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The Theory of Construction of Finite Semigroups II

Compositions of Semigroups, and Finite s-Decomposable Semigroups

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Contents

§ 0. Introduction.

The purpose of the present paper is to investigate construction of finite semilattices and compositions of semigroups which will play an important part in the theory of construction of finite s-decomposable semigroups. The theory of compositions of special semigroups is already included in the result obtained by Clifford [1].

A semilattice is the synonym of a commutative idempotent semigroup i.e. the multiplication system T satisfying

$$(\sigma\tau)\rho=\sigma(\tau\rho)\ ,\quad \sigma\tau=\tau\sigma\ ,\quad \sigma^2=\sigma$$

for all σ, τ, $\rho\in T$. We have known that a semigroup S is decomposed to a semilattice T, that is to say,[1]

$$(0.1)\qquad S=\sum_{\tau\in T}S_\tau\ ,\quad S_\tau^{\,2}\subset S_\tau\ ,\quad S_\tau S_\sigma\subset S_{\tau\sigma}=S_{\sigma\tau}$$

where this s-decomposition of S is greatest. (Cf. [3]) The study of a semilattice is indispensable for the theory of construction of a semigroup. In this paper we shall restrict ourselves to finite semilattices and we

1) $\sum$ denotes the set union.

shall show that all semilattices of order[2)] n are obtained if ones of order at most $n-1$ are done.

If T is of order $\geqq 2$ in the greatest s–decomposition of S, then S is called s–decomposable; and if T is of order 1, then S is called s–indecomposable. Further if S is homomorphic to a semilattice T such as (0.1), where this decomposition may not be greatest, then S is called *a composition of semigroups* $S_\tau(\tau \in T)$ *by* T. In case of no fear of confusion, we shall omit "by T." Now the following problems arise.

When a semilattice T and semigroups $S_\tau(\tau \in T)$ are given arbitrarily, does a composition of $S_\tau(\tau \in T)$ by T exist? How are all the compositions of $S_\tau(\tau \in T)$ by T constructed?

As it is difficult to treat a general case, the problems in this paper are restricted within a special case where T is finite. We shall show that there is always one composition at least if T is a finite chain or if both all S_τ and T are finite; but it does not always hold in the other cases. Furthermore we shall conclude finally that the study of s–decomposable semigroups is reduced to that of s–indecomposable semigroups.

§ 1. Compositions in the Case where T is of Order 2.

In these paragraphs § 1, § 2, we assume T to be a semilattice of order 2, i.e. $T=\{0, 1\}$ with multiplication $0^2=0$, $01=10=0$, $1^2=1$. By a composition S of two semigroups S_0 and S_1 (by $\{0, 1\}$), we shall mean $S=\sum_{i=1}^{2} S_i$ where $S_i^2 \subset S_i$, $S_0S_1 \subset S_0$, $S_1S_0 \subset S_0$.

1. Existence theorem. First we shall show a necessary and sufficient condition fulfilled by a composition S of semigroups S_0 and S_1. Let us denote by letters $x, y, z, \cdots$ elements of S_0 and by $\alpha, \beta, \cdots$ elements of S_1. The associative law we need in S is written as follows, since the law already holds in S_0 and S_1. Later we shall prove independency of these conditions.

(1. 1″) $(xy)\alpha = x(y\alpha)$,
(1. 2″) $\alpha(xy) = (\alpha x)y$,
(1. 3″) $(x\alpha)\beta = x(\alpha\beta)$,
(1. 4″) $\alpha(\beta x) = (\alpha\beta)x$,
(1. 5″) $(\alpha x)\beta = \alpha(x\beta)$,
(1. 6″) $(x\alpha)y = x(\alpha y)$.

If we let $\varphi_\alpha(x) = x\alpha$ and $\psi_\alpha(x) = \alpha x$, then (1. 1″)$\sim$(1. 6″) are formulated as

2) The order of a semigroup means the number of elements of the semigroup.

$$
\begin{array}{ll}
(1.1') & \varphi_\alpha(xy) = x\varphi_\alpha(y)\,, \\
(1.2') & \psi_\alpha(xy) = \psi_\alpha(x)y\,, \\
(1.3') & \varphi_\beta\varphi_\alpha(x) = \varphi_{\alpha\beta}(x)\,,^{3)} \\
(1.4') & \psi_\alpha\psi_\beta(x) = \psi_{\alpha\beta}(x)\,, \\
(1.5') & \varphi_\beta\psi_\alpha(x) = \psi_\alpha\varphi_\beta(x)\,, \\
(1.6') & \varphi_\alpha(x)y = x\psi_\alpha(y)\,.
\end{array}
$$

Let $\Phi_0 = \{\varphi_\alpha;\ \alpha \in S_1\}$ and $\Psi_0 = \{\psi_\alpha;\ \alpha \in S_1\}$. Φ_0 and Ψ_0 are semigroups with the usual multiplication of mappings. According to the terminology, (1.1')～(1.6') are expressed in other words:

(1.1) φ_α is a right translation of S_0,
(1.2) ψ_α is a left translation of S_0,
(1.3) S_1 is dually homomorphic to[4] Φ_0,
(1.4) S_1 is homomorphic to Ψ_0,
(1.5) every element of Φ_0 commutes with every element of Ψ_0,
(1.6) the substituted semigroup of S_0 by φ_α is equal to that of S_0 by ψ_α. (See [4])

Of course Φ_0 is a dually homomorphic image of S_1 into the right translation semigroup Φ, and Ψ_0 is a homomorphic image of S_1 into the left translation semigroup Ψ.

Conversely if there exist subsemigroups Φ_0 and Ψ_0 of Φ and Ψ respectively which fulfil (1.1)～(1.6), then a composition S of S_0 and S_1 is obtained. We see that Φ_0 and Ψ_0 exist in reality, for example, so is the set composed of only identical mapping [4]. In fact, if φ_α and ψ_α, for all $\alpha \in S_1$, are identical mappings of S_0, they satisfy the conditions (1.1)～(1.6). Thus we have

Theorem 1. *If semigroups S_0 and S_1 are arbitrarily given, there exists at least one composition S of S_0 and S_1. In order to construct S, we find subsemigroups Φ_0 and Ψ_0 of the translation semigroups Φ and Ψ respectively such that S_1 is dually homomorphic to Φ_0, and S_1 is homomorphic to Ψ_0, furthermore Φ_0 and Ψ_0 satisfy* (1.5) *and* (1.6). *Then the product of $x \in S_0$ and $\alpha \in S_1$ is defined as $x\alpha = \varphi_\alpha(x)$ and $\alpha x = \psi_\alpha(x)$.*

Since a pair of Φ_0 and Ψ_0 determines a composition S of S_0 and S_1, we must find all possible pairs of Φ_0 and Ψ_0 in order to obtain all compositions of the given S_0 and S_1, but it happens that different pairs (Φ_0, Ψ_0) and (Φ_0', Ψ_0') determine isomorphic compositions. The isomorphism problem of compositions is to be solved in the later paragraph.

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3) $\varphi_\beta\varphi_\alpha(x) = (\varphi_\beta\varphi_\alpha)(x) = \varphi_\beta(\varphi_\alpha(x))$.

4) We shall use the word "homomorphism to" as the synonym of "homomorphism onto."

2. Independency of six conditions. Let us give the six examples where each Example i satisfies the conditions (1. j), $j \neq i$, but does not (1. i). Regarding translations, see [4] and [5].

Example 1. Let $S_0 = \{a, b, c\}$ with multiplication $xy = a$ for all $x, y \in S_0$, and let $S_1 = \{d\}$, Now φ_d and ψ_d are defined as $\varphi_d(x) = c$ for all $x \in S_0$, and $\psi_d(x) = x$ for all $x \in S_0$. Then it is clear that φ_d is not a right translation of S_0.

Example 2. We may consider the dual form of Example 1.

Example 3. Take S_0, S_1, and ψ_d same as in Example 1, and let φ_d be a right translation such that $\varphi_d(a) = a$, $\varphi_d(b) = c$, $\varphi_d(c) = b$. Then φ_d is not idempotent.

Example 4. Consider the dual form of Example 3.

Example 5. Let S_0 and S_1 be same as in Example 1. φ_d and ψ_d are given such that

$$\varphi_d(a) = a, \quad \varphi_d(b) = \varphi_d(c) = b, \quad \psi_d(a) = \psi_d(c) = a, \quad \psi_d(b) = b.$$

Obviously
$$\varphi_d \psi_d \neq \psi_d \varphi_d.$$

Example 6. Let us define S_0, S_1, φ_d and ψ_d as following. Let $S_0 = \{a, b, c\}$ with multiplication

$$xy = \begin{cases} a & \text{for } y \neq b, \\ b & \text{for } y = b, \end{cases}$$

$S_1 = \{d\}$, $\varphi_d(x) = a$ for all $x \in S_0$, $\psi_d(a) = a$, $\psi_d(b) = \psi_d(c) = b$. Then the substituted semigroup of S_0 by φ_d is different from that by ψ_d. (See [4].) Thus we have

Theorem 2. *The conditions* (1. 1)$\sim$(1. 6) *are independent.*

§ 2. Various Propositions.

1. By a partially symmetric composition of S_0 and S_1 we mean a composition of S_0 and S_1 (by T of order 2) which fulfils $\varphi_\alpha = \psi_\alpha$ for all $\alpha \in S_1$. We have easily.

Theorem 3. *As far as a partially symmetric composition is concerned, the six conditions in* § 1 *are equivalent to*

(2. 1) $\varphi_\alpha(xy) = x\varphi_\alpha(y) = \varphi_\alpha(x)y$, for $x, y \in S_0$,

(2. 2) $\varphi_\alpha \varphi_\beta = \varphi_\beta \varphi_\alpha = \varphi_{\alpha\beta}$,

in other words, letting $\Phi_0 = \{\varphi_\alpha;\ \alpha \in S_1\}$, (2.1′) *$\varphi_\alpha$ is a right translation as well as a left translation of S_0,*

(2.2′) *S_1 is homomorphic to a commutative subsemigroup Φ_0 of $\Phi\cap\Psi$.*[5] *After such Φ_0 is gotten, the product of $x\in S_0$ and $\alpha\in S_1$ is given as: $x\alpha=\alpha x=\varphi_\alpha(x)$.*

Corollary 1. *If S_0 is commutative and S is partially symmetric, then the six conditions are equivalent to*

(2.3) $\varphi_\alpha(xy)=x\varphi_\alpha(y)$, $\varphi_\alpha\varphi_\beta=\varphi_\beta\varphi_\alpha=\varphi_{\alpha\beta}$.

Theorem 4. *A commutative composition S of the two commutative semigroups S_0 and S_1 is determined by a homomorphism of S_1 into the translation semigroup Φ of S_0.*

2. When $S_0^2=S_0$, the conditions (1.1)$\sim$(1.6) becomes simpler. The following lemma is pointed out without proof by A. H. Clifford in § 3 of [1].[6]

Lemma 1. *If $S_0^2=S_0$, then the condition* (1.5) *can be excluded, for it is naturally satisfied.*

Proof. $\varphi_\beta\psi_\alpha(z)=\varphi_\beta\psi_\alpha(xy)=\varphi_\beta(\psi_\alpha(x)y)=\psi_\alpha(x)\varphi_\beta(y)=\psi_\alpha(x\varphi_\beta(y))$ $\psi_\alpha\varphi_\beta(xy)=\psi_\alpha\varphi_\beta(z)$.

Theorem 5. *If S_0 is commutative and $S_0^2=S_0$, then a composition S of S_0 and any S_1 is partially symmetric. S is determined by $\Phi_0=\{\varphi_\alpha\,;\,\alpha\in S_1\}$ in which each φ_α fulfils* (2.3).

Proof. Any element z of S_0 is expressed as $z=xy$ for some $x, y\in S_0$. By commutativity of S_0, and the conditions in § 1.

$$\varphi_\alpha(z)=\varphi_\alpha(xy)=x\varphi_\alpha(y)=\varphi_\alpha(y)x=y\psi_\alpha(x)=\psi_\alpha(x)y=\psi_\alpha(xy)=\psi_\alpha(z)\ ,$$

hence S is partially symmetric.

