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Some Properties of $(n-1)$-Manifolds in the Euclidean n-Space

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S. S. Cairns showed that any polyhedral m-manifold in normal position in some Euclidean space may have an analytic approximation and some conditions are sufficient for a polyhedral m-manifold to be in normal position in some Euclidean space [2][1] and that especially for $m \leqq 4$ any polyhedral m-manifold may have such a subdivision that it is imbedded rectilinearly in normal position in some Euclidean space and therefore any polyhedral m-manifold $(m \leqq 4)$ has an analytic structure [3]. It is an open question[2] whether or not any polyhedral manifold may be suitably subdivided and imbedded rectilinearly in normal position in some Euclidean space and may have an analytic structure in this way.

In the first part of this paper we shall show, along the same line as S. S. Cairns, a condition that a polyhedral $(n-1)$-manifold in the Euclidean n-space is in normal position and therefore has an analytic structure. It is easily shown by examples that a polyhedral $(n-1)$-manifold P^{n-1} is not always in normal position in the Euclidean n-space even if P^{n-1} is a Brouwer manifold, that is, a manifold in which the star of any vertex of P^{n-1} is imbedded rectilinearly in the Euclidean $(n-1)$-space.

In the second part of this paper we shall define the curvatura integra for a polyhedral $(n-1)$-manifold P^{n-1} in regular position in the Euclidean n-space R^n and show that it remains unaltered during an isotopic deformation of P^{n-1} in R^n.

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1) The number in bracket indicates the number in the references.

2) After the preparation of this paper, I had an opportunity to notice the mimeographed note by J. Milnor [5] in which it was shown that R. Thom [6] constructed a polyhedral closed 8-manifold having no subdivision which is imbedded rectilinearly in normal position in any Euclidean space.

1. Differentiable appoximations of ($n-1$)-manifolds in the Euclidean n-space.

Let S be a set of points in the Euclidean n-space R^n. A k-plane H^k ($k \geqq 1$) in R^n will be called *transversal to* S if there exists a positive number ε such that the straight line through any two points of S makes an angle greater than ε with H^k. A k-plane $H^k(p)$ ($k \geqq 1$) through a point p of S will be called *transversal to* S *at* p if $H^k(p)$ is transversal to some neighborhood p on S.

Let M^m be a topological m-manifold in the Euclidean n-space R^n. We shall say that M^m is *in normal position* in R^n if it is possible to define through each point p of M^n an ($n-m$)-plane $H^{n-m}(p)$ which varies continuously with p and is transversal to M^m at p.

Under a *polyhedral* m-*manifold* P^m we shall mean a locally finite simplicial complex in some Euclidean space with the condition that every vertex of P^m is an inner point of its star on P^m which is a complex whose underlying space is homeomorphic to the m-simplex, that is, combinatorially equivalent to the m-simplex. (An equivalent condition would be that the link of every j-simplex, $0 \leqq j \leqq m$, should be combinatorially equivalent to the boundary of the ($m-j$)-simplex.)

Let P^m be a polyhedral m-manifold in the Euclidean n-space R^n. We shall say that P^m is *locally in normal position* in R^n if the star of every vertex on P^m is in normal position in R^n [3]. Then we shall obtain the following

Theorem 1. *Every polyhedral* ($n-1$)-*manifold* P^{n-1} *locally in normal position in the Euclidean* n-*space* R^n *is in normal position in* R^n.

Before we proceed to the proof of the theorem we shall prove some lemmas.

Lemma 1. *Let* p *be a point of* P^{n-1} *and let* $St(p)$ *be the star of* p *on* P^{n-1}. *Then a straight line* $H(p)$ *through* p *is transversal to* $St(p)$ *if and only if* $H(p)$ *is transversal to* P^{n-1} *at* p.

Proof. Let $U(p)$ be a neighborhood of p on P^{n-1} which is contained in $St(p)$. If $H(p)$ is transversal to $St(p)$, then $H(p)$ is transversal to $U(p)$. Therefore $H(p)$ is transversal to P^{n-1} at p.

Conversely if $H(p)$ is not transversal to $St(p)$, then there exist two

3) According to S. S. Cairns [2], a Brouwer manifold in general position in some Euclidean space R^n, that is, a Brouwer manifold such that the vertices on the star of its any simplex are linearly independent in R^n, is locally in normal position in R^n. But as shown already in 2), a polyhedral manifold locally in normal position is not always in normal position.

points q and r in $St(p)$ which determine a straight line $H(q, r)$ parallel to $H(p)$. If one of q and r is p, then $H(q, r)$ coincides with $H(p)$ and has a common point except for p with any neighborhood of p in P^{n-1}. Therefore $H(p)$ is not transversal to P^{n-1} at p.

