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On Knots and Periodic Transformations[1)]

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Introduction

Let T be a homeomorphism of the 2-sphere S^2 onto itself. If T is regular[2)] except at a finite number of points, then it is proved by B. v. Kerékjártó [11] that T is topologically equivalent to a linear transformation of complex numbers. Now let T be a homeomorphism of the 3-sphere S^3 onto itself. If T is regular except at a finite number of points, then it is known[3)] that the number of points at which T is not regular is at most two. Furthermore it is also known[4)] that if T is regular except at just two points, then T is topologically equivalent to the dilatation of S^3. Let T be sense preserving and regular except at just one point. Then whether or not T is equivalent to the translation of S^3 is not proved yet[5)]. Now let T be regular at every point of S^3. In general, in this case, T can be more complicated[6)] and there remain difficult problems[7)].

In this paper we shall be concerned with sense preserving periodic transformations of S^3 onto itself, which is a special case of regular transformations of S^3. Furthermore suppose that T is different from the identity and has at least one fixed point. Then it has been shown by P. A. Smith [19] that the set F of all fixed points of T is a simple closed curve. It is proved by D. Montgomery and L. Zippin [13] that generally T is not equivalent to the rotation of S^3 about F. It will naturally be conjectured[8)] that if T is semilinear, then T is equivalent to the rotation of S^3. In this case F is, of course, a polygonal simple

1) A part of this paper was published in [12]. See also the footnote 11).

2) A homeomorphism T of a metric space X onto itself is called regular at $p \in X$, if for each $\varepsilon > 0$ there exists $\delta > 0$ such that if $d(p, x) < \delta$, then $d(T^n(p), T^n(x)) < \varepsilon$ for every integer n.

3) See T. Homma and S. Kinoshita [9].

4) See T. Homma and S. Kinoshita [8] [9].

5) See also H. Terasaka [21].

6) See R. H. Bing [3] D. Montgomery and L. Zippin [13].

7) See, for instance, [4] Problem 40.

8) See D. Montgomery and H. Samelson [14].

closed curve in S^3 and D. Montgomery and H. Samelson [14] has proved[9] that if F is a parallel knot of the type $(p, 2)$ then F is trivial in S^3 provided the period of T is two.

Now let M be a closed 3-manifold without boundary and with trivial 1-dimensional homology group[10,11]. If k is a polygonal simple closed curve in M, then we can define the g-fold cyclic covering space $M_g(k)$ of M, branched along k. Then in § 1 it will be proved that the fundamental group of M is isomorphic to a factor group of that of $M_g(k)$. Furthermore a fundamental formula of the Alexander polynomial of k in M (see (6)), which is proved by R. H. Fox [6] for $M=S^3$, will be given.

Now let k_0 be a polygonal simple closed curve in S^3, whose 2-fold cyclic covering space $M_2(k_0)$ of S^3, branched along k_0, is homeomorphic to S^3. Then it will be proved in § 2 that (i) the determinant of the knot k_0 must be equal to the square of an odd number, (ii) the degree of the Alexander polynomial of k_0 is not equal to two and that (iii) almost all knots of the Alexander-Briggs' table[12] are not equivalent to k_0, where k_0 is considered as a knot in $M_2(k_0)$. Similarly if k_1 is a polygonal simple closed curve in S^3, whose 3-fold cyclic covering space $M_3(k_1)$ of S^3, branched along k_1, is homeomorphic to S^3, then it will be proved that (i) the degree of the Alexander polynomial of k_1 is not equal to two and that (ii) almost all knots of the Alexander-Briggs' table[12] are not equivalent to k_1, where k_1 is considered as a knot in $M_3(k_1)$.

If T is a periodic transformation of S^3 described above, then the orbit space M is a simply connected 3-manifold. Furthermore S^3 is the p-fold cyclic covering space of M, branched along F, where p is the period of T. Therefore, under the assumption that the well known Poincaré conjecture of 3-manifolds is true[13], the results of § 2 can be naturally applied to the position of F in S^3. (See Theorem 5 and Theorem 6).

§ 1.

1. In this section M will denote a closed 3-manifold without boundary and with trivial 1-dimensional homology group. Let k be an

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9) See also C. D. Papakyriakopoulos [15] T. Homma [10].