Corollary 2. *A composition S of two semilattices S_0 and S_1 is a semilattice.*

Proof. S_0 is commutative and idempotent so that $S_0^2=S_0$. By Theorem 5, we have $\varphi_\alpha=\psi_\alpha$ for all $\alpha\in S_1$, and so S is commutative. Idempotency of S is evident.

Theorem 6. *If S_0 has a two-sided unit, then a composition S of S_0*

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5) $\cap$ means intersection.

6) We remark "right" or "left" translations used by us correspond to "left" or "right" ones used by him respectively.

and any S_1 is determined by a homomorphism f of S_1 into S_0. In detail, their composition S is determined by

$$\varphi_\alpha(x)=xf(\alpha)\,,\qquad \psi_\alpha(x)=f(\alpha)x\,.$$

Proof. Since S_0 has a two-sided unit e, Φ and Ψ coincide with the inner right and left translation semigroups respectively [4]. Therefore, for any $\alpha\in S_1$, there are elements $f(\alpha)$ and $g(\alpha)$ in S_0 such that

$$\varphi_\alpha(x)=xf(\alpha)\,,\qquad \psi_\alpha(x)=g(\alpha)x\,.$$

Setting $x=e$ especially, $\varphi_\alpha(e)=ef(\alpha)=f(\alpha)$, $\psi_\alpha(e)=g(\alpha)e=g(\alpha)$. Since $\varphi_\alpha(e)=\varphi_\alpha(e)e=e\psi_\alpha(e)=\psi_\alpha(e)$ according to (1.6′), we have $f(\alpha)=g(\alpha)$. Next we shall prove that f is a homomorphism. By (1.1′) and (1.3′), for all $x\in S_0$,

$$xf(\alpha\beta)=\varphi_{\alpha\beta}(x)=\varphi_\beta\varphi_\alpha(x)=\varphi_\beta(xf(\alpha))=x\varphi_\beta(f(\alpha))=xf(\alpha)f(\beta)\,,$$

hence we have $f(\alpha\beta)=f(\alpha)f(\beta)$. Conversely it is easily seen that $\varphi_\alpha(x)=xf(\alpha)$ and $\psi_\alpha(x)=f(\alpha)x$ fulfil (1.1′)$\sim$(1.6′). Thus the theorem has been proved.

3. We provide the following condition for a semigroup S_0.

Condition A′. If $xa=xb$ for all x in S_0, then $a=b$.

This condition is stronger than Condition A in [1] due to Clifford. Let R be the set of all the inner right translations f_a of S_0:

$$R=\{f_a\,;\ a\in S_0\}\quad \text{where}\ \ f_a(x)=xa\,.\quad (\text{Cf. [4]})$$

Condition A′ means that the correspondence $a\rightarrow f_a$ is one to one. Of course $R\subset\Phi$. (cf. § 1) Now we can find a subsemigroup Φ_1 of Φ which contains R as a two-sided ideal. In detail, Φ_1 is defined as

$$\Phi_1=\{\varphi\,;\ \varphi\in\Phi,\ R\varphi\subset R\ \text{and}\ \varphi R\subset R\}\,.$$

Though it is our main purpose to point out that a composition of S_0 and a semigroup S_1 is constructed under some conditions simpler than (1.1)$\sim$(1.6) when S_0 satisfies Condition A′, we shall discuss the more general case i.e. the extension of S_0 in the sense of Clifford [1] under Condition A′. U denotes a semigroup with a two-sided zero 0^*, U^* the set of all non-zero elements of U, and S the extension of S_0 by U in the sense of Clifford. Further we denote by $(\Phi_1 : R)$ the difference semigroup of Φ_1 modulo R in the sense of Rees [2].

Theorem 7. *We assume that S_0 satisfies Condition A′. A dual* *homomorphism of U into $(\Phi_1 : R)$ determines completely an extension of S_0 by U in the Clifford's sense. In other words, if there is given a mapping $\alpha \rightarrow \varphi_\alpha$ of the elements of U^* into Φ_1 such that, for $\alpha, \beta \in U^*$, $\varphi_\alpha \varphi_\beta = \varphi_{\beta\alpha}$ if $\beta\alpha \in U^*$, and $\varphi_\alpha \varphi_\beta \in R$ if $\beta\alpha = 0^*$. Then there exists a unique extension S of the given S_0 by the given U such that $x\alpha = \varphi_\alpha(x)$ for $x \in S_0$, $\alpha \in U^*$.*

Proof. Since R is a two-sided ideal of Φ_1, for any $x \in S_0$ and any $\alpha \in U^*$, there is y in S_0 such that $f_x \varphi_\alpha = f_y$. This y is unique because S_0 satisfies Condition A′. Now, in order to determine the product αx of $\alpha \in U^*$ and $x \in S_0$, we define $\psi_\alpha(x)$ as follows: $y = \psi_\alpha(x)$. This definition is equivalent to

(2.4) $\quad \varphi_\alpha(z)x = z\psi_\alpha(x)$ for all $z \in S_0$, any $\alpha \in U^*$.

Then we shall prove

(2.5) $\quad \psi_\alpha(x_1 x_2) = \psi_\alpha(x_1)x_2$ for every $x_1, x_2 \in S_0$,

(2.6) $\quad \psi_{\alpha\beta} = \psi_\alpha \psi_\beta$,

(2.7) $\quad \varphi_\alpha \psi_\beta = \psi_\beta \varphi_\alpha$.

The proof of (2.5). Using (2.4),

$$z(\psi_\alpha(x_1)x_2) = (z\psi_\alpha(x_1))x_2 = (\varphi_\alpha(z)x_1)x_2 = \varphi_\alpha(z)(x_1 x_2) = z\psi_\alpha(x_1 x_2)$$

for all $z \in S_0$, whence $\psi_\alpha(x_1)x_2 = \psi_\alpha(x_1 x_2)$.
The proof of (2.6). Since $\varphi_{\alpha\beta} = \varphi_\beta \varphi_\alpha$, by (2.4),

$$z\psi_{\alpha\beta}(x) = \varphi_{\alpha\beta}(z)x = \varphi_\beta(\varphi_\alpha(z))x = \varphi_\alpha(z)\psi_\beta(x) = z\psi_\alpha(\psi_\beta(x))$$

for all $z \in S_0$. Hence $\psi_{\alpha\beta}(x) = \psi_\alpha \psi_\beta(x)$ for all $x \in S_0$.
The proof of (2.7). Since φ_α and φ_β are right translations of S_0,

$$z\psi_\beta(\varphi_\alpha(x)) = \varphi_\beta(z)\varphi_\alpha(x) = \varphi_\alpha(\varphi_\beta(z)x) = \varphi_\alpha(z\psi_\beta(x)) = z\varphi_\alpha(\psi_\beta(x))$$

for all $z \in S_0$, and so $\psi_\beta \varphi_\alpha(x) = \varphi_\alpha \psi_\beta(x)$ for every $x \in S_0$, whence we have $\psi_\beta \varphi_\alpha = \varphi_\alpha \psi_\beta$. Now we define the product αx as $\alpha x = \psi_\alpha(x)$ where $\alpha \in U^*$, $x \in S_0$. In the case where $\alpha \in U^*$, $\beta \in U^*$, and $\alpha\beta = 0$ in U^*, $\varphi_\beta \varphi_\alpha = \varphi_{\alpha\beta} \in R$ and hence there is a unique $y \in S_0$ such that $\varphi_\beta \varphi_\alpha = f_y$. Then we define the product $\alpha\beta$ of elements α and β of S as $\alpha\beta = y$. Thus the multiplication in the extension of S_0 by U is uniquely determined, for φ_α and ψ_α have been shown to satisfy the conditions due to Clifford (Cf. §4 in [1]), q. e. d.

If a semigroup S_0 satisfies Condition A′, a composition of S_0 and a semigroup S_1 by $T = \{0, 1\}$ is easily obtained as the result of the above theorem. It is stated as follows:

Corollary 3. *A dual-homomrphism of S_1 into Φ_1 determines completely a composition of S_0 and S_1. In detail, if we are given $\{\varphi_\alpha\,;\,\alpha\in S_1\}$ such that $\varphi_\alpha\varphi_\beta=\varphi_{\beta\alpha}$, α, $\beta\in S_1$, then there is only a composition of S_0 and S_1 such that $x\alpha=\varphi_\alpha(x)$ for $x\in S_0$, $\alpha\in S_1$.*

If we suppose the following Condition A$''$ in stead of Condition A$'$, we have similarly a theorem and a corollary parallel to the above theorem and the corollary.

Condition A$''$. If $ax=bx$ for all $x\in S_0$, then $a=b$.

4. Let $(S_0,\,U,\,\varphi_\alpha)$ denote the extension of S_0 by U as described in Theorem 7. Consider the extensions $(S_0,\,U,\,\varphi_\alpha)$ and $(S_0',\,U',\,\varphi'_{\alpha'})$ of S_0 and S_0' respectively. Under what condition is $(S_0,\,U,\,\varphi_\alpha)$ isomorphic to $(S_0',\,U',\,\varphi'_{\alpha'})$? However we do not expect to solve this problem completely here, but we shall find what condition is necessary and sufficient for existence of an isomorphism ζ of $S=(S_0,\,U,\,\varphi_\alpha)$ to $S'=(S_0'\,U',\,\varphi'_{\alpha'})$ such that $\zeta(S_0)=S_0'$, $\zeta(U^*)=U'^*$. At first suppose that such ζ exists. Let ξ be the contraction of ζ to S_0, i.e. $\xi(x)=\zeta(x)=x'$, $x\in S_0$. ξ is an isomorphism of S_0 to S_0', and let η be the isomorphism of U to U' such that η maps the zero of U to the zero of U' and $\eta(\alpha)=\zeta(\alpha)=\alpha'$ for $\alpha\in U^*$. Since ζ is an isomorphism, and $x\alpha\in S_0$,

$$\xi\varphi_\alpha(x)=\xi(x\alpha)=\zeta(x\alpha)=\zeta(x)\zeta(\alpha)=\xi(x)\eta(\alpha)=\varphi'_{\eta(\alpha)}\xi(x)$$

for every $x\in S_0$; and hence $\xi\varphi_\alpha=\varphi'_{\eta(\alpha)}\xi$ or

(2.8) $\quad\varphi'_{\eta(\alpha)}=\xi\varphi_\alpha\xi^{-1}$ for ever $\alpha\in U^*$.