If q and r are different from p, then the line-segments $\overline{pq}$ and $\overline{pr}$, which are determined by the points p and q and the points p and r respectively, are contained in $St(p)$. For any neighborhood $U(p)$ of p on P^{n-1}, all the points of $\overline{pq}$ and $\overline{pr}$ sufficiently near to p are contained in $U(p)$. Therefore there exist two points q' and r' in $U(p)$ which belong to $\overline{pq}$ and $\overline{pr}$ respectively and determine a straight line parallel to $H(q, r)$. Thus $H(p)$ is not transversal to P^{n-1} at p and the lemma is proved.

Lemma 2. *Let p be a point of P^{n-1}, let $St(p)$ be the star of p on P^{n-1} and let $H(p)$ be a straight line through p transversal to P^{n-1} at p. Then the straight line $H(q)$ through an inner point q of $St(p)$ and parallel to $H(p)$ is transversal to P^{n-1} at q.*

Proof. If we choose a neighborhood $U(q)$ of q on P^{n-1} which is contained in $St(p)$, then a straight line transversal to $St(p)$ is transversal to $U(q)$. Since $H(p)$ is, according to Lemma 1, transversal to $St(p)$, $H(p)$ is also transversal to $U(q)$. Thus the lemma is proved.

Lemma 3. *The totality of the straight lines transversal to P^{n-1} at a point p of P^{n-1} is, whenever there exists such a line, homeomorphic to an open $(n-1)$-cell.*

Proof. According to Lemma 1 we are only to prove that the totality of the straight lines through p and transversal to the star $St(p)$ of p on P^{n-1} is homeomorphic to an open $(n-1)$-cell.

First consider the case that $St(p)$ is composed of a single $(n-1)$-simplex s^{n-1}. Then we choose affine coordinates $(x_1 \cdots, x_n)$ in R^n such that $x_1 = \cdots = x_n = 0$ at p and $x_n = 0$ on the $(n-1)$-plane determined by s^{n-1}. Then a straight line through p is transversal to $St(p)$ if and only if it is through a point $(x_1 \cdots, x_{n-1}, x_n > 0)$. It is easy to see that the totality of the straight lines through p and a point $(x_1, \cdots, x_{n-1}, x_n > 0)$ is, regarding as a subset of the projective $(n-1)$-space composed of all the straight lines through p, is homeomorphic to an open $(n-1)$-cell.

Secondly consider the case that $St(p)$ is composed of two $(n-1)$-simplexes s_1^{n-1} and s_2^{n-1}. Then the following two cases are possible:

(1) s_1^{n-1} and s_2^{n-1} are parallel. In this case the situation is the same as above.

(2) s_1^{n-1} and s_2^{n-1} are not parallel. We choose affine coordinates

$(x_1, \dots, x_n)$ in R^n such that $x_1 = \dots = x_n = 0$ at p and $x_{n-1}=0$, $x_n \geqq 0$ on s_1^{n-1} and $x_n=0$, $x_{n-1} \geqq 0$ on s_2^{n-1}. Then it is easy to see that the totality of the straight lines through p and transversal to $St(p)$ is the totality of the straight lines which are determined by p and a point $(x_1, \dots, x_{n-2}, x_{n-1}>0, x_n>0)$. It is homeomorphic to an open $(n-1)$-cell.

Finally consider the case that $St(p)$ is composed of $(n-1)$-simplexes $s_1^{n-1}, \dots, s_k^{n-1}$ $(k \geqq 3)$ where s_i^{n-1} and s_{i+1}^{n-1} $(i=1, \cdots, k)$[4] have a common $(n-2)$-simplex. According to the second case, a transversal line to $s_i^{n-1} \cup s_{i+1}^{n-1}$ is a straight line through p and a point of the intersection of an open half space[5] determined by s_i^{n-1} and that determined by s_{i+1}^{n-1}.

It may be easily proved that for a straight line H through any two points of $St(p)$ either there exists an n-simplex s_i^{n-1} parallel to H or there exist two points q and r on s_i^{n-1} and s_{i+1}^{n-1} respectively for some i determining a straight line parallel to H.

