Kurata, Yoshiki
Osaka Math. J.
9 (1957), 189-193

Decompositions of Semi-Prime Rings and Jordan Isomorphisms

By Yoshiki KURATA

§ 1. Recently, in his paper [1], A. W. Goldie studied a minimal decomposition set of a semi-simple ring. In § 2, we shall consider a minimal decomposition set of a semi-prime ring which is an extension of that of semi-simple rings, and shall obtain a generalization of a theorem of M. Nagata [4], Proposition 34.

In § 3, we shall consider a Jordan isomorphism of a ring onto a semi-prime ring. Combining with the results in § 2. we shall obtain some generalizations of theorems due to I. Kaplansky ([2], Theorems 1 and 3) under the assumption that the prime rings are not of characteristic 2.

The writer is grateful to Mr. H. Tominaga for his kind advices.

§ 2. A ring R is called a *semi-prime ring* if R is isomorphic to a subdirect sum of prime rings, that is, if there exist prime ideals $\mathfrak{P}_\lambda$ $(\lambda \in \Lambda)$ in R such that $\bigcap_{\lambda\in\Lambda} \mathfrak{P}_\lambda = 0$. Whenever Λ_0 is a subset of Λ such that $\bigcap_{\lambda\in\Lambda_0} \mathfrak{P}_\lambda = 0$, R is isomorphic to a subdirect sum of prime rings $R/\mathfrak{P}_\lambda$ $(\lambda \in \Lambda_0)$. Such a representation of R is said to be *irredundant* if R is not isomorphic to a subdirect sum of any proper subset of these rings.

A semi-prime ring R has an irredundant representation if and only if R satisfies the following conditions: There exist prime ideals $\mathfrak{P}_\lambda$ $(\lambda \in \Lambda_0)$ in R such that $\bigcap_{\lambda\in\Lambda_0} \mathfrak{P}_\lambda = 0$ and $\bigcap_{\lambda\in\Lambda_1} \mathfrak{P}_\lambda \neq 0$ for any proper subset Λ_1 in Λ_0. A set of prime ideals which satisfies the above conditions is called a *minimal decomposition set for R* (we shall abbreviate it to *m. d. s.*). If we denote $\mathfrak{P}_\lambda^* = \bigcap_{\nu\in\Lambda_0,\ \nu\neq\lambda} \mathfrak{P}_\nu$, the latter condition may be replaced by $\mathfrak{P}_\lambda^* \neq 0$ for all $\lambda \in \Lambda_0$.

Lemma 1. *Let $\mathfrak{P}$ be a prime ideal of a ring R.*

(i) *If $\mathfrak{A}$ is a two-sided ideal of R, then we have either $\mathfrak{A} \subseteqq \mathfrak{P}$ or $r(\mathfrak{A})$*[1] $\subseteqq \mathfrak{P}$.

1) $r(\mathfrak{A})$ $(l(\mathfrak{A}))$ denotes the right (left) annihilator of $\mathfrak{A}$ in R. If R is semi-prime, then $r(\mathfrak{A}) = l(\mathfrak{A})$ for any two-sided ideal $\mathfrak{A}$ in R.

(ii) *If, in particular, R is semi-prime and $r(\mathfrak{P}) \neq 0$, then $\mathfrak{P}$ is a minimal prime ideal in R.*

Proof. The first part is obvious by the definition of prime ideals. To prove the second part we suppose that $\mathfrak{S}$ is a prime ideal in R such that $\mathfrak{S} \subseteqq \mathfrak{P}$. Then we have either $\mathfrak{P} \subseteqq \mathfrak{S}$ or $r(\mathfrak{P}) \subseteqq \mathfrak{S}$ by (i). The latter implies $r(\mathfrak{P})^2 = 0$, which is a contradiction because R has no non-zero nilpotent ideals.

Lemma 2. *Let $\{\mathfrak{P}_\lambda\}_{\lambda \in \Lambda_0}$ be an m. d. s. for a semi-prime ring R. Then $\mathfrak{P}_\lambda$ is a minimal prime ideal in R.*

Proof. It follows from Lemma 1 (ii).

The following theorem is a generalization of [4], Proposition 34.

Theorem 3. *A semi-prime ring R has at most one minimal decomposition set of prime ideals.*

Proof. Let $\{\mathfrak{P}_\lambda\}_{\lambda \in \Lambda_0}$ be an m. d. s. for R and let $\mathfrak{P}$ be a member of another m. d. s. for R. For each $\lambda \in \Lambda_0$, we have either $\mathfrak{P} \subseteqq \mathfrak{P}_\lambda$ or $r(\mathfrak{P}) \subseteqq \mathfrak{P}_\lambda$ by Lemma 1 (i). Since $\bigcap_{\lambda \in \Lambda_0} \mathfrak{P}_\lambda = 0$ and $r(\mathfrak{P}) \neq 0$, $\mathfrak{P} \subseteqq \mathfrak{P}_\lambda$ for some $\lambda \in \Lambda_0$. It follows from Lemma 2 that $\mathfrak{P} = \mathfrak{P}_\lambda$.