10) In this paper we shall use only the integral homology group.

11) In [12] M was supposed to be only a 3-manifold without boundary. Professor R. H. Fox kindly pointed out to me that "the linking number Link (k, x_i)" in [12] is not well-defined for an arbitrary 3-manifold M. Some propositions on knots in M turn out thereby to be erroneous, although it does not affect my main results in § 5 of [12].

12) See [1] [12].

13) Meanwhile, this conjecture turned out to be unnecessary. See R. H. Fox [7].

oriented polygonal simple closed curve in M and let V be a sufficiently small tubular neighbourhood of k in M. Then the boundary $\dot{V}$ of V is a torus. A *meridian* of V is by definiton a simple closed curve on $\dot{V}$ which bounds a 2-cell in V but not on $\dot{V}$. Let x be an oriented meridian of V. For each simple closed curve y which does not intersect k we can define the *linking number* Link (k, y) of k and y[14]. Then

$$\text{Link}\,(k, x) = \pm 1\,.$$

We may always suppose that x is so oriented that

$$\text{Link}\,(k, x) = 1\,.$$

It is easy to see that for each integer $p(\neq 0)$ x^p is not homotopic to 1.

We shall denote the fundamental group of $M-k$ by $F(M-k)$ or sometimes by $F(k, M)$. Now let $\{x, x_1, x_2, \cdots, x_n\}$ be a complete set of generators of $F(M-k)$, where x stands for the element of the fundamental group corresponding to the path x. Put

$$\text{Link}\,(k, X_i) = L(i) \qquad (i = 1, 2, \cdots, n)$$

and

$$x_i = x^{-L(i)} X_i\,.$$

Then $\{x, x_1, x_2, \cdots, x_n\}$ forms again a complete set of generators of $F(M-k)$. For each i

$$\text{(1)} \qquad \text{Link}\,(k, x_i) = 0\,.$$

Let $R_s = 1$ $(s = 1, 2, \cdots, m)$ be a complete system of defining relations of $F(M-k)$ with respect to $\{x, x_1, \cdots, x_n\}$. Then the symbol

$$\text{(2)} \qquad \{x, x_1, \cdots, x_n : R_1, R_2, \cdots, R_m\}$$

will be called a *presentation*[15] of $F(M-k)$. It is easy to see that

$$\{x, x_1, \cdots, x_n : x, R_1, \cdots, R_m\}$$

is a presentation of $F(M)$. x being equal to unity, this presentation can be transformed to the following one:

$$\text{(3)} \qquad \{x_1, x_2, \cdots, x_n : \hat{R}_1, \hat{R}_2, \cdots, \hat{R}_m\}\,,$$

where $\hat{R}_s$ is obtained by deleting x from R_s.

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14) See [17] § 77.

15) See R. H. Fox [6].

2. Let $w \in F(k,M)$. Then w is written as a word which consists of at most $x, x_1, \cdots, x_n$. Let $f(w)$ be an integer which is equal to the exponent sum of w, summed over the element x. By (1) it is easy to see that f is a homomorphism of $F(k,M)$ onto the set of all integers. Now put

$$F_g(k, M) = \{w \in F(k, M) \mid f(w) = 0 \pmod{g}\} ,$$

where g is a positive integer. Then $F_g(k, M)$ is a normal subgroup of $F(k, M)$. Therefore there exists uniquely the *g-fold cyclic covering space* $\tilde{M}_g(k)$[16] *of* $M-k$, whose fundamental group is isomorphic to $F_g(k, M)$. Since x is a meridian of V, we can also define the *g-fold cyclic covering space* $M_g(k)$ *of* M, *branched along* k[17]. For each g $M_g(k)$ is a closed 3-manifold without boundary.