From this fact we may show that a straight line $H(p)$ through p is transversal to S if and only if $H(p)$ is through some point of $\bigcap_{i=1}^{k} K(s_i^{n-1})$ where $K(s_i^{n-1})$ is an open half space of the $(n-1)$-plane determined by s_i^{n-1}. The totality of the above mentioned line, if any, is an open $(n-1)$-cell. Thus the lemma is proved.

Proof of Theorem 1. Let ε be a positive constant less than $\frac{1}{n}$. Let s^j be any j-simplex of P^{n-1} and let s^{n-1} be any $(n-1)$-simplex of P^{n-1} which belongs to the star of s^j on P^{n-1}. We choose barycentric coordinates $(u_0, u_1, \cdots, u_{n-1})$ on s^{n-1} such that $u_{j+1} = \cdots = u_{n-1} = 0$ on s^j. Let $N_{s^{n-1}}(s^j)$ be the set of points whose barycentric coordinates $(u_0, \cdots, u_{n-1})$ satisfy the following inequalities:

$$\varepsilon \leqq u_0, \cdots, \varepsilon \leqq u_j,\ 0 \leqq u_{j+1} \leqq \varepsilon, \cdots, 0 \leqq u_{n-1} \leqq \varepsilon .$$

We shall define $N(s^j)$ by

$$N(s^j) = \sum_{s^{n-1} \in St(s^j)} N_{s^{n-1}}(s^j)$$

where $St(s^j)$ is the star of s^j on P^{n-1}.

As $St(s^j)$ is a combinatorial $(n-1)$-cell, $N(s^j)$ is also a combinatorial $(n-1)$-cell and its boundary is a combinatorial $(n-2)$-sphere. Thus

4) $k+1$ is regarded as 1.

5) If affine coordinates $(x_1, \cdots, x_n)$ in R^n are chosen such that $x_n=0$ on an $(n-1)$-plane K^{n-1} (an $(n-1)$-simplex s^{n-1}), then the open half space determined by $K^{n-1}(s^{n-1})$ is the set of points $(x_1, \cdots, x_{n-1}, x_n>0)$ or the set of points $(x_1, \cdots, x_{n-1}, x_n<0)$.

P^{n-1} is covered by these closed $(n-1)$-dimensional regions which are disjoint from each other except eventually for common faces.

We shall define transversal lines on $N(s^j)$ step by step by induction on the dimension of the simplexes of P^{n-1}.

The initial step of induction is to define transversal lines on $N(s^o)$ of every vertex s^o of P^{n-1}. According to the hypothesis of the theorem, we may define a straight line $H(s^o)$ which passes through s^o and is transversal to the star of s^o at s^o. Let p be any point of $N(s^o)$ and let $H(p)$ be the straight line through p and parallel to $H(s^o)$.

According to Lemma 2, $H(p)$ is transversal to P^{n-1} at p. Then we define a transversal line $H(p)$ through p on $N(s^o)$ by the requirement

$$H(p) \parallel H(s^o) .$$

If transversal lines $H(p)$ are defined on every $N(s^k)$ $(k<j)$, the general step of induction is to extend the definition of $H(p)$ over $N(s^j)$ where s^j is any j-simplex of P^{n-1}.

Let t^j be the set of points where all the barycentric coordinates for s^j exceed ε. Then $H(p)$ is already defined on the closure of s^j-t^j by the hypothesis of induction.

First we shall extend the definition of $H(p)$ over t^j. According to Lemma 3 a straight line $H(p)$ through an inner point p of the star of s^j is transversal to P^{n-1} at p if only if the straight line through the origin O of R^n and parallel to $H(p)$, regarded as an element of the projective $(n-1)$-space composed of all the lines through O, falls into an open $(n-1)$-cell $D(t^j)$. Thus we obtain a mapping[6] φ of the boundary of t^j into $D(t^j)$. As $D(t^j)$ is an open $(n-1)$-cell, we may extend φ to the mapping from t^j into $D(t^j)$. Thus $H(p)$ is defined on t^j.

Next we shall define $H(p)$ over the other part of $N(s^j)$. Let s^{n-1} be an $(n-1)$-simplex in the star of s^j on P^{n-1}. Let t^{n-1} be the set of points where all the barycentric coordinates exceed ε. Let t'^j be the bounding simplex of t^{n-1} parallel to t^j. Let t''^{n-j-2} be the bounding simplex of t^{n-1} opposite to t'^j. Consider any point q on t^j and denote by $B^{n-j-1}_{s^{n-1}}(q)$ the intersection of $N_{s^{n-1}}(s^j)$ with the $(n-j-1)$-plane determined by q and t''^{n-j-2} and define $B^{n-j-1}(q)$ as follows:

$$B^{n-j-1}(q) = \sum_{s^{n-1}\in St(s^j)} B^{n-j-1}_{s^{n-1}}(q) ,$$

where $St(s^j)$ is the star of s^j on P^{n-1}.