REMARK. From the above proof, we see that an m. d. s. for a semi-prime ring R (if there exists) consists of all prime ideals of R with non-zero right (left) annihilators.

Let S be the complete direct sum of prime rings R_λ $(\lambda \in \Lambda)$. Then, for $x_\lambda \in R_\lambda$, (x_λ) will signify the element of S whose λ–component is x_λ. Now let R_λ'' be the subring of S consisting of all elements with zeros in all ν–th places for $\nu \neq \lambda$. Identifying R_λ'' with R_λ, we obtain $S = R_\lambda \oplus R_\lambda^\circ$ for all $\lambda \in \Lambda$, where R_λ° is an ideal of S consisting of all elements with zero in the λ–th place. A subring R of S is a subdirect sum of the prime rings R_λ $(\lambda \in \Lambda)$, if the set of its λ–th components is equal to R_λ for each $\lambda \in \Lambda$.

Theorem 4. *$\{R \cap R_\lambda^\circ\}_{\lambda \in \Lambda}$ is an m. d. s. for a subdirect sum R of R_λ $(\lambda \in \Lambda)$ if and only if $R \cap R_\lambda \neq 0$ for all $\lambda \in \Lambda$.*

Proof. The ideal $R \cap R_\lambda^\circ$ is a prime ideal of R, since $R/(R \cap R_\lambda^\circ) \cong (R + R_\lambda^\circ)/R_\lambda^\circ \cong R_\lambda$. Evidently we have $\bigcap_{\lambda \in \Lambda} (R \cap R_\lambda^\circ) = R \cap \bigcap_{\lambda \in \Lambda} R_\lambda^\circ = 0$, and $(R \cap R_\lambda^\circ)^* = R \cap (R_\lambda^\circ)^* = R \cap R_\lambda \neq 0$.

The converse part is clear.

Next, we assume that R is a special subdirect sum[2] of prime rings R_λ $(\lambda \in \Lambda)$, that is, R contains R_λ'' for all $\lambda \in \Lambda$. In this case we have $R = R_\lambda \oplus (R \cap R_\lambda^\circ)$, and hence we have

Corollary. *If R is a special subdirect sum of prime rings R_λ $(\lambda \in \Lambda)$, R has an m.d.s. $\{R \cap R_\lambda^\circ\}_{\lambda \in \Lambda}$.*

Furthermore, by Theorem 3, $\{R \cap R_\lambda^\circ\}_{\lambda \in \Lambda}$ is the unique m.d.s. for R, and hence the totality of R_λ which is equal to $(R \cap R_\lambda^\circ)^*$ exhausts the unique prime components in our special subdirect sum representation of R.

§3. A mapping φ of a ring R into another ring R' is called a *Jordan homomorphism* of R into R' if it satisfies the following conditions:

$$(1) \qquad \varphi(x+y) = \varphi(x) + \varphi(y)\,,$$

$$(2) \qquad \varphi(xy+yx) = \varphi(x)\varphi(y) + \varphi(y)\varphi(x)$$

for all x and y in R. In case R' is not of characteristic 2 ($2x'=0$ implies $x'=0$), (2′) is equivalent to

$$(2') \qquad \varphi(x^2) = \varphi(x)^2\,.$$

If, in particular, φ is one-to-one, then we shall say it is a *Jordan isomorphism.*

In his paper [5], M. F. Smiley proved that a Jordan homomorphism of a ring onto a prime ring which is not of characteristic 2 is either a homomorphism or an anti-homomorphism. In this section, we shall consider a Jordan isomorphism of a ring R onto a semi-prime ring R'. Throughout this section, we assume that R' is a semi-prime ring which is represented as a subdirect sum of prime rings R_λ' $(\lambda \ni \Lambda)$.

Theorem 5. *Let φ be a Jordan isomorphism of a ring R onto R'. Suppose that R_λ' is not of characteristic 2 for each $\lambda \in \Lambda$. Then R is also semi-prime and isomorphic to a subdirect sum of prime rings each of which is either isomorphic or anti-isomorphic to some of R_λ'.*

Proof. The mapping φ_λ: $R \ni x \to \varphi(x)_\lambda \in R_\lambda'$ is an onto Jordan homomorphism by our assumptions, where $\varphi(x) = (\varphi(x)_\lambda)$. By a theorem obtained by M. F. Smiley, φ_λ is either a homomorphism or an anti-homomorphism. If we denote the kernel of φ_λ by $\mathfrak{P}_\lambda$, then $\mathfrak{P}_\lambda$ is a two-sided ideal in R and φ_λ induces either an isomorphism or an anti-isomorphism of $R/\mathfrak{P}_\lambda$ onto R_λ'. Hence $\mathfrak{P}_\lambda$ is a prime ideal in R. More-

2) See [3], §9.

over, if $x \in \bigcap_{\lambda \in \Lambda} \mathfrak{P}_\lambda, \varphi(x)_\lambda = 0$ for all $\lambda \in \Lambda$, and therefore $\varphi(x) = 0$. Since φ is one-to-one, $x = 0$. Hence R is isomorphic to a subdirect sum of the prime rings $R/\mathfrak{P}_\lambda$ $(\lambda \in \Lambda)$.