Conversely suppose that ξ is an isomorphism of S_0 to S_0', η is an isomorphism of U to U' and (2.8) is fulfilled. f_x and $f'_{\xi(x)}$ denote inner right translations of S_0 and S_0' respectively. By the definition given in the proof of Theorem 7, $y=\psi_\alpha(x)$ implies $f_x\varphi_\alpha=f_y$, and so

$$f'_{\xi(x)}\varphi'_{\eta(\alpha)}=\xi f_x\xi^{-1}\xi\varphi_\alpha\xi^{-1}=\xi f_x\varphi_\alpha\xi^{-1}=\xi f_y\xi^{-1}=f'_{\xi(y)}$$

or we get $\xi(y)=\psi'_{\eta(\alpha)}\xi(x)$ and then $\xi\psi_\alpha(x)=\psi'_{\eta(\alpha)}\xi(x)$ for every $x\in S_0$. Hence $\xi\psi_\alpha=\psi'_{\eta(\alpha)}\xi$, that is,

(2.9) $\quad\psi'_{\eta(\alpha)}=\xi\psi_\alpha\xi^{-1}$ for every $\alpha\in U^*$. It is clear by [4] that if φ_α is a right translation, $\xi\varphi_\alpha\xi^{-1}$ is so; and if ψ_α is a left translation $\xi\psi_\alpha\xi^{-1}$ is so. Now, a mapping ζ of S to S' is defined as follows.

$$\zeta(z)=\begin{cases}\xi(z) & \text{if}\quad z\in S_0\,,\\ \eta(z) & \text{if}\quad z\in U^*\,.\end{cases}$$

We shall prove that ζ is an isomorphism. It goes without saying that it is one to one. Firstly, for any $x \in S_0$ and any $\alpha \in U^*$,

$$\zeta(x)\zeta(\alpha)=\xi(x)\eta(\alpha)=\varphi'_{\eta(\alpha)}\xi(x)=\xi\varphi_\alpha\xi^{-1}\xi(x)=\xi\varphi_\alpha(x)=\xi(x\alpha)=\zeta(x\alpha)\ ;$$

and similarly $\zeta(\alpha x)=\zeta(\alpha)\zeta(x)$ by means of (2.9). Next, if $\alpha, \beta \in U^*$ with $\alpha\beta \in S_0$, then $\eta(\alpha)\eta(\beta) \in S_0'$ and

$$\xi(x)\xi(\alpha\beta)=\xi(x(\alpha\beta))=\xi(\varphi_{\alpha\beta}(x))=\xi(\varphi_\beta\varphi_\alpha(x))\ ,$$
$$\xi(x)\eta(\alpha)\eta(\beta)=\varphi'_{\eta(\alpha)\eta(\beta)}\xi(x)=\varphi'_{\eta(\beta)}\varphi'_{\eta(\alpha)}\xi(x)=(\xi\varphi_\beta\xi^{-1})(\xi\varphi_\alpha\xi^{-1})\xi(x)=\xi\varphi_\beta\varphi_\alpha(x)\ .$$

Since $\zeta(\alpha\beta)=\xi(\alpha\beta)$, $\zeta(\alpha)\zeta(\beta)=\eta(\alpha)\eta(\beta)$, we get $\xi(x)\zeta(\alpha\beta)=\xi(x)\zeta(\alpha)\zeta(\beta)$ for every $x \in S_0$ i.e. every $\xi(x) \in S_0'$. Hence $\zeta(\alpha\beta)=\zeta(\alpha)\zeta(\beta)$ because of Condition A$'$. As f and g are isomorphisms, the proof is not required in the case where $x, y \in S_0$ or $\alpha, \beta \in U^*$ with $\alpha\beta \in U^*$. Therefore we have

Theorem 8. *Let ξ and η be isomorphisms of S_0 and U to S_0' and U' respectively, and let $S=(S_0, U, \varphi_\alpha)$ and $S'=(S_0', U', \varphi'_{\alpha'})$. In order that between S and S' there is an isomorphism which preserves ξ and η on S_0 and U respectively, it is necessary and sufficient that $\varphi'_{\alpha'}=\xi\varphi_\alpha\xi^{-1}$ for every $\alpha \in U^*$, where $\alpha'=\eta(\alpha)$.*

§ 3. Fundamental Properties of a Semilattice.

In this paragraph a semigroup means a semilattice which is not assumed to be finite. Let a, b be elements of a semilattice S. If $a \geqq b$ is defined to mean $ab=a$, S is a partly ordered set in which ac is a least upper bound of a and c for any $a, c \in S$. Conversely if S is a partly ordered set admitting a least upper bound of any pair of elements, then S is a semilattice under the multiplication ac defined to mean the least upper bound of a and c. The following lemma has been already known. See [6] or Ex. 1, p. 18 in [8].

Lemma 2. *A semilattice is characterized by a partly ordered set in which there is a least upper bound of any pair of elements.*

By the way, the condition "$ab=a$" is equivalent to "$xb=a$ for some x". We shall often see an inequality $a>b$ which means $a \geqq b$ and $a \neq b$; and if neither $a \geqq b$ nor $a \leqq b$, a and b are said to be incomparable; otherwise a and b are called comparable. Of course we may say that a semilattice is a partly ordered set in which there is a least upper bound of any finite number of elements.

A homomorphism of a semilattice S to a semilattice S' is defined as a mapping of S onto S' which preserves multiplication; if a homomorphism is one to one, it is called an isomorphism of S to S' (or between S and S'); especially we mean by an automorphism an isomorphism of S onto S itself.

Lemma 3. *An isomorphism of a semilattice S to a semilattice S' is characterized by a mapping f of the partly ordered set S onto the partly ordered set S' which fulfils the following conditions.*

(3.1) *f is one to one,*

(3.2) *$x \geqq y$ in S if and only if $f(x) \geqq f(y)$ in S', where* (3.2) *is equivalent to* (3.2′) *under* (3.1) :

(3.2′) *$x > y$ in S if and only if $f(x) > f(y)$ in S'.*

Proof. Suppose that f is an isomorphism of S onto S'. From $x \geqq y$, $x = yz$ for some $z \in S$, so that $f(x) = f(y)f(z)$; hence $f(x) \geqq f(y)$. Conversely suppose that f is a mapping which fulfils (3.1) and (3.2). It is well known that such a mapping must preserve least upper bounds whenever they exist [8]. Hence f is an isomorphism between S and S'. Since f is one to one, $x = y$ if and only if $f(x) = f(y)$; accordingly, under the condition (3.1), $x > y$ if and only if $f(x) > f(y)$. Thus (3.1) and (3.2) are equivalent to (3.1) and (3.2′).

However a homomorphism is not characterized by a mapping f of S onto S' satisfying

(3.3) $x \geqq y$ in S implies $f(x) \geqq f(y)$ in S';

also an isomorphism is not characterized by a one-to-one mapping f of S onto S' satisfying (3.3). The following example shows this fact.

Example. Two semilattices S and S' are given as

S

	a	b	c	d	e
a	a	a	a	a	a
b	a	b	a	a	a
c	a	a	c	c	c
d	a	a	c	d	c
e	a	a	c	c	e

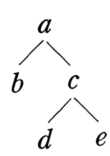


Fig. 1.

S'

	a'	b'	c'	d'	e'
a'	a'	a'	a'	a'	a'
b'	a'	b'	a'	b'	a'
c'	a'	a'	c'	c'	c'
d'	a'	b'	c'	d'	c'
e'	a'	a'	c'	c'	e'

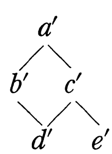


Fig. 2.

Let f be a mapping of $S = \{a, b, c, d\}$ onto $S' = \{a', b', c', d'\}$ such that

$f(a)=a'$, $f(b)=b'$, $f(c)=c'$, $f(d)=d'$. Although f is one-to-one and $x \geqq y$ implies $f(x) \geqq f(y)$ for $x, y \in S$, f is not an isomorphism between the two semilattices, for

$$f(bd)=f(a)=a' \neq b'=b'd'=f(b)f(d)\,.$$

The following lemma is very easily proved.

Lemma 4. (3.4) *A subsemigroup of a semilattice is a semilattice.*

(3.5) *The non-empty intersection of subsemilattices of a semilattice is a subsemilattice.*

(3.6) *A homomorphic image of a semilattice is also so.*

As well known, an ideal I of a semilattice S is a subset which satisfies $IS \subset I$. Obviously

Lemma 5. *A subset I of a semilattice S is an ideal if and only if $x \in I$ and $x \leqq y$ imply $y \in I$.*

Corollary 4. (3.7) *The non-empty intersection of ideals of S is an ideal of S.*

(3.8) *The set union of ideals of S is also an ideal of S.*

Especially the ideal $\{x\,;\, a \leqq x\}$ is called a principal ideal, denoted by I_a. It is shown that the correspondence $a \to I_a$ is one to one and I_{ab} is the intersection of I_a and I_b [6]. Let $\mathfrak{I}$ be the system composed of all principal ideals of S. The multiplication in $\mathfrak{I}$ is defined as intersection of principal ideals. Then S is isomorphic onto $\mathfrak{I}$. Thus a semilattice is faithfully represented by the system of some subsets of a set under the multiplication of set-intersection [7]. We may say in other words as follows.

Theorem 9. *Let f_a be an inner translation of a semilattice S: $f_a(x) = xa = ax$ for any $x \in S$. The inner translation semigroup $R=\{f_a\,;\, a \in S\}$ of a semilattice S is isomorphic to S. In other words, a semilattice satisfies Condition A' in § 2.*

Now we shall define a terminology "*a cut*." Let S be a semilattice which contains two elements at least, and let a be an element of S which is not greatest. Further let $S_1=\{x\,;\, x \leqq a\}$ and $S_0=S-S_1$ which means the set S_0 of all elements belonging to S but not to S_1.

Lemma 6. *S_1 is a subsemilattice of S, and S_0 is an ideal of S.*

Proof. If x and y are elements of S_1, $ax=ay=a$ and $a(xy)=(ax)y = ay=a$ whence $xy \in S_1$. By (3.4) of Lemma 4, S_1 is a subsemilattice of S. Next we shall prove that $b \not\leq a$ implies $bx \not\leq a$ for all $x \in S$. Suppose $bx \leq a$, then $b \leq bx \leq a$. This contradicts with $b \not\leq a$. Thus it has been proved that S_0 is an ideal of S.

We have obtained a decomposition of S to a semilattice $T=\{0, 1\}$: $S=S_0 \cup S_1$. This decomposition of S is called *a cut of the semilattice S from a*; and further S_0 and S_1 are called the *upper class* and the *lower class* in the cut of S from a respectively. If $a \not\leq b$, then there is a cut of S such that a is contained in the upper class S_0 and b in the lower class S_1. In fact, a cut of S from b is so.

§ 4. Finite Semilattice.

In this paragraph S denotes a finite semilattice. If $a > b$ and $a > x > b$ for no x, we say "*a covers b*." The structure of S is graphically represented by the so-called "diagram" [8] in which a segment is drawn from a to b whenever a covers b, and a is placed higher than b or we write sometimes a to be placed on the left-side of b whenever $a > b$. Since S is finite, it has the greatest element and minimal elements; the former is called a zero 0.

1. By the hight $d[x]$ of an element x of S we mean the maximum length d of chains $x_0 < x_1 < x_2 < \cdots < x_d = x$ where x is greatest, x_0 is minimal, and x_i covers $x_{i-1} (i=1, \cdots, d)$. Of course $d[x]=0$ if and only if x is a minimal element. From the definition of height, we have

Lemma 7. *$y > x$ implies $d[y] > d[x]$.*

Proof. Let $d=d[x]$. We have a chain

$x_0 < x_1 < \cdots < x_d = x < x_{d+1} < \cdots < x_{d+k} = y$ for a certain k, $k \geqq 1$.
Hence $d[x]=d < d+k \leqq d[y]$, so $d[x] < d[y]$.

When neither $x \leqq y$ nor $x \geqq y$, we say that two elements x and y are mutually incomparable. From Lemma 7, we directly have

Corollary 5. *$d[y]=d[x]$ implies that either $y=x$, or y and x are incomparable.*

By the dimension or height $d[S]$ of S we mean the maximum of heights of elements of S: $d[S]=\operatorname{Max}_{x \in S} d[x]$. So far as S contains two elements at least, $d[S] \geqq 1$. Also the dimension $d[U]$ of a subsemilattice U of S is similarly considered.

Lemma 8. *$d[S]=d[x]$ if and only if $x=0$.*

Proof. From $x<0$ for every non-zero $x\in S$, it follows that $d[x]< d[0]$. Therefore $d[0]$ is the maximum of $d[x]$ for $x\in S$.

Lemma 9. *Let $d[a]=m$. There are elements x_i $(i=0, 1, \cdots, m)$ of S such that $d[x_i]=i$ $(i=0, 1, \cdots, m)$ and x_i covers x_{i-1} $(i=1, \cdots, m)$.*

Proof. By the definition of $d[a]$, there is a chain

$$x_0<x_1<\cdots<x_m=a$$

where x_0 is minimal and x_i covers x_{i-1} $(i=1, \cdots, m)$. It is clear that $d[x_i]\geqq i$ since there exists a chain $x_0<x_1<\cdots<x_i$. Suppose $j=d[x_i]>i$, then we must have

$$y_0<y_1<\cdots<y_j=x_i<x_{i+1}<\cdots<x_m=a$$

so that $d[a]\geqq j+m-i>m$ because $j>i$, contradicting with $d[a]=m$.