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6) We shall always denote a mapping as "a continuous mapping".

As q ranges over t^j, the set $B^{n-j-1}(q)$ fills out $N(s^j)$ in a one-to-one continuous way. If now q is any point of t^j and p is any point of $B^{n-j-1}(q)$, then $H(p)$ will mean the straight line through p parallel to $H(q)$. According to Lemma 2, $H(p)$ is transversal to P^{n-1} at p. Thus the definition of $H(p)$ on $N(s^j)$ is completed, and the theorem is proved.

Let M^m be a topological m-manifold in some Euclidean space R^n and let ε be a given positive number. A differentiable m-manifold V^m in R^n [7] is said to be *an ε-approximation of M^m* in R^n, if V^m is homeomorphic to M^m and the distance of the corresponding points of this homeomorphism is less than ε. If a polyhedral m-manifold P^m is in normal position in some Euclidean space R^n, there exists, according to S. S. Cairns [2], an analytic manifold in R^n which is an ε-approximation of P^m in R^n. Therefore we obtain the following

Theorem 2. *Under the same conditions as Theorem* 1, *there exists, for any given positive number ε, an analytic manifold in R^n which is an ε-approximation of P^{n-1} in R^n.*

Next we shall say that a topological m-manifold in the Euclidean n-space R^n is in regular position in R^n if there exist unit vectors $v_1(p), \cdots, v_{n-m}(p)$ through every point p of M^m such that $v_1(p), \cdots, v_{n-m}(p)$ vary continuously with p and that the $(n-m)$-plane spanned by these vectors is tranversal to M^m at p.

Let M^m be a compact m-manifold in regular position in R^n and let ε be a given positive number. Then M^m may be imbedded, according to H. Whitney [8], in a $(n-m)$-parameter family of manifolds $M(t_1, \cdots, t_{n-m})$, $|t_i|<1$ $(i=1, \cdots, n-m)$, with the following conditions:

(1) $M(0, \cdots, 0)=M$.

(2) $M(t_1, \cdots, t_{n-m})$ is analytic if $(t_1, \cdots, t_{n-m}) \neq (0, \cdots, 0)$.

(3) Each manifold $M(t_1, \cdots, t_{n-m})$ is an ε-approximation of M^m in R^n.

(4) If $p^*=Z(p; t_1, \cdots, t_{n-m})$ is the point of $M(t_1, \cdots, t_{n-m})$ corresponding to the point of M^m, Z is a continuous function of its arguments, and the points p^* fill out an open set R containing M in a one-to-one way; hence no two of the manifolds intersect. The point p^* for p fixed fill out a portion of an $(n-m)$-plane $H^{n-m}(p)$ through p.

Theorem 3. *Let P^{n-1} be a compact polyhedral $(n-1)$-manifold locally*

7) For the definition of a "differentiable manifold in R^n", see H. Whitney [7].

in normal position in the Euclidean n-space R^n and let ε be any given positive number. Then P^{n-1} may be imbedded in a one parameter family of manifold $M(t)$ $(|t|<1)$ which satisfy the above conditions (1)-(4).

Proof. According to Theorem 1, P^{n-1} is in normal position in R^n. Therefore there exists, through every point p of P^{n-1}, a straight line $H(p)$ transversal to P^{n-1} at p which varies continuously with p. It is well known that every $(n-1)$-manifold in the Euclidean n-space R^n is orientable and divides R^n into two domains D_1 and D_2.

To show that P^{n-1} is in regular position in R^n, we are only to orient the straight line $H(p)$ at any point p of P^{n-1} from D_1 to D_2.