$F(\tilde{M}_g(k))$ and $F(M_g(k))$ are calculated from $F(k, M)$ as follows: Let (2) be a presentation of $F(k,M)$. Put

$$x_{ij} = x^j x_j x^{-j} . \qquad \begin{pmatrix} i = 1, 2, \cdots, n \\ j = 0, 1, \cdots, g-1 \end{pmatrix}$$

Since $f(R_s) = 0$ for every s $(s = 1, 2, \cdots, m)$, $x^t R_s x^{-t}$ $(t = 0, 1, \cdots, g-1)$ is expressible by a word which consists of at most x_{ij} and x^g. We denote it by notations

$$x^t R_s x^{-t} = \tilde{R}_{st} .$$

Then

$$(4) \qquad \{x^g,\ x_{ij} : \tilde{R}_{st}\}$$

is a presentation of $F(\tilde{M}_g(k))$ and

$$(5) \qquad \{x^g,\ x_{ij} : x^g,\ \tilde{R}_{st}\}$$

is one of $F(M_g(k))$.

Theorem 1. *$F(M)$ is isomorphic to a factor group of $F(M_g(k))$.*

Proof. Let (3) and (5) be presentations of $F(M)$ and $F(M_g(k))$, respectively. Let G be a group whose presentation is given by

$$\{y^g,\ y_i,\ y_{ij} : y^g,\ \tilde{R}_{st}(y^g,\ y_{ij}) ,\ y_{ij} y_i^{-1}\} .$$

This presentation can be transformed to the following one:

$$\{y_i : \hat{R}_s(y_i)\} .$$

16) See, for instance, [17].

17) See, for instance, H. Seifert [18].

Therefore $F(M)$ is isomorphic to G. On the other hand it is easy to see that G is isomorphic to a factor gronp of $F(M_g(k))$. Thus $F(M)$ is isomorphic to a factor group of $F(M_g(k))$, and our proof is complete.

3. Now let (2) be a presentation of $F(M-k)$. Replace the multiplication by the addition and put

$$jx \pm x_i - jx = \pm x^j x_i . \qquad \begin{pmatrix} i=1, 2, \cdots, n \\ j=0, \pm 1, \pm 2, \cdots \end{pmatrix}$$

Furthermore suppose that the addition is commutative. Then for each relation $R_s=1$ $(s=1, 2, \cdots, m)$ we have a relation $\bar{R}_s=0$, which is a linear equation of x_i. From these linear equations we can make the *Alexander matrix*, whose (s, i)-th term is the coefficient of x_i in $\bar{R}_s=0$. If we put $x=1$ in the Alexander matrix, then we have a matrix which gives the 1-dimensional homology group $H_1(M)$ of M. Since $H_1(M)$ is trivial by our assumption $m \geqq n$.

If two oriented knots k_1 and k_2 in M are equivalent to each other, then $F(k_1, M)$ and $F(k_2, M)$ are *directly isomorphic*[18]. It was proved by J. W. Alexander [2] that if two *indexed groups*[18] are directly isomorphic to each other, then the elementary factors different from unity of the Alexander matrices and also their products $\Delta(x, k_i, M)$ $(i=1, 2)$ are the same each other. Of course they are determined up to factors $\pm x^p$, where p is an integer. $\Delta(x, k, M)$ will be called the *Alexander polynomial of k in M.* Clearly $\Delta(1, k, M) = \pm 1$. It should be remarked that $\Delta(x, k, M_g(k))$[19] is also defined from (4) replacing x^g by x.

It can be proved that

$$(6) \qquad \Delta(x, k, M_g(k)) = \prod_{j=0}^{q-1} \Delta(\sqrt[g]{x}\ \omega_j, k, M),$$

where $\omega_j = \cos\frac{2\pi j}{g} + i \sin\frac{2\pi j}{g}$. This is known for the case $M=S^3$[20]. But as the proof of the latter depends essentially only on the following equation of determinants:

$$\begin{vmatrix} a_1 & a_2 & \cdots & a_g \\ xa_2 & a_1 & \cdots & a_{g-1} \\ \cdots & \cdots & \cdots & \cdots \\ xa_g & xa_3 & \cdots & a_1 \end{vmatrix} = \prod_{j=0}^{q-1} f(\sqrt[g]{x}\ \omega_j),$$

18) See J. W. Alexander [2].

19) We use the same symbol to a knot k in M and the knot which is the set of all branch points of $M_g(k)$. $\Delta(x, k, M_g(k))$ is the Alexander polynomial of k in $M_g(k)$, if $\Delta(1, k, M_g(k)) = \pm 1$. See also R. H. Fox [6].