Corollary 6. *Let $d[S]=n$. There is a chain*

$$x_0<x_1<\cdots<x_i<\cdots<x_n=0$$

where x_0 is minimal and x_i covers x_{i-1} $(i=1, \cdots, n)$.

Lemma 10. *$x<y$ and $d[y]=d[x]+1$ imply that y covers x. But the converse is not true.*

Proof. Suppose that y does not cover x. There is z at least such that $x<z<y$.
By Lemma 7, $d[x]<d[z]<d[y]$ or $d[x]+1\leqq d[z]<d[y]$.
Hence we have $d[x]+1<d[y]$, contradicting with the assumption. Falseness of the converse is shown by the example which has already been given in § 3, Fig. 1. Though a covers b, $d[a]=2$, $d[b]=0$.

Suppose that two finite semilattices S and S' are isomorphic and let f be that mapping of S onto S'. If 0 is a zero in S, $f(0)$ is a zero in S'. By Lemma 3, we have easily

Lemma 11. *If x is minimal in S, then $f(x)$ is minimal in S'. If x covers y, then $f(x)$ covers $f(y)$.*

Lemma 12. $d[x]=d[f(x)]$.

Let f be an automorphism of a semilattice S.

Lemma 13. *$x\neq f(x)$ implies $x\nless f(x)$ and $x\ngtr f(x)$.*

Proof. Suppose $x<f(x)$. Then $d[x]<d[f(x)]$ by Lemma 7. This contradicts with Lemma 12. Similarly $x>f(x)$ is also false.

Lemma 14. *Let x and y be two incomparable elements. If a covers x and y, then a is the least upper bound of x and y.*

Proof. Since a is an upper bound of x and y, $x \leqq xy \leqq a$. From the assumption that a covers x, it follows that either $x = xy$ or $xy = a$. On the other hand, since x and y are incomparable, we must have $xy = a$, that is, a is the least upper bound of x and y.

We obtain the following lemma immediately from the uniqueness of a least upper bound.

Lemma 15. *Let x and y be two incomparable elements. If a covers x and y, and also b covers x and y, then $a = b$.*

2. Special Semilattices. If the dimension of a finite semilattice S is 1, S is called an *elementary semilattice.* In such a semilattice, 0 covers every element different from 0.

Lemma 16. *Let S be a finite semilattice of order $\geqq 2$. S is an elementary semilattice if and only if the multiplication is given as*

$$xy = \begin{cases} x & \text{if } x = y, \\ 0 & \text{if } x \neq y. \end{cases}$$

Proof. Suppose, at first, that S is an elementary semilattice. It is sufficient only to discuss a product xy of $x \neq 0$ and $y \neq 0$ such that $x \neq y$. In an elementary semilattice, 0 covers every element different from 0, so that $x \neq y$, $x \neq 0$, and $y \neq 0$ imply that x and y are incomparable. By Lemma 14, 0 is the least upper bound of x and y: $xy = 0$.

Conversely, suppose that the multiplication is given as

$$xy = x \text{ if } x = y; \quad xy = 0 \text{ if } x \neq y.$$

From this, it follows that $x < 0$ for every $x \neq 0$, and so we have no chain such as $x < y < 0$. Hence $d[S] = 1$. The associative law is easily proved.

Remark. We can define an elementary semilattice as the above-mentioned multiplication system, even if it is not finite.

Theorem 10. *The structure of a finite elementary semilattice is completely determined by its cardinal number. In other words, there is isomorphically only one elementary semilattice of any given cardinal, and the multiplication is defined as in Lemma* 16.

Theorem 11. *A finite semilattice S is a lattice if and only if S has the least element i.e. unit.*

Proof. Assume that S has the least element 1. For any elements a and b of S, the two subsets $\{x;\ x \leqq a\}$ and $\{y;\ y \leqq b\}$ are subsemilattices of S by Lemma 6, and the two sets intersect, for 1 at least belongs to both. Since the intersection D is a subsemilattice of S (cf. (3.5) of Lemma 4), D contains the greatest element of D. Hence it has proved that there is the greatest lower bound of a and b. Conversely it is clear that a finite lattice has the least element.

Theorem 12. *If and only if $x \neq y$ implies $d[x] \neq d[y]$, then the partly ordered set S is a chain i.e. a linearly ordered set.*

Proof. It is clear that if S is a chain there is a one-to-one correspondence between the elements of S and their heights. We shall prove the converse. If the correspondence $x \to d[x]$ is one-to-one, the only one element x_0, the height of which is 0, is obviously minimal in S. Let x_1 be the only one element such that $d[x_1]=1$. If there is $y < x_1$, then $d[y] < d[x_1]$ by Lemma 7, so that $y = x_0$. Assume that $x_0 < x_1 < \cdots < x_{i-1}$ where $d[x_j]=j$ $(j=0, 1, \cdots, i-1)$, then the only one element x_i whose height is i covers x_{i-1}. Because the existence of elements y_j $(j=0, 1, \cdots, i-1)$, $y_0 < y_1 < \cdots < y_{i-1}$, whose heights are j $(j=0, 1, \cdots, i-1)$ respectively, is assured by Lemma 9, and, for any $\{y_j\}$ such as above, we have $y_j = x_j$ $(j=0, 1, \cdots, i-1)$ by the assumption of the one-to-one mapping $x \to d[x]$. Therefore $x_0 < x_1 < \cdots < x_{i-1} < x_i$. Repeating this procedure, we have $x_0 < x_1 < \cdots < x_i < \cdots < x_n = 0$ where x_i covers x_{i-1} $(i=1, \cdots, n)$, and at last all the elements of S are picked up successively. The proof of the theorem has been completed.

§ 5. Translations of a Semilattice.

1. As defined in [4] or [1], a right translation φ of a semilattice S is a mapping of S into itself satisfying

$$\varphi(xy) = x\varphi(y) \quad \text{for every } x,\ y \in S.$$

Since we consider S as a semilattice, we need no distinction between "right" and "left", and hence φ fulfils

(5.1) $\quad \varphi(xy) = x\varphi(y) = \varphi(y)x = \varphi(yx) = y\varphi(x) = \varphi(x)y\,.$

Lemma 17. (5.2) *φ is idempotent:* $\varphi^2 = \varphi$

(5.3) *any translation φ commutes with any translation ψ:* $\varphi\psi = \psi\varphi$.

(5.4) $\quad \varphi\psi(x) = \varphi(x)\psi(x)\,.$

Proof. From $\varphi(x) = \varphi(x^2) = x\varphi(x)$, we get (5.2) :

$$\varphi^2(x)=\varphi(x\varphi(x))=\varphi(x)\varphi(x)=\varphi(x) \text{ by } (5.1).$$

The proof of (5.3) is included in Lemma 1. (5.4) is proved in the following manner: $\varphi\psi(x)=\varphi\psi(x^2)=\varphi(x\psi(x))=\varphi(x)\psi(x)$.

Lemma 18. *A translation of a semilattice S is a homomophism of S into itself.*

Proof. Using (5.1) and idempotency of φ,

$$\varphi(xy)=\varphi^2(xy)=\varphi(x\varphi(y))=\varphi(x)\varphi(y).$$

Lemma 19. *If φ is a translation of a semilattice S, then*

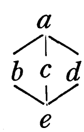
(5.5) $\varphi(x)\geqq x$ *for all* $x\in S$,

(5.6) $x\geqq y$ *implies* $\varphi(x)\geqq\varphi(y)$.

Proof. From $x=x^2$, $\varphi(x)=\varphi(x^2)=x\varphi(x)\geqq x$. If $x\geqq y$ i.e. $x=xy$, then $\varphi(x)=\varphi(xy)=x\varphi(y)\geqq\varphi(y)$.

Remark. Let us consider a mapping $\varphi_0=\begin{pmatrix}abcde\\abcae\end{pmatrix}$[7] in the semilattice S:

S	a	b	c	d	e
a	a	a	a	a	a
b	a	b	a	a	b
c	a	a	c	a	c
d	a	a	a	d	d
e	a	b	c	d	e



This φ_0 is a homomorphism of S into itself and satisfies (5.2), (5.5), and (5.6), but φ_0 is not a translation. (cf. [4]) Consequently the converse of Lemma 18 is not true, and the three conditions (5.2), (5.5), and (5.6) are not sufficient condition for φ to be a translation.

We add that the conditions (5.5) and (5.6) are equivalent to

$$\varphi(xy)\geqq x\varphi(y) \quad \text{for every } x,\ y.$$

2. Translation Semigroup. According to Lemma 17, the translation semigroup Φ of S is a semilattice under the multiplication $\varphi\psi$. Denote by $\varphi\geqq\psi$ the ordering in the semilattice Φ, that is, $\varphi\psi=\varphi$. $\varphi(S)$ is the set of all images $\varphi(x)$ of $x\in S$ under the translation φ. Then we have

Lemma 20. *The following three inequalities are equivalent.*

(5.7) $\varphi\geqq\psi$.

7) $\begin{pmatrix}x_1 & x_2 & \cdots & x_n\\ \varphi(x_1) & \varphi(x_2) & \cdots & \varphi(x_n)\end{pmatrix}$ means a mapping which associates x_i with $\varphi(x_i)$.

(5.8) $\varphi(x) \geqq \psi(x)$ for all $x \in S$.

(5.9) $\varphi(S) \subset \psi(S)$.

Proof. (5.7)→(5.8): Since $\varphi\psi = \varphi$, $\varphi(x) = \varphi(\psi(x)) \geqq \psi(x)$.
(5.8)→(5.9): Take any $z \in \varphi(S)$, then $z = \varphi(x)$ for some $x \in S$. Using (5.8) i.e. $\varphi(x) = \varphi(x)\psi(x)$ and Lemma 17, we get

$$z = \varphi(x) = \varphi(x)\psi(x) = \varphi\psi(x) = \psi\varphi(x) = \psi(\varphi(x)) \in \psi(S)$$

whence $\varphi(S) \subset \psi(S)$.
(5.9)→(5.7): By (5.9), for any x, there is y such that $\varphi(x) = \psi(y)$ and so $\psi(\varphi(x)) = \psi^2(y) = \psi(y) = \varphi(x)$. Hence we have $\psi\varphi = \varphi$.

Let us consider a mapping which associates $\varphi \in \Phi$ with the subset $\varphi(S)$ of S. From (5.9) and (5.7) of Lemma 20, we get directly

Corollary 7. *$\varphi(S) = \psi(S)$ implies $\varphi = \psi$, that is, $\varphi \to \varphi(S)$ is on-to-one.*

Corollary 8. *$\varphi(S) = S$ if and only if φ is the identical mapping of S.*

Lemma 21. *$\varphi\psi(S) = \varphi(S) \cap \psi(S)$ where $\cap$ means the intersection.*

Proof. By Lemma 20, $\varphi\psi(S) \subset \varphi(S)$ and $\varphi\psi(S) \subset \psi(S)$ since $\varphi\psi \geqq \varphi$ and $\varphi\psi \geqq \psi$. Hence $\varphi\psi(S) \subset \varphi(S) \cap \psi(S)$. On the other hand, letting any $z \in \varphi(S) \cap \psi(S)$, $z = \varphi(x) = \psi(y)$ for some x and $y \in S$. Then $z = \varphi(x) = \varphi^2(x) = \varphi\psi(y) \in \varphi\psi(S)$ whence $\varphi(S) \cap \psi(S) \subset \varphi\psi(S)$. This completes the proof.

Combining the above lemmas, we have

Theorem 13. *The translation semigroup Φ of a semilattice S is also a semilattice with unit under the ordering defined by one of* (5.7), (5, 8), *and* (5.9). *Φ is ismorphic to a lattice composed of some subsets of S with multiplication of intersection.*

3. Finite Case. In particular, the translation semigroup of a finite semilattice is finite and has a unit. Accordingly we have by Theorem 11

Theorem 14. *The translation semigroup Φ of a finite semilattice S is a lattice.*

Theorem 15. *If S is a finite lattice, then Φ is isomorphic to S. Conversely if S is a finite semilattice and Φ is ismorphic to S, then S is a lattice.*

Proof. If S is a finite lattice, S has a unit and hence Φ coincides with the inner translation semigroup R (Cf. [4]). By Theorem 9, R is isomorphic to S, after all Φ is isomorphic to S. Conversely if S is a finite semilattice, Φ is a lattice by Theorem 14, and, since Φ is assumed to be isomorphic to S, S is a lattice.