2. Curvatura integra and isotopy of $(n-1)$-manifolds in the Euclidean n-space.

Let P^{n-1} be a compact polyhedral $(n-1)$-manifold in regular position in the Euclidean n-space R^n. Then we may define through every point p of P^{n-1} a unit vector $v(p)$ which varies continuously with p and transversal to P^{n-1} at p. Making correspond the point p to $v(p)$, we shall obtain a mapping φ of P^{n-1} into the unit sphere of R^n. As P^{n-1} is orientable, we may define the degree of φ. Then we obtain the following

Lemma 4. *Under the above conditions, the degree of φ is independent of $v(p)$ defined on P^{n-1} provided that $v(p)$ varies continuously with p and is transversal to P^{n-1} at p.*

Proof. Let $v(p)$ and $w(p)$ be transversal unit vectors on P^{n-1} and let φ and ψ be the mappings of P^{n-1} to the unit sphere S^{n-1} of R^n defined by $v(p)$ and $w(p)$ respectively. To prove the lemma, we shall show that there exists a mapping $\Phi(p, t)$ of $P^{n-1}\times I$[8] into S^{n-1} such that $\varphi(p)=\Phi(p, 0)$, $\psi(p)=\Phi(p, 1)$ and the unit vector $v_t(p)$ through p and parallel to the unit vector $\overrightarrow{O\cdot\Phi(p, t)}$ through the origin O of R^n is transversal to P^{n-1} at p.

We shall define $\Phi(p, t)$ on $P^{n-1}\times 0\cup P^{n-1}\times 1$ so that $\Phi(p, 0)=\varphi(p)$ and $\Phi(p, 1)=\psi(p)$. Then we extend to the other part of $P^{n-1}\times I$ by induction on the dimension of the simplexes of P^{n-1}.

First we shall define Φ on $s^o\times I$ for every vertex of s^o on P^{n-1}. Let $T(s^o)$ be the totality of the unit vectors transversal to P^{n-1} at s^o. As $\varphi(s^o)$ and $\psi(s^o)$ fall into $T(s^o)$ which is an open $(n-1)$-cell by Lemma 3, we may extend the mapping Φ over $s^o\times I$ so that $\Phi(p, t)$ falls

8) I is the unit interval $[0, 1]$.

into $T(s^0)$, that the unit vector $v_t(p)$ through p and parallel to the unit vector $\overrightarrow{O\cdot\Phi(p, t)}$ through O is transversal to P^{n-1} at p.

If Φ is defined on $s^k \times I$ for every k-simplex $(k<j)$ of P^{n-1}, then we shall extend Φ to $s^j \times I$ for any j-simplex of P^{n-1}. Let $T(s^j)$ be the totality of the unit vectors through O transversal to the star $St(s^j)$ of s^j. Φ is already defined on the boundary of $s^j \times I$. If p is an element of the boundary of $s^j \times I$, then $v_t(p)$ is transversal to P^{n-1} at p. As $T(s^j)$ is an open $(n-1)$-cell, we may extend the mapping Φ over $s^j \times I$ so that $\Phi(p, t)$ falls into $T(s^j)$, that is, that the unit vector through p and parallel to the unit vector $\overrightarrow{O\cdot\Phi(p, t)}$ through O is transversal to P^{n-1} at p. Thus the lemma is proved.

Now we shall define the *curvatura integra* $d(P^{n-1})$ of P^{n-1} in R^n as the degree of the mapping φ.

If M^m is a differentiable m-manifold in some Euclidean space R^n, then, according to S. S. Cairns [1], M^m may be so triangulated into cells (σ^m) that the vertices of each m-cell determine a non singular m-simplex and that the totality of simplexes so determined is a polyhedral manifold P^m homeomorphic to M^m in such a way that the corresponding m-cells have identical vertices and that the tangent m-plane to M^m at any point of a cell σ^m of (σ^m) differs arbitrarily small in its direction from the m-plane determined by the simplex corresponding to σ^m. We shall call P^m a Cairns' approximation of M^m in R^n.

Now let p and q be the corresponding points on P^m and M^m respectively. If $H(p)$ is the straight line through p parallel to the normal line to M^m at q and P^m is a sufficiently close approximation of M^m, then $H(p)$ is, according to S. S. Cairns [2], transversal to P^m as required by the definition of normal lines. Therefore we obtain the following

Theorem 4. *If M^{n-1} is a compact differentiable manifold in the Euclidean n-space R^n, then the usual curvatura integra of M^{n-1} in R^n coincides with the curvatura integra of a polyhedral $(n-1)$-manifold which is a Cairns' approximation sufficiently close to M^{n-1} in R^n.*

Let P^m and Q^m be a polyhedral m-manifolds in some Euclidean space R^n. Then we shall say that P^m and Q^m are congruent (or isotopic) in R^n, if there exists an orientation preserving semi-linear homeomorphism ψ of R^n which satisfies $\psi(P^m)=Q^m$. Then there exists, according to V. K. A. M. Gugenheim [4], a semi-linear homeomorphism $\Phi(p, t) =(\phi_t(p), t)$ of $P^m \times I$ into $R^n \times I$ such that $\phi_t(p)$ is a semi-linear homeomorphism of P^m into R^n and that $\phi_0(P^m)=P^m$ and $\phi_1(P^m)=Q^m$. Then we shall obtain the following