20) See R. H. Fox [6].

where $f(y)=a_1+a_2y+\cdots+a_gy^{g-1}$, the proof for the general case is the same as for the case $M=S^3$ and is omitted.

As a special case of (6) we have

$$\Delta(1, k, M_g(k)) = \prod_{j=0}^{g-1} \Delta(\omega_j, k, M) . \tag{7}$$

$\Delta(1, k, M_g(k)) \neq 0$ if and only if the 1-dimensional Betti number $p_1(M_g(k))=0$. If $p_1(M_g(k))=0$, then $|\Delta(1, k, M_g(k))|$ is equal to the product of 1-dimensional torsion numbers. In this case if $|\Delta(1, k, M_g(k))|=1$, then $M_g(k)$ has no torsion number.

§ 2.

1. Let k_0 be a simple closed curve in the 3-sphere S^3 and M_2 the 2-fold cyclic covering space of S^3, branched along k_0. In No. 1 and 2 we assume that M_2 is homeomorphic to the 3-sphere and the position of k_0 in M_2 will be studied. These results will be used later in § 3. In No. 1 we prove only the following

Theorem 2. *The determinant of k_0 in M_2 must be equal to the square of an odd number.*

Proof. Let $\Delta(x)=\sum_{r=1}^{2n} a_r x^r$ be the Alexander polynomial of k_0 in S^3. Since the determinant d_0 of k_0 in M_2 is the product of torsion numbers of the 1-dimensional homology group of the 2-fold cyclic covering space of M_2, branched along k_0, it follows from (7) that

$$d_0 = |\Delta(1)\,\Delta(-1)\,\Delta(i)\,\Delta(-i)| .$$

By our assumptions $|\Delta(1)|=1$ and $|\Delta(-1)|=1$. Put

$$a = a_0-a_2+a_4-\cdots+(-1)^n a_{2n} ,$$
$$b = a_1-a_3+a_5-\cdots+(-1)^{n-1}a_{2n-1} .$$

Suppose first that n is even. Then

$$\Delta(i) = a_0+a_1i-a_2-a_3i+\cdots-a_{2n-2}-a_{2n-2}i+a_{2n} .$$

Since $a_r=a_{2n-r}$, $\Delta(i)=a$. Therefore $\Delta(-i)=a$. Then we have $d_0=a^2$.

Now suppose that n is odd. Then

$$\Delta(i) = a_0+a_1i-a_2-a_3i+\cdots+a_{2n-2}+a_{2n-1}i-a_{2n} .$$

Since $a_r=a_{2n-r}$, we have

$$\Delta(i) = bi+a_n(i)^n .$$

Therefore

$$\Delta(-i) = b(-i) + a_n(-i)^n = -(bi + a_n(i)^n)\,.$$

Thus we have

$$d_0 = -(bi + a_n(i)^n)^2 = (b \pm a_n)^2\,.$$

Since the determinant of a knot is always an odd number, our proof is complete.

2. Now let $\Delta(x)$ be the Alexander polynomial of k_0 in S^3 and $\Delta_2(x)$ that of k_0 in M_2. Then by (6)

$$\Delta_2(x) = \Delta(\sqrt{x})\,\Delta(-\sqrt{x})\,. \tag{8}$$

Therefore the degree of $\Delta(x)$ is equal to that of $\Delta_2(x)$.

Suppose first that the degree of $\Delta(x)$ is 2. Put

$$\Delta(x) = ax^2 + bx + a\,,$$

where $a \neq 0$ and we may assume that $2a + b = 1$. Then by (8)

$$\Delta_2(x) = a^2x^2 + (2a^2 - b^2)\,x + a^2\,.$$

Furthermore $4a^2 - b^2 = \pm 1$, which means that $2a - b = \pm 1$. From this it follows that $2a = 1$ or $2a = 0$. Since $a \neq 0$ and a is an integer, this is a contradiction. Thus we have proved that *the degree of* $\Delta_2(x)$ *is not equal to* 2.