4. Elementary Semilattice. Let us consider Φ of an elementary semilattice which is not necessarily finite.

Theorem 16. *The translation semigroup* Φ *of an elementary semilattice* S *consists of* φ *defined as* $\varphi(x)=x$ *or* 0. Φ *is isomorphic to the lattice composed of all subsets of* S *which contain* 0 *with multiplication of intersection.*

Proof. At first, let φ be a translation of S. We have

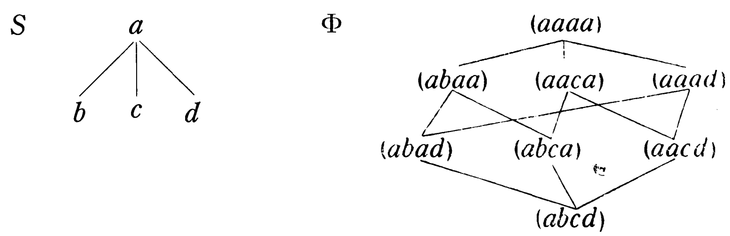
$$\varphi(x)=\varphi(x^2)=x\varphi(x)=\begin{cases} x & \text{if } x=\varphi(x) \\ 0 & \text{if } x\neq\varphi(x) \end{cases} \tag{5.10}$$

because S is an elementary semilattice. (Cf. Lemma 16)
Hence $\varphi(x)=x$ or 0, in particular, $\varphi(0)=0$.

Conversely we shall prove that such φ is a translation of S. If $x\neq y$, then $\varphi(xy)=\varphi(0)=0$, while $x\varphi(y)=0$ because $\varphi(y)=y$ or 0. If $x=y$ and $\varphi(x)=0$, then $\varphi(x^2)=\varphi(x)=0$ and $x\varphi(x)=0$; if $x=y$ and $\varphi(x)\neq 0$, then $\varphi(x^2)=\varphi(x)=x$ and $x\varphi(x)=x^2=x$. After all we have $\varphi(xy)=x\varphi(y)$. For any subset M containg 0, there is φ such that $\varphi(S)=M$. By Corollary 7 and Lemma 21 or Lemma 20, we see that Φ is isomorphic to the lattice of all subsets containing 0. The proof of the theorem has been finished.

Corollary 9. *Let* S *be the elementary semilattice of order* $n+1$. *Then the translation semigroup* Φ *is of order* 2^n *and of dimension* n *and* Φ *contains* $\dfrac{n!}{(n-i)!\,i!}$ *elements heights of which are* i.

Example.



where, for example, $(abad)$ means $\begin{pmatrix} abcd \\ abad \end{pmatrix}$.

5. Construction of Translations. Let S_0 be a finite semilattice and let S_1 be a semilattice which consists of only one element p: $S_1=\{p\}$;

and S denotes a composition of S_0 and S_1 by $T=\{0, 1\}$. (Cf. § 1) By Corollary 2 in § 2, we have known that S is also a semilattice. Let $\varphi_p(x)=xp$ for $x\in S_0$, and $\bar{\varphi}_p(x)=xp$ for $x\in S$; φ_p is a translation of S_0 and $\bar{\varphi}_p$ is an inner translation of S.

For an element $a\in S$ which fulfils $\bar{\varphi}_p(a)=a$, we choose a translation ξ of S_0 such that

$$(5.11) \qquad \varphi_p\xi=\varphi_p f_a$$

where f_a is a translation of S_0: $f_a(x)=xa$, $x\in S_0$. There exists certainly one ξ at least, for we can take f_a as ξ. For such ξ, a mapping φ of S into itself is defined as follows.

$$(5.12) \qquad \varphi(x)=\begin{cases}\xi(x) & \text{if } x\in S_0\,,\\ \bar{\varphi}_p(a)=a & \text{if } x=p\,.\end{cases}$$

We must prove that $\varphi(x)$ is a translation of S. At first if $x=y=p$, $\varphi(p^2)=\varphi(p)=a=pa=p\varphi(p)$; if both x and y belong to S_0, it is clear that $\varphi(xy)=x\varphi(y)$ because ξ is a translation of S_0. Lastly, we prove $\varphi(xp)=x\varphi(p)$ for $x\in S_0$. Since $xp\in S_0$ and $\xi\varphi_p=\varphi_p\xi$ by Lemma 17, and since $\bar{\varphi}_p$ is a translation of S, we have

$$(5.13) \qquad \begin{cases}\varphi(xp)=\xi(xp)=\xi(\varphi_p(x))=\varphi_p\xi(x)\\ x\varphi(p)=x\bar{\varphi}_p(a)=\bar{\varphi}_p(xa)=\varphi_p(xa)=\varphi_p f_a(x)\,.\end{cases}$$

According to (5.11), we obtain $\varphi(xp)=x\varphi(p)$. Thus every a and ξ fulfilling (5.11) construct a translation φ. Conversely if φ is any translation of S, we denote $a=\varphi(p)$ and let ξ be a contraction of φ to S_0. The equality $\bar{\varphi}_p(a)=ap=a$ follows from $\varphi(p^2)=\varphi(p)=p\varphi(p)$. Since $\varphi(xp)=x\varphi(p)$, we have (5.11) under the consideration of (5.13); and consequently any φ is determined by suitable a and ξ.

Summarizing the above description,

Theorem 17. *Every translation of the composition S of semilattices S_0 and $S_1=\{p\}$ is determined by an element a of S and a translation ξ of S_0 which satisfy*

$$\bar{\varphi}_p(a)=a \quad \text{and} \quad \varphi_p\xi=\varphi_p f_a\,.$$

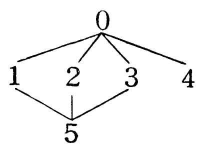
If we denote by $(a\,;\xi)$ a translation of S which a and ξ determine, we get easily

Lemma 22. $(a\,;\xi)=(b\,;\eta)$ *if and only if* $a=b$ *and* $\xi=\eta$.

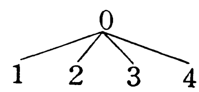
The translations of S seem to depend on S_0 and p according to Theorem 17, but we point out the following remark.

Remark (5.14). If S is a composition of S_0 and $\{p\}$ and at the same time a composition of S_0' and $\{p'\}$. The translation semigroup constructed from S_0 and $\{p\}$ coincides with that constructed from S_0' and $\{p'\}$. For, the translation semigroup of S depends only on S itself.

Remark (5.15). In (5.11) all the translations of S_0 can not be ξ, namely, not become a contraction of a translation φ to S_0. For example,



S



S_0

S is a composition of S_0 and $\{5\}$, and φ_5 is $\begin{pmatrix}0&1&2&3&4\\0&1&2&3&0\end{pmatrix}$. Although $\begin{pmatrix}0&1&2&3&4\\0&1&2&0&0\end{pmatrix}$ is a translation of S_0, it can not be a contraction of a translation of S.

§ 6. Decompositions of a Semilattice.

1. In construction of finite semilattices we shall meet the problem of seeking for all the representations or all the homomorphisms of a finite semilattice S into another finite semilattice S'. In order to solve this problem, it is important to find all the decompositions of S (Cf. [3]) or all the congruence relations in S.

In the following lemma, S is not assumed to be finite.

Lemma 23. *If $u\sim v$ is a congruence relation in a semilattice S, then $x\sim y$ for all x, y in the interval $[u, uv]$ or $[v, uv]$ where $[u, uv]$ denotes the subset $\{z\,;\, u\leqq z\leqq uv\}$.*

Proof. We treat the case $[u, uv]$, the other case treated similarly. From $u\sim v$, we get $u=u^2\sim uv$ and so $x=ux\sim(uv)x=uv$ since $x\in[u, uv]$. Similarly $y\sim uv$, whence $x\sim y$.

If a semilattice S is homomorphic to a semilattice T, we have a decomposition of S: $S=\sum_{\tau\in T} S_\tau$. By Lemma 23, each S_τ has the property that it contains with a, b, $a<b$, all elements between a and b; S_τ is said to be *convex*.

2. Terminology. Before the main discussion, we shall define new terms which are applied to also a general case. Let H be a semigroup, and I be a proper ideal of H. Suppose that I is homomorphic to a semigroup L where

(6.1) $I=\sum_{\tau\in L} I_\tau$

and the decomposition is defined by δ. Now an equivalence relation $x\sim y$ in H is defined as follows.

$x\sim y$ means that either (6.2) or (6.3) holds.

(6.2) there is τ such that $x\in I_\tau$ and $y\in I_\tau$.

(6.3) $x=y\in H-I$.

If the equivalence relation $x\sim y$ is a congruence relation, then we say that the decomposition (6.1) δ of I or the homomorphism of the ideal I to L is *extensible to H.* In such a case, the factor semigroup G of H due to the congruence relation is called *(I, L)-semigroup of H* or *(I, δ)-semigroup of H.* Let δ_0 denote the decomposition of I in which each class is composed of only one element, and let δ_1 one which gathers all the elements of I into a class. Then (I, δ_0)-semigroup of H is isomorphic to H, and (I, δ_1)-semigroup of H is the difference semigroup of H modulo I. In the former, δ_0 is said to be trivially extensible to H. Whenever H is homomorphic to another semigroup H' with a proper ideal K', there is an ideal I of H such that the homomorphism of I to K' is extensible to H.

Next we shall explain another term. Let K and K' be proper ideals of semigroups H and H' respectively. Suppose that H is homomorphic to H' and K is isomorphic to K' under the homomorphism of H to H'. Then we say that the homomorphism of H to H' *fixes* the ideal K (or the ideal K'), and that the decomposition of H *isolates* K.

3. Again come back to a finite semilattice S and suppose that S is homomorphic to a semilattice $T=\{0, 1, \cdots, m\}$, $m\geqq 2$ where 0 is the zero. Then ξ denotes the decomposition of S, $S=\sum_{i=0}^{m} S_i$, where each S_i is a finite convex subsemilattice, and, in particular, S_0 is an ideal of S. Let T' be any proper ideal of T and K be the inverse image of T' under the homomorphism of S to T. K is a proper ideal of S and we have the decomposition δ of K

$$\delta:\quad K=\sum_{i\in T'} S_i\,.$$

Then we get

Lemma 24. *The homomorphism of K to T' is extensible to H.*

Proof. The relation $x\sim y$ is defined accrding to (6.2) and (6.3). For $x\in S_i$, $y\in S_i$, $i\in T'$ and $z\in S_j$, $j\in T$, it holds that $xz\in S_iS_j\subset S_l$ and

$yz \in S_iS_j \subset S_l$ for some $l \in T'$ because S is homomorphic to T and K is an ideal of S. In the case of (6.2), $x \sim y$ implies $xz \sim yz$; in the case of (6.3), $x \sim y$, or $x = y$, implies $xz = yz$. Hence the relation is a congruence relation. Thus the lemma has been proved.

So we can consider (K, T')-semilattice G of S. Let η denote the decomposition of S given by the homomorphism of S to G. Then $\eta \geqq \xi$ in the sense of §5 in [3], and hence G is homomorphic to T. The homomorphism of G to T fixes the ideal T', in other words, the decomposition of G isolates T'.

Lemma 25. *(K, T')-semilattice of S is homomorphic to T fixing T'.*

Consequently we must solve the following problems.

(6.4) Find all decompositions of ideals I which are extensible to S.

(6.5) Find all decompositions of S which isolate I.

4. Let I be a proper ideal of a finite semilattice S. We assume that there is a decomposition of S isolating the ideal I, and then we denote the congruence relation by $x \sim y$, that is,

(6.6) if $x \sim y$ and x or y belongs to I, then $x = y$.