Theorem 5. *If P^{n-1} and Q^{n-1} are compact polyhedral $(n-1)$-manifolds in regular position in the Euclidean n-space R^n, then $d(P^{n-1})=d(Q^{n-1})$.*

Proof. Let $M^{n-1}(t)$ be an analytic approximation lying in the outer part[9] of P^{n-1} which is defined in Theorem 3 for some fixed $t\neq 0$. Let P'^{n-1} be a Cairns' approximation of $M^{n-1}(t)$ that is contained in R[10] but does not intersect P^{n-1}.

If $H(p)$ is a transversal line through every point p of P^{n-1} which is defined in the condition (4) of Theorem 3, then $H(p)$ intersects P'^{n-1} only at one point. It is easily shown that there exists a conguence T' between P^{n-1} and P'^{n-1} by which every point p of P^{n-1} corresponds to the point $H(p)\cap P'^{n-1}$.

In the same way as P^{n-1}, we shall define a polyhedral $(n-1)$-manifold Q'^{n-1} such that it is in the neighborhood of Q^{n-1} and is lying in the outer part of Q^{n-1} and a transversal line $H'(q)$ through every point q of Q^{n-1} intersects Q'^{n-1} only at one point. Let T'' be a congruence between Q^{n-1} and Q'^{n-1} by which every point q of Q^{n-1} correspond to $H'(q)\cap Q'^{n-1}$.

Let T be a conguence between P^{n-1} ank Q^{n-1} and let p be any point of P^{n-1}. Making correspond every line-segment $\overline{p.T'(p)}$ to the line-segment $\overline{T(p)\ T''(T(p))}$, we obtain a semi-linear isomorphism between the polyhedron $\tilde{P}=\bigcup_{p\in P^{n-1}}\overline{P\cdot T'(p)}$ and the polyhedron $\tilde{Q}=\bigcup_{p\in Q^{n-1}}\overline{q\cdot T''(q)}$ which is easily extended to a congruence F between P and Q.

Corresponding to T and F, we get a semi-linear homeomorphism $\Psi(p, t)=(\phi_t(p), t)$ and $\Phi(p, u, t)=(\pi_t(p, u), t)$ of $P^{n-1}\times I$ and $\tilde{P}\times I=P\times I\times I$ into $R^n\times I$ respectively such that $\phi_0(P^{n-1})=P^{n-1}$, $\phi_1(P^{n-1})=Q^{n-1}$, $\pi_0(\tilde{P})=\tilde{P}$, $\pi_1(\tilde{P})=\tilde{Q}$ and $\phi_t(p)$ and $\pi_t(p, u)$ are semi-linear homeomorphisms of P^{n-1} and $\tilde{P}$ into R^n respectively.

Making correspond to every point p of $\phi_t(P^{n-1})$ a vector $v_t(p)$ defined by $\overrightarrow{\pi_t(p, 0)\ \pi_t(p, 1)}$, we obtain a vector field $\{v_t(p)\}$ on $P_t^{n-1}=\phi_t(P^{n-1})$. Then we obtain a mapping φ_t of P_t^{n-1} into the unit sphere of R^n by the help of this vector field. As $\phi_t(p)$ and $\pi_t(p, u)$ vary continuously with their arguments and t, the degree of mapping φ_t remains unaltered when t varies from 0 to 1. Since the degree of φ_0 and the degree of φ_1 coincide with $d(P^{n-1})$ and $d(Q^{n-1})$ respectively, the theorem is thus proved.

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9) P^{n-1} divides R^n into two domains. One of the closures of these domains is compact and the other is not compact. By the "outer part" of P^{n-1} we shall mean the domain whose closure is not compact.

10) For the definition of R, see the condition (4) of Theorem 3.

Since a Cairns' approximation of $M^{n-1}(t)$ sufficiently near to it is isotopic to P^{n-1}, we obtain from Theorem 4 and 5 the following

Corollary. *If P^{n-1} is a compact polyhedral $(n-1)$-manifold in regular position in R^n and if $M^{n-1}(t)$ is the manifold defined in Theorem* 3, *then the usual curvatura integra of $M^{n-1}(t)$ $(t \neq 0)$ in R^n is equal to the curvatura integra of P^{n-1} in R^n.*

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