Now suppose that the degree of $\Delta_2(x)$ is 4. Put

$$\Delta(x) = ax^4 + bx^3 + cx^2 + bx + a\,,$$

where $a \neq 0$ and we may assume that $2a + 2b + c = 1$. Then by (8)

$$\begin{aligned}\Delta_2(x) = a^2x^4 + (2ac - b^2)\,x^3 + (2a^2 - 2b^2 + c^2)\,x^2 \\ + (2ac - b^2)\,x + a^2\,.\end{aligned}$$

Furthermore $4a^2 + 4ac + c^2 - 4b^2 = \pm 1$, which means that $2a - 2b + c = \pm 1$. From this it follows that $4b = 2$ or $4b = 0$. Since b is an integer, $4b = 2$ is a contradiction. Therefore $b = 0$ and $c = 1 - 2a$. Thus we have proved that if the degree of $\Delta_2(x)$ is 4, then $\Delta_2(x)$ must be limited to the following form:

$$a^2x^4 - 2a(2a-1)\,x^3 + (6a^2 - 4a + 1)\,x^2 - \cdots\,.$$

By the same way it can be seen easily that if the degrees of $\Delta_2(x)$

are 6 and 8, then $\Delta_2(x)$ must be limited to the following forms, respectively:

$$\begin{aligned} &a^2x^6-(b^2+2a^2)x^5+(4b^2-a^2-2b)\,x^4 \\ &\qquad -(6b^2-4a^2-4b+1)\,x^3+\cdots, \\ &a^2x^8-(b^2-2ac)\,x^7+(c^2+2b^2-4a^2-4ac+2a)\,x^6 \\ &\qquad -(4c^2-b^2+2ac-2c)\,x^5 \\ &\qquad +(6c^2-4b^2+6a^2+8ac-4c-4a+1)\,x^4-\cdots. \end{aligned}$$

From these we have the following

Theorem 3. *All knots of the Alexander-Briggs' table, except for the cases* 8_9 *and* 8_{20}, *are not equivalent to* k_0 *in* M_2.

3. Now let k_1 be a simple closed curve in S^3 and M_3 the 3-fold cyclic covering space of S^3, branched along k_1. In No. 3 we assume that M_3 is homeomorphic to the 3-sphere and the position of k_1 in M_3 will be studied.

Let $\Delta(x)$ be the Alexander polynomial of k_1 in S^3 and $\Delta_3(x)$ that of k_1 in M_3. Then by (6)

$$\Delta_3(x)=\Delta(\sqrt[3]{x})\,\Delta(\omega_1\sqrt[3]{x})\,\Delta(\omega_2\sqrt[3]{x}), \tag{9}$$

where $\omega_1=\dfrac{-1+\sqrt{3}i}{2}$ and $\omega_2=\dfrac{-1-\sqrt{3}i}{2}$.

Suppose first that the degree of $\Delta(x)$ is 2. Put

$$\Delta(x)=ax^2+bx+a,$$

where $a\neq 0$ and we may assume that $2a+b=1$. Then by (9)

$$\Delta_3(x)=a^3x^2+(b^3-3a^2b)\,x+a^3.$$

Furthermore $2a^3-3a^2b+b^3=\pm 1$, which means that $a-b=\pm 1$. From this is follows that $3a=2$ or $3a=0$. Since $a\neq 0$ and a is an integer, this is a contradiction. Thus we have proved that *the degree of* $\Delta_3(x)$ *is not equal to* 2.

By the same way as that of No. 2 we have the following.

Theorem 4. *All knots of the Alexander-Briggs' table, except for the cases* 5_1, 7_1, 8_{10} *and* 9_{47}, *are not equivalent to* k_1 *in* M_3.

§ 3.

Now let T be a sense preserving (of course semilinear) periodic transformation of S^3 onto itself. Furthermore let T be different from the identity and have at least one fixed point. Then the set F of all fixed points of T is a simple closed curve[21]. Suppose that p is the minimal number of the set of all positive period of T. It is easy to see that T is primitive[22]. T acts locally as a rotation about F[23]. Then, if we identify the points

$$x,\ T(x)\ , \cdots ,\ T^{p-1}(x)$$

in S^3, we have an orientable 3-manifold M. It is easy to see that M is simply connected. Since T acts locally as a rotation about F in S^3, we can see that S^3 is the p-fold cyclic covering space of M, branched along F.