Now let us consider an element a of S which has the property that

(6.7) $a \bar{\in} I$ and $a < y$ imply $y \in I$.

(6.7) is equivalent to (6.7′) if S^* denotes the difference semigroup of S modulo I where 0^* denotes the zero of S^* and x^* the image of an element x of $S-I$ into S^*.

(6.7′) a^* is covered by 0^* in S^*.

Such an element a will be called an element of S *covered by the ideal I.*

Lemma 26. *If a is an element of S covered by I, $a \sim x$ implies $x \leqq a$.*

Proof. It is sufficient to prove that $ax \not> a$ if $a \sim x$, because always $ax \geqq a$. Suppose $ax > a$ for some $x \in S$, then $ax \in I$ by (6.7), while $a \sim x$ implies $a \sim ax$, arriving at $ax = a$ because of (6.6). This is contradictory with the assumption $ax > a$. Therefore we have proved $ax \not> a$ i.e. $ax = a$, or $x \leqq a$.

Lemma 27. *Let a be an element covered by I. If $a \sim x$, then x is incomparable with any element b which is incomparable with a, in other words, $x \leqq b$ implies $b \leqq a$ or $a \leqq b$.*

Proof. By Lemma 26, $a \sim x$ implies $x \leqq a$. Since a is incomparable with b, we see $b \not< x$, $a < ab$, and hence $ab \in I$ by (6.7). We must prove $x \not< b$. Suppose $x < b$ for some $x \sim a$. From $a \sim x$, we get $ab \sim xb = b$ and so $ab = b$ by (6.6), which means that $a \leqq b$, contradicting with the assumption. Thus the proof has been finished.

For each element u of the finite semilattice S, $C(u)$ is defined as the set of all elements x of S such that

(6.8) $x \leqq u$,

(6.9) $x \leqq b$ implies $b \leqq u$ or $u \leqq b$.

$C(u)$ is not empty, for it contains u at least.

Summarizing Lemmas 23, 26, and 27, we have

Lemma 28. *Let I be a proper ideal of a finite semilattice S, and let $a_1, a_2, \cdots, a_\lambda$ be all the elements covered by I. If a congruence relation $x \sim y$ is defined in S such that* (6.6) *holds, then a subset $S_{a_i} = \{z;\ a_i \sim z\}$ satisfies the following conditions.*

(6.10) *S_{a_i} contains a_i,*

(6.11) *$S_{a_i} \subset C(a_i)$,*

(6.12) *S_{a_i} is convex.*

We add that any S_{a_i} does not intersect with S_{a_j} $(i \neq j)$ since a_i and a_j are incomparable.

5. On the other hand, the converse of the above lemma holds.

Lemma 29. *Let I be a proper ideal of a finite semilattice S, and let $a_1, \cdots, a_\lambda$ be all the elements covered by I. For each a_i we choose arbitrarily a subset S_{a_i} of S which satisfies the three conditions* (6.10), (6.11) *and* (6.12). *Let us define $x \sim y$ as follows. $x \sim y$ and $x \neq y$ if and only if both x and y belong to a same S_{a_i}. Then the relation becomes a congruence relation in S.*

Proof. We shall prove that $a_i \sim x$ implies $a_i z \sim xz$ or $a_i z = xz$ for any $z \in S$. Since $x \leqq xz$ and $x \leqq a_i$ because of (6.10), (6.11), and (6.8), xz and a_i are comparable: $xz \geqq a_i$ or $xz \leqq a_i$ (Cf. (6.11) and (6.9)). In the case $xz \geqq a_i$, we get directly $xz \geqq a_i z$ while $x \leqq a_i \leqq a_i z$ and $z \leqq a_i z$ lead to $xz \leqq a_i z$; hence $xz = a_i z \in I$. In the case $xz \leqq a_i$: Since $x \leqq xz \leqq a_i$ and $a_i \sim x$, the condition (6.12) shows $a_i \sim xz$, while $a_i z = a_i$ because $z \leqq xz \leqq a_i$; hence we have $xz \sim a_i z$. The proof of this lemma has been completed.

6. Theorem 18. *Let S be any finite semilattice. Find a sequence of ideals I_i of S*

$$I_0 \subset I_1 \subset I_2 \subset \cdots \subset I_\mu$$

where $I_i = I_{i-1} \cup \sum_{j=1}^{\lambda_i} S_{ij}$ $(i=1, \cdots, \mu)$ and we choose S_{ij} such that $a_{i1}, \cdots, a_{i\lambda_i}$ are covered by I_{i-1} and

(6.10′) $a_{ij} \in S_{ij}$,

(6.11′) $S_{ij} \subset C(a_{ij})$,

(6.12′) *S_{ij} is convex.*

Then we have a decomposition of S, $S = \sum_{\substack{1 \leqq j \leqq \lambda_i \\ 0 \leqq i \leqq \mu}} S_{ij}$, where $\lambda_0 = 0$, $I_0 = S_{00}$. Conversely any decomposition of S is obtained by such a process.

§ 7. Construction of Finite Semilattices.

With respect to construction of semilattices, we have already a few fundamental theorems, for example, Theorem 4 and Corollary 2. Consider the difference semigroup S^* of a semilattice S modulo a proper ideal I. Since a semilattice fulfils Condition A', S is constructed from I, S^* and a system of suitable translations of I (Cf. Theorems 7, 8). But our important problem is to discuss how to describe construction-method and isomorphism-condition in simple words as possible.

All the semilattices of order $\leqq 5$ were obtained in [9]. In this paragraph we shall discuss how all the finite semilattices are theoretically constructed. We shall show two methods: one by induction on the order, the other by induction on the dimension.

1. First Method. Let S_0 be any semilattice of order $n-1$ and φ be any translation of S_0. S denotes a composition of S_0 and a new one-element semilattice $\{p\}$ determined by φ: $px = xp = \varphi(x)$. Then S is a semilattice of order n in which p is a minimal element. Possibility of construction of such a semilattice S is assured by (5.2) of Lemma 17, Theorem 4, and Corollary 2. Conversely let S be any semilattice of order n and p be a minimal element of S. Consider a cut of S from p, where S_0 is the upper class, and the lower class S_1 is $\{p\}$, i.e. S is a composition of S_0 and $\{p\}$ (Cf. § 4)

Theorem 19. *A translation φ of a finite semilattice S_0 determines a composition S of S_0 and $\{p\}$.*

Accordingly, if all the semilattices S_0 of order $n-1$ are given, we can construct all semilattices S of order n by means of all the translations of all S_0, since the translations have been obtained by Theorem 17. S is denoted by $(S_0;\ \varphi, p)$. It happens that $(S_0;\ \varphi, p)$ and (S_0', φ', p') are isomorphic. We shall find a necessary and sufficient condition.

(1) First consider the case where there is an isomorphism f of $(S_0;\ \varphi, p)$ to $(S_0';\ \varphi', p')$ such that $f(S_0)=S_0'$, so $f(p)=p'$. Then, since Theorem 8 is applied to this case,[8] we have

Lemma 30. *Let f be an isomorphism of S_0 to S_0'. $S=(S_0;\ \varphi, p)$ is isomorphic to $S'=(S_0';\ \varphi', p')$ under the extension of f to S, if and only if $\varphi'=f\varphi f^{-1}$.*

(2) Secondly suppose that f is an isomorphism of $S=(S_0;\ \varphi, p)$ to $S'=(S_0';\ \varphi', q)$, but S_0 and S_0' are not isomorphic under f. Then p is not mapped to q but to p' in S_0', and q is not mapped by f^{-1} to p but to q' in S_0. Since p and q are minimal, p' and q' are also so. Let $\bar{S}_0=S-\{p\}-\{q'\}$, $\bar{S}_0'=S'-\{p'\}-\{q\}$. $\bar{S}_0$ and $\bar{S}_0'$ are ideals of S and S' respectively, and $\bar{S}_0$ is isomorphic to $\bar{S}_0'$ by f. Then $S=\bar{S}_0\cup\{q'\}\cup\{p\}$, $S'=\bar{S}_0'\cup\{p'\}\cup\{q\}$ where $q'p\in\bar{S}_0$, $p'q\in\bar{S}_0'$ and so S and S' are decomposed to an elementary semilattice of order 3. By Theorem 8, denoting $S_0=(\bar{S}_0;\ \bar{\varphi}, q')$, $S_0'=(\bar{S}_0';\ \bar{\varphi}', p')$, we have as a necessary and sufficient condition

(7.1) $\bar{\varphi}'=f\varphi f^{-1}$, $\varphi'=f\bar{\varphi}f^{-1}$,

where φ and φ' are considered as translations of $\bar{S}_0$ and $\bar{S}_0'$. Consequently we have the following lemma.

Lemma 31. *Let $S_0=(\bar{S}_0;\ \bar{\varphi}, q')$, $S_0'=(\bar{S}_0';\ \bar{\varphi}', p')$, and let f be an isomorphism of $\bar{S}_0$ to $\bar{S}_0'$, and, φ and φ' be translations of $\bar{S}_0$ and $\bar{S}_0'$, respectively. If and only if $\bar{\varphi}=f^{-1}\varphi' f$ and $\bar{\varphi}'=f\varphi f^{-1}$, then there is an isomorphism h of $S=(S_0;\ \varphi, p)$ to $S'=(S_0';\ \varphi', q)$ such that h is the extension of f, and $h(p)=p'$, $h(q')=q$.*

Summarizing the two lemmas,

Theorem 20. *$(S_0;\ \varphi, p)$ is isomorphic to $(S_0';\ \varphi', q)$ if and only if either*

(7.2) *S_0 is isomorphic to S_0' by a mapping f, and $\varphi'=f\varphi f^{-1}$,*

or (7.3) *there are ideals $\bar{S}_0$, $\bar{S}_0'$, and minimal elements q', p' such that $S_0=(\bar{S}_0;\ \bar{\varphi}, q')$, $S_0'=(\bar{S}_0';\ \bar{\varphi}', p')$, and, $\bar{S}_0$ is isomorphic to $\bar{S}_0'$ under f, q' is mapped to q, p is mapped to p' under f, and*

8) U in Theorem 8 is considered here as the semilattice of order 2.

$$\bar{\varphi}=f^{-1}\varphi' f\,,\quad \bar{\varphi}'=f\varphi f^{-1}\,.$$

By the way, if $S_0=S_0'$, f is considered as an automorphism of S_0.

2. Second Method. Let S be a finite semilattice whose dimension n is larger than 1. By Lemma 9 there certainly exists an element b of S such that $d[b]=n-1$. I denotes the set of all elements a such that $d[a]\geqq n-1$. Since $d[0]=n$ by Lemma 8, I contains 0, and I is not only a proper subset but also an ideal of S. Because, for $x\in I$, $y\in S$, $d[xy]\geqq d[x]\geqq n-1$ by Lemma 7. Let $S^*=(S:I)$ be the difference semigroup of S modulo I in Rees' sense [2]. We denote by x^* the image of x of S into S^* under the homomorphism of S to S^*. All the elements of I are mapped to a zero 0^* of S^*: $z^*=0^*$ for all $z\in I$. Of course S^* is a semilattice by Lemma 4, and the correspondence $x\to x^*$ is one to one as far as $d[x]<n-1$. According to Lemma 10, I is a subsemilattice of dimension 1.

Lemma 32. *I is an elementary semilattice.* (cf. § 4)

Lemma 33. *If $d[x]\neq n-1$, $x<y$ in S, then $x^*<y^*$ in S^*. Conversely if $x^*<y^*$, $d[y]<n-1$ in S^*, then $x<y$ in S.*

Proof. Let us, first, prove the former half. We may assume $d[x]<n-1$, for $d[x]=n$ implies $x=0$ and so there is no $y>x$. In the case where $d[y]\geqq n-1$, this theorem is clear. It is sufficient to treat only a case $d[x]<d[v]<n-1$. (See Lemma 7). Now $y=xz$ for some $z\in S$, where we see $d[z]<n-1$. For, $d[z]\leqq d[y]<n-1$. Since S is homomorphic to S^*, $y^*=x^*z^*$ so that $x^*\leqq y^*$ in S^*. But we conclude $x^*\neq y^*$ from $x\neq y$ because of the one-to-one correspondence $x\to x^*$ in the range of x, $d[x]<n-1$. Hence we obtain $x^*<y^*$ in S^*.

