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On the Graphs of Knots

By Takeshi YAJIMA and Shin'ichi KINOSHITA[1)]

1. Introduction.

A projection of a knot or a link on a 2-dimensional plane divides the plane into several regions. It is a frequently used method to separate these regions into two classes, white regions and black regions, in the study of the knot theory. Simplifying this method, C. Bankwitz [1] introduced the notion of graph of knots in his study of alternating knots. R. J. Aumann also used the graph of knots in [3].

The present note contains a method of a graphical treatment of knots and links, and, as an application, a sufficient condition is given that a knot is amphibious. Other applications have been given in [5].

2. The graph of knots.

Let π be a regular normed projection[2)] of a knot K on a 2-dimensional sphere S. If π has n double points $D_1, D_2, \cdots, D_n$, then it divides S into $n+2$ regions, each of which is homeomorphic to an open disk. Now, separate these regions into two classes $\mathfrak{A}$ and $\mathfrak{B}$, in such a way that each segment of π, i.e. an arc from a double point to the next one, is always the common boundary of a region of $\mathfrak{A}$ and that of $\mathfrak{B}$. We assume for convenience that the point at infinity is contained in a region of class $\mathfrak{B}$.

Let $A_1, A_2, \cdots, A_\alpha$ be the regions of class $\mathfrak{A}$. Take points $P_i \in A_i$ $(i=1, 2, \cdots, \alpha)$ and connect these points by n non-intersecting arcs $d_1, d_2, \cdots, d_n$ in such a way that each d_k corresponds to D_k $(k=1, 2, \cdots, n)$ and P_i and P_j are connected by d_k, if and only if A_i and A_j have a common double point D_k on their boundaries.

The regions of class $\mathfrak{A}$ in all can be considered as a projection of a surface spanning K which is twisted 180° at each double point of π.

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1) The authors express our gratitude to Professor Hidetaka Terasaka for his kind suggestions.

2) See [2], p. 6.

From this consideration, we define the sign of d_k $(k=1, 2, \cdots, n)$ in such a way that it is $+$ or $-$ according as the surface is twisted at D_k as the right screw or the left one, respectively (Fig. 1).

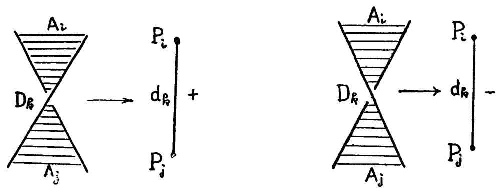


Fig 1

Then we have a linear graph of α vertices $P_1, P_2, \cdots, P_\alpha$, and n signed arcs $d_1, d_2, \cdots, d_n$; each vertex of which corresponds to a region of $\mathfrak{A}$, and each signed arc corresponds to a normed double point. We call this linear graph the *graph* of the projection π and denote it by $g(\pi)$ (c.f. 5. (1) for links).

From the same consideration about the class $\mathfrak{B}$, we get another graph $g'(\pi)$. We call this the *dual graph* of $g(\pi)$. We get the dual graph $g'(\pi)$ directly from a given graph $g(\pi)$ as follows. Let $B_1', B_2', \cdots, B_\beta'$ be the regions of $S-g(\pi)$. Take points $Q_i \in B_i'$ $(i=1, 2, \cdots, \beta)$ and connect these points by n non-intersecting arcs $d_1', d'_2, \cdots, d_n'$ in such a way that each d_k' corresponds to d_k of $g(\pi)$ $(k=1, 2, \cdots, n)$ and Q_i and Q_j are connected by d_k', if and only if d_k is the common boundary of B_i' and B_j'. Each arc d_k' takes the sign opposite to the sign of d_k. Thus we have a graph $g'(\pi)$ which consists of vertices $Q_1, Q_2, \cdots, Q_\beta$ and arcs $d_1', d_2', \cdots, d_n'$.

The following examples show several graphs of the trefoil knot.

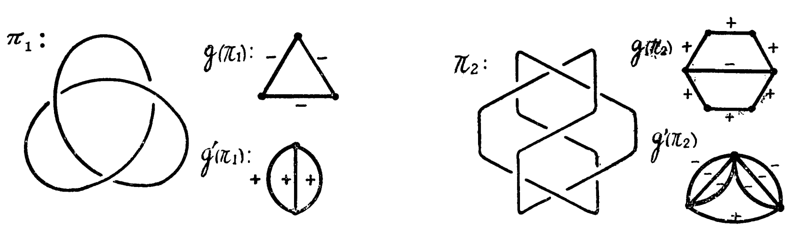


Fig. 2

REMARK 1. For a given projection π of a knot or a link, we have two graphs $g(\pi)$ and $g'(\pi)$. Conversely, for a given graph, whichever $g(\pi)$ or $g'(\pi)$, there is a uniquely determined projection π.

