap C_1$ are opposite faces of the 4-cell $D_i$, there exists a locally flat 2-cell $e_i$ in $D_i$ whose boundary is

$$(k'_i \cap D_i) \cup (k \cap D_i) \cup \{(f(F) - \bigcup_{j=1}^{n} C_j) \cap \text{bdry } D_i\} .$$

By our assumption that $k$ is null-equivalent there exists a locally flat 2-cell $d_1$ such that $d_1 \subset C_1$, the interior of the 2-cell $d_1$ is contained in Int $C_1$ and the boundary of the 2-cell $d_1$ is $k$.

Then

$$h(F) = \bigcup_{i=1}^{n} d_i \cup \bigcup_{i=2}^{n} e_i \cup \{f(F) \cap (E^4 - \bigcup_{i=1}^{n} C - \bigcup_{i=2}^{n} D_i)\}$$

is a locally flat connected surface homeomorphic to $f(F)$. It is easy to define a homeomorphism $h$ as an approximation of $f$. This completes the proof of the first step and hence the proof of Theorem 1.

As remarked before, Theorem 2 is a special case of Theorem 1. But Theorem 2 can be proved directly using the method of the proof of Theorem 1. For, in this case, the existence of $d_1$, in the proof, is almost obvious and our assumption is used only to verify it.

The proof of Theorem 3 is done by the same way. In this case, since $f(F)$ is a connected surface with boundary, we can draw the narrow 4-cell $D_i$ $(i=1, 2, \cdots, n)$, in the proof of Theorem 1, from the neighborhood of the singularity $p_i$ to the boundary of $f(F)$. No special reference to the singularity $p_1$ is necessary.

As far as the converse of Theorem 1 is concerned the problem is still open.

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