Conversely assume $x^*<y^*$ and $d[y]<n-1$. There is $z^*\in S^*$ such that $y^*=x^*z^*$ consequently $y=xz$ in S, where we see $d[z]<n-1$, $d[x]<n-1$, and we can easily show $y\neq x$. Therefore we have $x<y$. Thus this lemma has been proved.

As consequence of the above lemma, we have

Lemma 34. *Let $d[x]<n-1$. If and only if x is minimal in S, x^* is minimal.*

Lemma 35. *Let $d[S]=n$. If and only if $x_0<x_1<\cdots<x_m$ in S where x_0 is minimal, $d[x_m]<n-1$, and x_i covers x_{i-1} $(i=1,\cdots,m)$, then $x_0^*<x_1^*<\cdots<x_m^*$ in S^* such that x_0^* is minimal, $x_m^*\neq 0^*$, and x_i^* covers x_{i-1}^* $(i=1,\cdots,m)$.*

Immediately from these lemmas,

Theorem 21. *Let S be a finite semilattice whose dimension is $n>1$. If S is given, then I and S^* are uniquely determined in the above mentioned manner, and the height of each element x^* of S^* is given as $d[0^*]=n-1$, $d[x^*]=d[x]$ if $0\neq x\in S$. Consequently $d[S^*]=n-1$.*

Conversely, suppose that an elementary semilattice I and a semilattice S^*, whose dimension is $n-1$, are given. Since I satisfies Condition A' in § 2, Theorem 7 is applied to this case and hence S is obtained as the extension of I by S^* in the Clifford's sense. However we should remark that the following condition is added.

Lemma 36. *For any non-zero $x\in I$, there is a non-zero $\alpha\in S^*$ such that $\varphi_\alpha(x)=x$.*

For, since $d[x]=n-1$, there is non-zero $a^*=\alpha\in S^*$ such that $d[a]=n-2$, $a<x$, so $ax=x$. Conversely if this condition is satisfied, then Lemma 35 makes us see that all the elements of I are of height $\geqq n-1$ in the extension of I by S^*.

Theorem 22. *Suppose that the following three factors are given: a semilattice S^* of dimension $n-1$, an elementary semilattice I, and a system of the translations φ_α of I, where $0^*\neq\alpha\in S^*$, as seen in Theorem 7 and Lemma 36.*[9] *Then a semilattice S is uniquely determined such that S contains the ideal I all the elements of which are of height $\geqq n-1$ and the difference semigroup $(S:I)$ is isomorphic to S^*.*

For simplicity, we shall denote $S=(I, S^*, \varphi_\alpha)$. Thus we have a construction method by induction on the dimension, while there remains the isomorphism problem of the above extensions, but Theorem 8 solves this problem. Let us consider $S=(I, S^*, \varphi_\alpha)$ and $S'=(I', S'^*, \varphi'_{\alpha'})$. Suppose that S is isomorphic to S'. By Lemma 12, I is isomorphic to I' and so S^* is isomorphic to S'^* under any isomorphism ζ of S to S'. We can apply Theorem 8 to any ζ.

Theorem 23. *(I, S^*, φ_α) is isomorphic to $(I', S'^*, \varphi'_{\alpha'})$ if and only if*

(7.4) *there is an isomorphism ξ of I to I',*

(7.5) *there is an isomorphism η of S^* to S'^*,*

(7.6) *$\varphi'_{\eta(\alpha)}=\xi\varphi_\alpha\xi^{-1}$ for every non-zero $\alpha\in S^*$.*

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9) We note that the sense of the mark $*$ of U^* in Theorem 7 differs from that of $*$ of S^* here.

We shall be able to state the condition (7.6) in other words because of speciality of I. For simplicity, we assume $I=I'$, $S^*=S'^*$ without loss of generality. Firstly let us research automorphisms of a finite elementary semilattice I. Reminding us of Lemma 11, we have easily the following lemma, in which I may be not finite.

Lemma 37. *A mapping ξ of I onto itself is an automorphism of I if and only if ξ maps 0 to itself and causes a permutation of all the non-zero elements.*

Next, we shall find a relation between translations φ_α and $\xi\varphi_\alpha\xi^{-1}$ of I. According to Theorem 16, $\varphi_\alpha(x)=0$ or x for $x\in I$. For φ_α the subsets X_α and Y_α of I are defined as

$$X(\varphi_\alpha)=\{x;\ \varphi_\alpha(x)=0,\ x\in I\}\ ,\quad Y(\varphi_\alpha)=I-X(\varphi_\alpha)=\{y;\ \varphi_\alpha(y)=y,\ y\in I\}$$

where $X(\varphi_\alpha)$ is non-empty.

Lemma 38. *$x\in X(\varphi_\alpha)$ implies $\xi(x)\in X(\xi\varphi_\alpha\xi^{-1})$,*
$y\in Y(\varphi_\alpha)$ implies $\xi(y)\in Y(\xi\varphi_\alpha\xi^{-1})$.

Proof. If $x\in X(\varphi_\alpha)$, then $\xi\varphi_\alpha\xi^{-1}\xi(x)=\xi\varphi_\alpha(x)=\xi(0)=0$; if $y\in Y(\varphi_\alpha)$, then $\xi\varphi_\alpha\xi^{-1}\xi(y)=\xi\varphi_\alpha(y)=\xi(y)$.

Therefore we have

Corollary 10. *(I, S^*, φ_α) is isomorphic to $(I, S^*, \varphi_{\alpha'})$ if and only if*

(7.7) *there is an automorphism ξ of I,*

(7.8) *there is an automorphism η of S^*,*

(7.9) $X(\varphi_{\eta(\alpha)})=\xi(X(\varphi_\alpha))$ *for all non-zero $\alpha\in S^*$.*

§8. Compositions in the Case where T is Finite.

We shall again investigate compositions of semigroups in succession to §1. Let $T=\{\tau_0, \tau_1, \cdots, \tau_{n-1}\}$ be a semilattice of order n, where τ_0 is the zero, and suppose that there is given a system of semigroups S_τ, $\tau\in T$, which are not necessarily finite. Let us consider a composition S of S_τ $(\tau\in T)$ by T. As far as the construction method is concerned, we wish to apply Theorem 1 to this case repeatedly.

1. Suppose that a composition S of S_τ by T is obtained. Let τ_{n-1} be any minimal element of T and let $T=(T_0, \gamma, \tau_{n-1})$ where $T_0=\{\tau_0, \tau_1, \cdots, \tau_{n-2}\}$ is the upper class in the cut of T from τ_{n-1}, and γ is a right translation of T_0: $\gamma(\tau)=\tau\tau_{n-1}$ for $\tau\in T_0$, Letting

(8.1) $S_0=\sum_{i=0}^{n-2} S_{\tau_i}$,

S_0 is an ideal of S and we have a decomposition $S=S_0 \cup S_{\tau_{n-1}}$. We can consider S as a composition of the two semigroups S_0 and $S_{\tau_{n-1}}$.

Here we shall use similar notations as in §1: denote by $x, y, \cdots$ elements of S_0, by $\alpha, \beta, \cdots$ elements of $S_{\tau_{n-1}}$, by Φ and Ψ the right and left translation semigroups of S_0 respectively. Further, σ denotes the homomorphism of S_0 to T_0 due to the decomposition (8.1). Then the right translations φ_α of S_0 used in construction of a composition fulfils $\sigma\varphi_\alpha=\gamma\sigma$ because

$$\sigma(\varphi_\alpha(x))=\sigma(x\alpha)=\sigma(x)\,\sigma(\alpha)=\sigma(x)\tau_{n-1}=\gamma(\sigma(x)) \quad \text{for all} \quad x\in S_0 .$$

$\bar{\Phi}$ denotes the set of all right translations φ of S_0 which fulfil $\sigma\varphi=\gamma\sigma$, $\bar{\Psi}$ the set of all left translations ψ of S_0 which fulfil $\sigma\psi=\gamma\sigma$. By Theorem 1, we have easily the following theorem, in which, however, there is enough ground for improvement. We wish to describe the condition by means of a method of no induction.

Theorem 24. *In order that there is a composition of S_τ by T, it is necessary and sufficient that*

(8.2) *for a cut of T from any minimal element τ_{n-1}, there exists a composition S_0 of S_τ, $\tau\in T_0$, by the upper class T_0,*

(8.3) *the following subsemigroups $\bar{\Phi}$ and $\bar{\Psi}$ are not empty.*

$$\bar{\Phi}=\{\varphi;\ \sigma\varphi=\gamma\sigma,\ \varphi\in\Phi\}\ ,\quad \bar{\Psi}=\{\psi;\ \sigma\psi=\gamma\sigma,\ \psi\in\Psi\}\ .$$

(8.4) *there are subsemigroups Φ_0 and Ψ_0 of $\bar{\Phi}$ and $\bar{\Psi}$ respectively which fulfil the following conditions:*

(8.4.1) *there is a dual homomorphism $\alpha\to\varphi_\alpha$ of $S_{\tau_{n-1}}$ to Φ_0, and there is a homomorphism $\alpha\to\psi_\alpha$ of $S_{\tau_{n-1}}$ to Ψ_0,*

(8.4.2) $\varphi_\alpha\psi_\beta=\psi_\beta\varphi_\alpha$ *for all* $\varphi_\alpha\in\Phi_0$, $\psi_\beta\in\Psi_0$,

(8.4.3) $\varphi_\alpha(x)y=x\psi_\alpha(y)$ *for all* $x, y\in S_0$, *all* $\varphi_\alpha\in\Phi_0$, $\psi_\alpha\in\Psi_0$. *Any composition of S_τ by T is determined by φ_α and ψ_α as above-mentioned.*

Thus we have seen that the existence of a composition of S_τ by T is generally not assured. (See the later example.)

2. The case where T is a chain. Suppose that $T=\{\tau_0, \tau_1, \cdots, \tau_{n-1}\}$ is particularly a finite chain, and each S_τ is not necessarily finite.

Theorem 25. *If T is a finite chain, and semigroups S_τ $(\tau\in T)$ are arbitrarily given, then there exists a composition S of S_τ by T.*

Proof. We shall use induction on the order n and use the same notations as in the proof of Theorem 24. If S_0 is assumed to be obtained, $\bar{\Phi}$ and $\bar{\Psi}$ are not empty, because, since γ is an identical mapping of T_0 in the present case, $\bar{\Phi}$ and $\bar{\Psi}$ contain not only the identical mapping of S_0 but some inner translations of S_0. Φ_0 and Ψ_0 exist certainly, for example, we may choose as Φ_0 and Ψ_0 one-element semigroup which is composed of only the identical mapping. The existence of composition of S_τ by T is proved by Theorem 24, for Φ_0 and Ψ_0 fulfil the conditions.

Remark. If all S_τ are finite, we can choose as Φ_0 and Ψ_0 semigroups which are composed of some inner right translations and some inner left translations of S_0 respectively.