REMARK 2. For the sake of simplicity we may omit one of the signs $+$ or $-$ from graphs.

REMARK 3. Let π and π^* be projections of knots K and K^* respectively, which are mirror images to each other. Then $g(\pi)$ and $g(\pi^*)$ are of the same type and have the opposite signs. We call $g(\pi^*)$ the *conjugate graph* of $g(\pi)$.

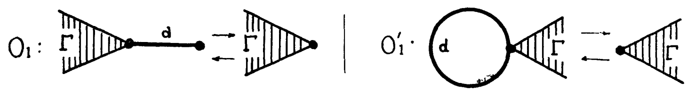
REMARK 4. R. J. Aumann [3] defined the canonical form K' from a graph $g(\pi)$ of a knot K. K' is a knot equivalent to K situating on an orientable 2–manifold T. In general $T-K'$ is not connected. However, we can prove that at least one of $T-K'$ and $T'-K''$ is connected where K'' is the canonical form from $g'(\pi)$ and T' is the orientable 2–manifold with respect to K''.

3. The operations on graphs.

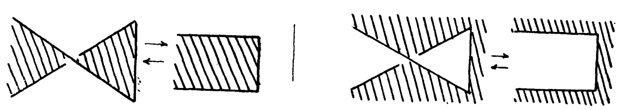
If the intersection of a connected subcomplex Γ of a graph $g(\pi)$ with the remaining subcomplex $\overline{g(\pi)-\Gamma}$ consists of at most two vertices, then we call Γ a *block* of $g(\pi)$. A single arc is the smallest block of graphs. If blocks $\Gamma_1, \cdots, \Gamma_k$ have two common vertices then we say that these blocks are *connected in parallel*. If blocks Γ_1 and Γ_2 have a common vertex which does not belong to any others, then we say that Γ_1 and Γ_2 are *connected in a series*. A common vertex of blocks will be called a *joint*.

It is well known that every deformation of a knot is equivalent to a combination of the fundamental deformation Ω_1, Ω_2 and Ω_3 in [2][3]. These fundamental deformations can be translated into the operations on graphs as follows. Since there are two graphs $g(\pi)$ and $g'(\pi)$ for a projection π of a knot, we have two operations on graphs O_i and O_i' for a deformation Ω_i. Such operations will be called the *dual operations* of each other.

(3.1) *If a graph $g(\pi)$ consists of a block Γ and an arc d whose intersection is a single joint, then d can be omitted. If a graph $g(\pi)$ consists of a block Γ and a closed arc d whose intersection is a single joint, then d can be omitted. The converse operations are also true.*

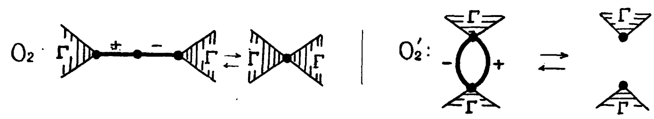


These statements are verified by the following projections.

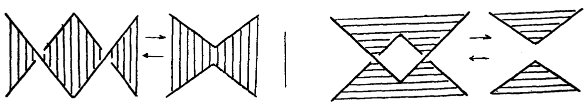


3) See [2], p. 7.

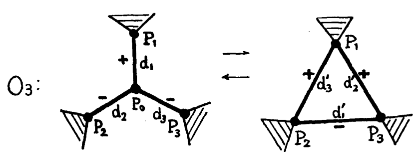
(3. 2) *If a graph* $g(\pi)$ *consists of a block* Γ *and two arcs* d_1, d_2 *in opposite signs which are connected in a series* Γ, d_1, d_2, Γ, *then the arcs can be omitted affixing the joints of* Γ. *If a graph* $g(\pi)$ *consists of a block* Γ *and two arcs* d_1, d_2 *in opposite signs which are connected in parallel, then the arcs can be omitted. The converse operations are also true.*



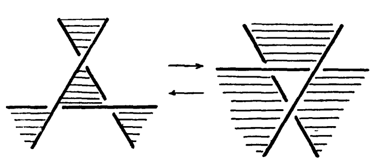
The following projections verify the statements.