REMARK. In case R' is a special subdirect sum of R_λ' $(\lambda \in \Lambda)$, our assumption for the characteristic is nothing but to say that R' is not of characteristic 2.

Now, we suppose that R' is a special subdirect sum of prime rings R_λ' $(\lambda \in \Lambda)$ and is not of characteristic 2, and suppose that φ is a Jordan isomorphism of a ring R onto R'. Then as is shown in Theorem 5, R is semi-prime, that is, there exist prime ideals $\mathfrak{P}_\lambda$ $(\lambda \in \Lambda)$ in R such that $\bigcap_{\lambda \in \Lambda} \mathfrak{P}_\lambda = 0$. Evidently, $R = \mathfrak{P}_\lambda^* \oplus \mathfrak{P}_\lambda$ corresponding to $R' = R_\lambda' \oplus (R' \cap R_\lambda'^{\circ})$. Then we can see that R is also a special subdirect sum of prime rings $\mathfrak{P}_\lambda^*$ $(\lambda \in \Lambda)$.[3] And so, by Theorem 3 and Corollary to Theorem 4, $\{\mathfrak{P}_\lambda\}_{\lambda \in \Lambda}$ is the unique m. d. s. for R. Accordingly, if we denote $\mathfrak{P}_\lambda^*$ by R_λ, then the totality of R_λ exhausts the unique prime components in our special subdirect sum representation of R. On the other hand, the totality of R_λ' are those of R'. Now we shall prove the following theorem which corresponds to [2], Theorem 3.

Theorem 6. *Under the above situation, the prime components of R and R' can be paired off in such a way that φ is an isomorphism or an anti-isomorphism of each pair.*

Proof. $\varphi(R_\lambda) = \varphi(\mathfrak{P}_\lambda)^* = (R' \cap R_\lambda'^{\circ})^* = R_\lambda'$. Hence, φ is a Jordan isomorphism of R_λ onto R_λ' and thus our proof is completed by [5].

Finally, we shall prove the following theorem which corresponds to [2], Theorem 1.

Theorem 7. *Let φ, R and R' be as in Theorem 5. Then φ induces in $V_R(R)$[4] an isomorphism onto $V_{R'}(R')$.*

Proof. Let x be in $V_R(R)$, and let y_λ' be in R_λ'. Taking an element y in R with $\varphi_\lambda(y) = y_\lambda'$, we have

$$\begin{aligned} &\varphi(x)_\lambda \cdot y_\lambda' = \varphi_\lambda(x) \cdot \varphi_\lambda(y) \\ &= \begin{cases} \varphi_\lambda(xy) = \varphi_\lambda(yx) & \text{if } \varphi_\lambda \text{ is a homomorphism,} \\ \varphi_\lambda(yx) = \varphi_\lambda(xy) & \text{if } \varphi_\lambda \text{ is an anti-homomorphism,} \end{cases} \\ &= \varphi_\lambda(y) \cdot \varphi_\lambda(x) = y_\lambda' \cdot \varphi(x)_\lambda , \end{aligned}$$

which proves $\varphi(V_R(R)) \subseteq V_{R'}(R')$.

3) See [3], Theorem 15.

4) $V_R(R)$ denotes the center of R.

Conversely, let $\varphi(x)$ be an arbitrary element in $V_{R'}(R')$. Then $\varphi_\lambda(x)\cdot\varphi_\lambda(y)=\varphi_\lambda(y)\cdot\varphi_\lambda(x)$ for any y in R. As φ_λ is either a homomorphism or an anti-homomorphism, by the last equality one will readily see $\varphi(xy)=\varphi(yx)=\varphi(x)\cdot\varphi(y)$. This completes our proof.

(Received September 26, 1957)

Reference

[1] A. W. Goldie: Decompositions of semi-simple rings, J. London Math. Soc., **31** (1956), 40–48.

[2] I. Kaplansky: Semi-automorphisms of rings, Duke Math. J., **14** (1947), 521–525.

[3] N. H. McCoy: Subdirect sums of rings, Bull. Amer. Math. Soc., **53** (1947), 856–877.

[4] M. Nagata: On the theory of radicals in a ring, J. Math. Soc. Japan, **3** (1951), 330–343.

[5] M. F. Smiley: Jordan homomorphisms onto prime rings, Trans. Amer. Math. Soc., **84** (1957), 426–429.

Department of Mathematics,
Hokkaido University