Now we assume that the Poincaré conjecture is true[13]. Then M is a 3-sphere.

First we consider the case $p=2$. Since S^3 is the 2-fold cyclic covering space of M, branched along F, we can apply the results of §2 to the position of F in S^3. Therefore we have the following

Theorem 5. *Let T be a periodic transformation described above. Furthermore suppose that the period of T is* 2. *Then, under the assumption that the Poincaré conjecture is true*[13], *we have that*

- (i) *the determinant of F must be equal to the square of an odd number,*
- (ii) *the degree of the Alexander polynomial of F is not equal to* 2 *and that*
- (iii) *all knots of the Alexander-Briggs' table, except for the cases* 8_9 *and* 8_{20}, *are not equivalent to F.*

Now we consider the case $p=3$. Since S^3 is the 3-fold cyclic covering space of M, branched along F, we have the following

Theorem 6. *Let T be a periodic transformation described above. Furthermore suppose that the period of T is* 3. *Then, under the assumption that the Poincaré conjecture is true*[13], *we have that*

- (i) *the degree of the Alexander polynomial of F is not equal to* 2 *and that*
- (ii) *all knots of the Alexander-Briggs' table, except for the cases* 5_1, 7_1, 8_{10} *and* 9_{47}, *are not equivalent to F.*

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21) See P. A. Smith [17].
22) See P. A. Smith [18].
23) See D. Montgomery and H. Samelson [12].

References

[1] J. W. Alexander and G. B. Briggs: On types of knotted curve, Ann. Math. 28 (1927), 562–586.

[2] J. W. Alexander: Topological invariants of knots and links, Trans. Amer. Math. Soc. 30 (1928), 275–306.

[3] R. H. Bing: A homeomorphism between the 3-sphere and the sum of two horned sphere, Ann. Math. 56 (1952), 354–362.

[4] S. Eilenberg: On the problems of topology, Ann. Math. 50 (1949), 247–260.

[5] R. H. Fox: Free differential calculus II, Ann. Math. 59 (1954), 196–210.

[6] R. H. Fox: Free differential calculus III, Ann. Math. 64 (1956), 407–419.

[7] R. H. Fox: On knots whose points are fixed under a periodic transformation of the 3-sphere, Osaka Math. J. 10 (1958).

[8] T. Homma and S. Kinoshita: On a topological characterization of the dilatation in E^3, Osaka Math. J. 6 (1954), 135–144.

[9] T. Homma and S. Kinoshita: On homeomorphisms which are regular except for a finite number of points, Osaka Math. J. 7 (1955), 29–38.

[10] T. Homma: On Dehn's lemma for S^3, Yokohama Math. J. 5 (1957), 223–244.

[11] B. v. Kerékjártó: Topologische Charakterisierung der linearen Abbildungen, Acta Litt. ac. Sci. Szeged 6 (1934), 235–262.

[12] S. Kinoshita: Notes on knots and periodic transformations, Proc. Japan Acad. 33 (1957), 358–361.

[13] D. Montgomery and L. Zippin: Examples of transformation groups, Proc. Amer. Math. Soc. 5 (1954), 460–465.

[14] D. Montgomery and H. Samelson: A theorem on fixed points of involutions in S^3, Can. J. Math. 7 (1955), 208–220.

[15] C. D. Papakyriakopoulos: On Dehn's lemma and the asphericity of knots, Ann. Math. 66 (1957), 1–26.

[16] K. Reidemeister: Knotentheorie, Berlin (1932).

[17] H. Seifert and W. Threlfall: Lehrbuch der Topologie, Leipzig (1935).

[18] H. Seifert: Ueber das Geschlecht von Knoten, Math. Ann. 110 (1935), 571–592.

[19] P. A. Smith: Transformations of finite period II, Ann. Math. 40 (1939), 497–514.

[20] P. A. Smith: Fixed points of periodic transformations, Appendix B in Lefschetz, Algebraic topology (1942).

[21] H. Terasaka: On quasi-translations in E^n, Proc. Japan Acad. 30 (1954), 80–84.