(3. 3) *If vertices* P_1, P_2, P_3 *are connected with a vertex* P_0 *by arcs* d_1, d_2, d_3 *respectively where one of the arcs is in opposite sign from the others, then* P_0 *can be omitted and each* d_i $(i=1, 2, 3)$ *replaced by the arc* d_i' *connecting the vertices* P_j, P_k *with the opposite sign to* d_i, *where the suffices* j, k *are different from* i *and from each other. The converse operation is also true.*

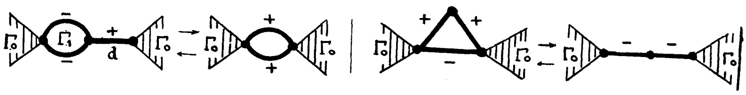


In this case O_3 and O_3' coincide. The corresponding projection is as follows.

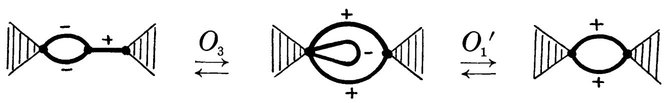


From the fundamental operations we have the following corollaries.

(3. 4) *If a graph* $g(\pi)$ *consists of two blocks* Γ_0, Γ_1 *and an arc* d *which are connected in a series* Γ_0, Γ_1, d, Γ_0 *where* Γ_1 *consists of two arcs of the same sign which are connected in parallel and* d *is in opposite sign from* Γ_1, *then* d *can be omitted and* Γ_1 *changed the sign. The converse and their dual operations are also true.*

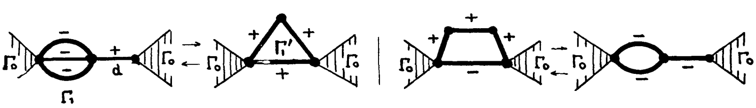


Proof.

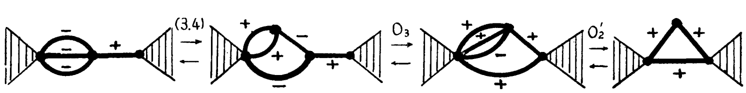


The statement of the right hand side is proved by the correspnding dual operations of the left one.

(3. 5) *If a graph $g(\pi)$ consists of two blocks Γ_0, Γ_1 and an arc d which are connected in a series Γ_0, Γ_1, d, Γ_0 where Γ_1 consists of three arcs of the same sign connected in parallel and where d is in opposite sign to Γ_1, then d can be omitted and Γ_1 replaced by the triangular block Γ_1' of the opposite sign from Γ_1. The converse and their dual operations are also true.*



Proof.



The statement of the right hand side is proved by the corresponding dual operations of the left one.

The above-mentioned operations on graphs correspond to the local deformations of knots. We have further operations on graphs, which correspond to the deformations in the global. They are the consequence of the fundamental operations on graphs.

(3. 6) *Every graph $g(\pi)$ of a knot is equivalent to its dual graph $g'(\pi)$ by the fundamental operations.*

Proof[4]. Suppose $g(\pi)$ is a graph of the class $\mathfrak{A}$, and B_∞ is the region of $S-\pi$ containing the point at infinity. Let D_1 be a double point of π on the boundary of B_∞, and D_2 be a consecutive double point of π along the boundary of B_∞. Let s be a bounding segment of B_∞ whose boundaries are D_1 and D_2. Let D_1', D_2' be two points on s, and

4) We can also prove (3, 6) by a pure graphical method.

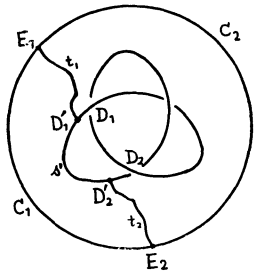


Fig. 3

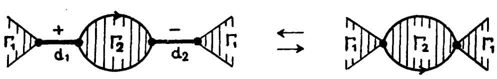
s' be a sub-segment of s whose boundaries are D_1' and D_2'.

We can suppose that π is contained in a circle C. If we take two points E_1, E_2 on the circle C, then C is divided into two arcs C_1 and C_2. We can connect E_1 with D_1' and E_2 with D_2' by disjoint arcs t_1 and t_2 respectively where t_1 and t_2 do not intersect π and C except their boundaries. Put $\pi_i = \pi - s' + (C_i + t_1 + t_2)$, $(i=1, 2)$. Either $C_1+t_1+t_2$ or $C_2+t_1+t_2$ can be deformed into s' in $S-\pi+s'$. Assume the former. Then we have $g(\pi_1)=g(\pi)$ and $g(\pi_2)=g'(\pi)$. Since π_1 is equivalent to π_2 by the deformation of C_1 into C_2, $g(\pi_1)$ is equivalent to $g(\pi_2)$ by the fundamental operations. Hence $g(\pi)$ and $g'(\pi)$ are equivalent.

(3.7) *If a graph $g(\pi)$ consists of blocks $\Gamma_1, \cdots, \Gamma_k$ which are connected in parallel, then we can deform a block in a cyclic order.*

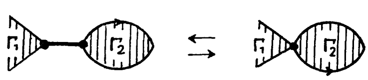
Since $g(\pi)$ is situated on a sphere, the statement is evident.

(3.8) *If a graph $g(\pi)$ consists of two blocks Γ_1, Γ_2 and two arcs d_1, d_2 in opposite signs such that they are connected in a series Γ_1, d_1, Γ_2, d_2, Γ_1, then the arcs can be omitted with a rotation of Γ_2 in 180° around the axis through the joints of Γ_2. The converse operation is also true.*



This statement corresponds to the deformation Ω_5 in [2][5].

(3.9) *If a graph $g(\pi)$ consists of two blocks Γ_1, Γ_2 and an arc d in any signs such that they are connected in a series Γ_1, d, Γ_2, then d can be omitted with a rotation of Γ_2 in 180°. The converse operation is also true.*



This is the operation corresponding to the deformation Ω_4 in [2][6], and the proof may be omitted. However, from O_1, O_2 and (3.8) it is easy to prove the statement by the graphical method.

5) See [2], p. 11.
6) See [2], p. 10.

4. Applications.

To trace the process of deformations of knots is a main purpose of the graphical representation of knots. For example, the equivalence of π_2 and π_1 (Fig. 2) is proved as follows.

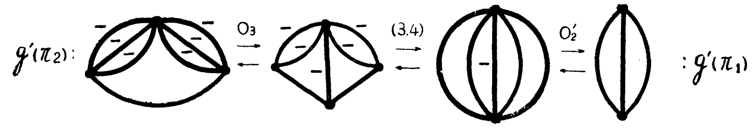


Fig. 4

Thus the deformation of knots, therefore the equivalence of knots, is explained schematically by the graphical representation of knots.

The graphical representation of knots also make a contribution to the study of construction of knots. For example, as the union[7] of knots 3_1 and 4_1 in the table of Alexander-Briggs[8], we have the following possible cases.

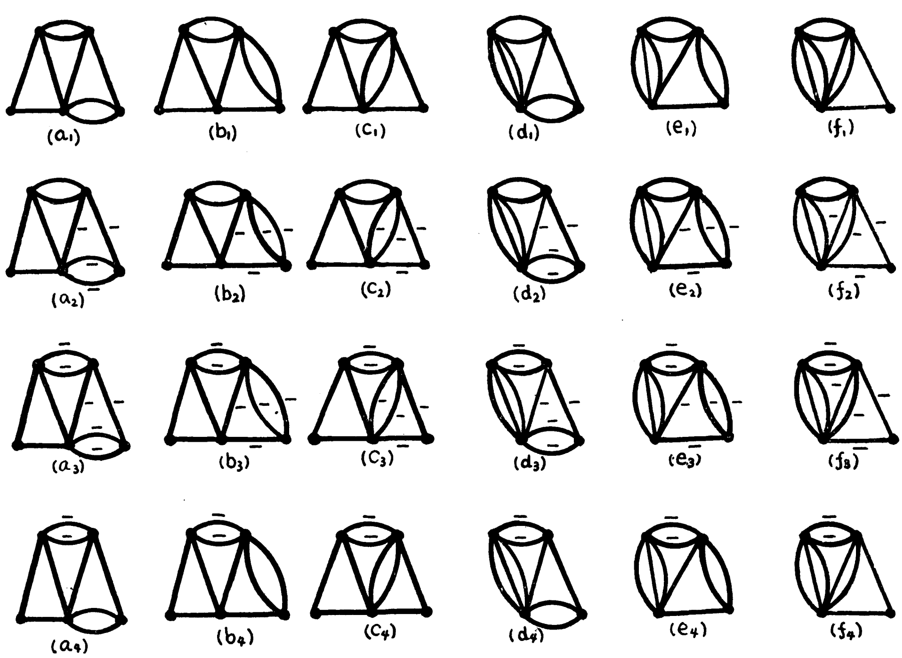


Fig. 5

7) See [5]. p. 134.
8) See [2], p. 70.

(a_i) and (b_i) are equivalent by the following operation.

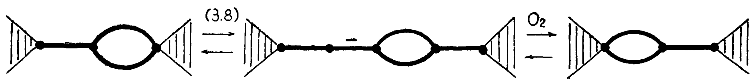


Fig. 6

(d_i) and (e_i) are also equivalent by the same operation.

From the corollary (3. 5) we have

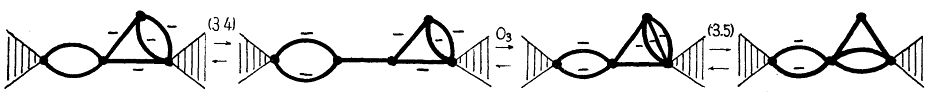


Fig. 7

which shows that (a_2) and (c_4), (a_4) and (c_2), (d_2) and (f_4), (d_4) and (f_2) are equivalent to each other respectively.

From the same corollary we have

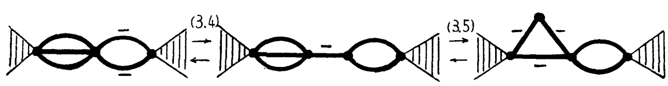


Fig. 8

which shows that (a_4) and (d_3), (f_4) and (c_3), (d_4) and (a_3), (c_4) and (f_3) are equivalent to each conjugate graph of the other, respectively.

In the consequence we have

$(a_1)\sim 9_{30}$ of Alexander-Briggs's table,
$(c_1)\sim 9_{25}$ " ,
$(d_1)\sim 9_{22}$ " ,
$(f_1)\sim 9_{36}$ " ,
$(a_2)\sim(c_4)\sim(f_3)\sim 9_{44}$ " ,
$(c_2)\sim(a_4)\sim(d_3)\sim 9_{45}$ " ,
$(d_2)\sim(f_4)\sim(c_3)\sim 9_{42}$ " ,
$(f_2)\sim(d_4)\sim(a_3)\sim 9_{43}$ " .

Further application of graphs to the knot-theory is the following

Theorem. *If $g(\pi)$ and $g'(\pi)$ are of the same type and have the opposite signs, then the original knot K of π is amphibious, that is to say, K is equivalent to its mirror image K^**[9].

9) A necessary condition for the amphibious knot is given in [2], p. 31. But this is not a sufficient condition. Recently H. Schubert [4] has given a necessary and sufficient condition for the amphibious knot which have the two-bridge representation.

Proof. Since $g(\pi)$ and $g'(\pi)$ are of the same type and have the opposite signs, $g(\pi^*)$ and $g'(\pi)$ coincide. Therefore, from (3.6) π and π^* are equivalent. Hence K and K^* are equivalent.

From this theorem we can verify that the knots 4_1, 6_3, 8_3, 8_9, 8_{12}, 8_{17}, and 8_{18} of Alexander-Briggs's table are amphibious.

5. Remarks for links.

(1) In the case of links, the construction of the graph must be modified as follows: For a connected projection π of a link L, we construct $g(\pi)$ by the same manner as the graph of knots. If π is not connected, then there exist some regions which are not simply connected. Suppose A_i is such a region and there exist r components $\pi^1, \pi^2, \cdots, \pi^r$ of π, which intersect the boundary of A_i. Instead of the point P_i in the construction of the graph of knots, we take r points $P_{i1}, P_{i2}, \cdots, P_{ir}$ and presume that P_{ij} is the representative of A_i with respect to π^j $(j=1, 2, \cdots, r)$. Then we have the graph in the same manner as in the case of knots.

(2) In the operation O_2, let P_1 and P_2 be the joints of Γ with the arcs d_1 and d_2 respectively, and P_3 be the joint of d_1 with d_2. P_1 does not coincide with P_2 in the case of knots. However P_1 and P_2 may coincide in the case of links. In this case, d_1 and d_2 can be omitted and P_3 remained without cancelling.

The other operations are also valid in the case of links.

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