Remark. Even if T is an infinite chain, a composition S of S_τ $(\tau \in T)$ exists. Let x_τ, y_τ be elements of S_τ. The multiplication $x_\tau \cdot x_\mu$ in S is defined as follows.

$$x_\tau \cdot x_\mu = \begin{cases} x_\tau x_\tau & \text{if } \tau = \mu \\ x_{max(\tau,\mu)} & \text{if } \tau \neq \mu \end{cases}$$

Let us prove associative law: $(x_\tau y_\mu) z_\nu = x_\tau (y_\mu z_\nu)$.
In the case where τ, μ, ν are all distinct,

$$(x_\tau \cdot x_\mu) \cdot x_\nu = x_{max(\tau,\mu)} \cdot x_\nu = x_{max(\tau,\mu,\nu)} = x_\tau \cdot x_{max(\mu,\nu)} = x_\tau \cdot (x_\mu \cdot x_\nu)$$

In the case where only two are equal,

$$\begin{array}{ll}
\text{if } \tau > \mu, & (x_\tau y_\tau) z_\mu = x_\tau y_\tau = x_\tau (y_\tau z_\mu) \\
\text{if } \tau < \mu, & (x_\tau y_\tau) z_\mu = z_\mu = x_\tau z_\mu = x_\tau (y_\tau z_\mu) \\
\text{if } \tau > \mu, & (x_\tau z_\mu) y_\tau = x_\tau y_\tau = x_\tau (z_\mu y_\tau) \\
\text{if } \tau < \mu, & (x_\tau z_\mu) y_\tau = z_\mu y_\tau = z_\mu = x_\tau z_\mu = x_\tau (z_\mu y_\tau) \\
\text{if } \tau > \mu, & (z_\mu x_\tau) y_\tau = x_\tau y_\tau = z_\mu (x_\tau y_\tau) \\
\text{if } \tau < \mu, & (z_\mu x_\tau) y_\tau = z_\mu y_\tau = z_\mu = z_\mu (x_\tau y_\tau)
\end{array}$$

In the case where $\tau = \mu = \nu$, that law is clear. Thus the proof is finished.

3. The case where every S_τ is finite. The following theorem is obtained as a special case of Theorem 1 of Yamada's paper [11].

Theorem 26. *If a semilattice T and finite semigroups S_τ $(\tau \in T)$ are arbitrarily given, then there is a composition of S_τ $(\tau \in T)$ by T. Of course T may be infinite.*

Although the existence of a composition is thus assured in the case where T is finite, there remains the question if there exist Φ_0 and Ψ_0

fulfilling (8. 4. 1), (8. 4. 2), and (8. 4. 3), when each S_τ is finite.

If we get an affirmative answer to the problem, all the compositions of S_{τ_i} $(i=0, 1, \cdots, n-1)$ by T will be constructed by the successive procedure as stated in **1.**

We can not solve the problem completely here, and shall discuss it in another paper. If T has a special property, then the problem is affirmed.

Theorem 27. *If the translation semigroup Φ of T_0 is composed of the identical mapping and all the inner translations of T_0, then the problem is affirmed.*

Proof. If γ is an inner translation of T_0, $\gamma(\tau)=\tau\tau_i$ for a suitable $\tau_i \in T_0$. Let e_i be an idempotent element of S_{τ_i}: $\sigma(e_i)=\tau_i$. Then Φ_0 and Ψ_0 are defined as the sets of only one φ and ψ respectively:

$$\varphi(x)=xe_i, \quad \psi(x)=e_i x \qquad \text{for} \quad x \in S_0;$$

if γ is an identical mapping, then

$$\varphi(x)=x, \quad \psi(x)=x \qquad \text{for} \quad x \in S_0.$$

We see that Φ_0 and Ψ_0 satisfy the conditions of Theorem 23.

Corollary. *If T_0 is a finite lattice or a semilattice of order* 3, *then the problem is affirmed.*

Corollary. *In $T=(T_0, \gamma, \tau_{n-1})$, if the minimal element τ_{n-1} is covered by only one element, then Theorem* 27 *holds.*

Proof. Let τ_i be the only one element which covers τ_{n-1}. Since τ_{n-1} is minimal, $\tau_{n-1} < \tau\tau_{n-1}$ for any $\tau \in T_0$. Using the assumption, we can easily prove that $\tau_i \leqq \tau\tau_{n-1}$ and hence $\tau\tau_i \leqq \tau\tau_{n-1}$, while we get $\tau\tau_{n-1} \leqq \tau\tau_i$ from $\tau_{n-1} < \tau_i$. Hence we obtain $\gamma(\tau)=\tau\tau_{n-1}=\tau\tau_i$ for all $\tau \in T_0$, namely γ is an inner translation of T_0.

Corollary. *Even if T is a finite lattice, Theorem* 27 *holds.*

Proof. Let τ_{n-1} be the least element of T, then γ is the identical mapping of T_0 and hence Theorem 27 holds.

Any semilattice of order at most 5 is either a lattice or a semilattice having minimal element which is covered by only one element. (Cf. [9] or § 10) If T is of order at most 5, Theorem 27 holds.

4. Especially if S_τ $(\tau \in T)$ are all finite s-indecomposable semigroups, every composition S of S_τ by a finite semilattice T has T as the greatest s-homomorphic image of S. (Cf. [3]) On the other hand, if an s–decomposable semigroup S is a set-union of s–indecomposable subsemigroups in its greatest s–decomposion. (Cf. [3]) Hence we have

Theorem 28. *Any finite s–decomposable semigroup S is obtained as a composition of finite s–indecompsable semigroups S_τ ($\tau \in T$) by a finite semilattice T.*

As far as construction of compositions is concerned, Theorems 24 and 27 etc. are, of course, applied to this case, but we remark the following properties.

Lemma 39. *Suppose that a semigroup S_0 is decomposed to a semilattice T_0: $S_0 = \sum_{\tau \in T_0} S_{0\tau}$, and S_1 is an s–indecomposable semigroup. The translations φ_α, ψ_α of S_0 which determine a composition of S_0 and S_1 satisfy the following condition.*

For $x, y \in S_0$, $\sigma(x) = \sigma(y)$ implies $\sigma(\varphi_\alpha(x)) = \sigma(\psi_\beta(y)) = \sigma\psi_\alpha(x)) = \sigma(\psi_\beta(y))$ for every $\alpha, \beta \in S_1$, where σ is the homomorphism $S_{0\tau} \in x \rightarrow \tau \in T_0$.

Proof. According to the proof of Theorem 24, $\sigma\varphi_\alpha$ and $\sigma\psi_\alpha$ are translations of the semilattice T_0, and hence the set $\Phi_0 = \{\sigma\varphi_\alpha;\ \alpha \in S_1\}$ is a subsemilattice of the translation semilattice of T_0 by Theorem 13 and Lemma 4, while Φ_0 must be one-element semilattice because S_1 is s–indecomposable. Therefore $\sigma\varphi_\alpha(x) = \sigma\varphi_\beta(x)$ for every $\alpha, \beta \in S_1$ and $x \in S_0$. Combining this result with Lemma 16 in [3], we obtain the present lemma.

In this paper we let the study of s–indecomposable semigroups untouched. More precise research of construction of finte s–decomposable semigroups will be performed after the theory of finite s–indecomposable semigroups is completed. Finite s–indecomposable semigroups will be discussed in Part III$\sim$Part VI.

5. Remark. Unless all S_τ are finite, Theorem 26 is not always valid, even if all S_τ are s–indecomposable. Although we see this fact from Theorem 24, we shall verify it by an example of the three semigroups, which have no composition. (Cf. [12])

Let $T = \{0, 1, 2\}$ be a semilattice with multiplication

$$nm = 0 \text{ for } n \neq m, \quad n^2 = n, \text{ where } n, m = 0, 1, 2;$$

let $S_0 = \{1, 2, \cdots, n, \cdots\}$ be an additive semigroup of all positive integers, and let S_1 and S_2 denote the semigroups composed of only p and q respectively:

$$S_1 = \{p\}, \quad S_2 = \{q\} \quad \text{where} \quad p^2 = p, \quad q^2 = q.$$

Since an idempotent translation of S_0 is only the identical mapping according to Example 1 of § 3 in [5], a composition U_0 of S_0 and S_1 is nothing but S_0 with a two-sided unit p adjoined. (See Theorem 1) Meanwhile, the identical mapping is only one idempotent translation of U_0 because U_0 has a unit. (Cf. [4]); and so we can find no idempotent translation φ of U_0 such that $\varphi(p) \in S_0$. Consequently it is concluded that there is no composition of S_0, S_1, and S_2 by the given $T=\{0, 1. 2\}$ (Cf. Theorem 24). Further we add that S_0 is s-indecomposable, for S_0 is shown to be i-indecomposable as follows. Let $g(n)$ be the greatest i-homomorphic image of n of S_0. Since $n=\underbrace{1+\cdots+1}_{n}$, we have $g(n)=g\underbrace{(1)+\cdots+g}_{n}(1)=g(1)$.

§ 9. The Isomorphism Problem of Compositions.

1. In the final paragraph of composition theory of semigroups, we shall call the isomorphism problem to account, that is, the problem to discuss a condition for compositions $\sum S_\tau$ and $\sum S_\tau'$ to be isomorphic, and it is convenient to consider the problem in connection with the greatest s-decomposition. Here the problem of isomorphism between s-indecomposable semigroups remain unsolved, which will be argued in another paper. In this paragraph, S_τ and S_τ' are not necessarily finite.

2. First of all, let us add a few theorems to the preceding paper [3] for the preparation of the argument of the title.

Theorem 29. *If two semigroups S and S' are isomorphic, then the greatest μ-homomorphic images of S and S' are isomorphic.*

Proof. Let T and T' be the greatest μ-homomorphic images of S and S' respectively. Since T' is considered as a μ-homomorphic image of S: $S \rightarrow S' \rightarrow T'$, and so T is homomorphic to T'. (Cf. [3]) Similarly T is a homomorphic image of S'. Suppose that the homomorphism of T to T' is not an isomorphism, then it is concluded that T is a greater μ-homomorphic image of S' than T'. This arrives at the contradiction with the assumption that T' is the greatest μ-homomorphic image of S'.

Corollary 11. *Suppose that two semigroups S and S' are isomorphic. If S is s-indecomposable, then S' is also so.*

Theorem 30. *Let T and T' be the greatest s-homomorphic images of the semigroups S and S' respectively*:

(9.1) $S=\sum\limits_{\tau \in T} S_\tau$ (9.2) $S'=\sum\limits_{\tau' \in T'} S'_{\tau'}$, *where S_τ and $S'_{\tau'}$ are all s–indecomposable.* (*See Theorem* 7 *in* [3]) *If S and S' are isomorphic, then there is an isomorphism $\tau\to\tau'$ between T and T' such that S_τ and $S'_{\tau'}$ are isomorphic under the isomorphism between S and S'.*

Proof. Let f be an isomorphism of S to S' and let $f(S_\tau)$ denote the image of S_τ under f. Since the mapping, which associates the elements of $f(S_\tau)$ with τ, is a homomorphism of S' to T, we get an s–decomposition $S'=\sum_{\tau'\in T} f(S_\tau)$. This is the greatest s–decomposition of S' by Theorem 7 in [3], since $f(S_\tau)$ is s–indecomposable because of Corollary 11. Consequently the s–decomposition $S'=\sum_{\tau\in T} f(S_\tau)$ must coincide with (9. 2). Then there is an isomorphism $\tau\to\tau'$ between T and T' such that $S'_{\tau'}=f(S_\tau)$. Thus the theorem has been proved.

Corollary 12. *Suppose that the greatest s–decompositions of two semigrouos S and S' are given: $S=\sum_{\tau\in T} S_\tau$, $S'=\sum_{\tau'\in T'} S'_{\tau'}$. If S and S' are isomorphic, there is an isomorphism f of T to T' such that, for any subsemilattice U of T, $S_U=\sum_{\tau\in U} S_\tau$ is isomorphic to $S_{U'}=\sum_{\tau'\in U'} S_{\tau'}$ where $U'=\{f(\tau)\,;\ \tau\in U\}$.*

3. Suppose that $T=\{0, 1\}$ is a semilattice where $0^2=01=10=0$, $1^2=1$, and the semigroups S_i and S_i' are isomorphic ($i=0, 1$). Let S be the composition of S_0 and S_1 constructed by the translations φ_α and ψ_α of S_0 for $\alpha\in S_1$, and S' be the composition of S_0' and S_1' constructed by φ_β' and ψ_β' of S_0' for $\beta\in S_1'$. Then we have, if exists,

Lemma 40. *We assume that any isomorphism between S and S' causes isomorphisms between S_i and S_i' ($i=0, 1$). The compositions S and S' are isomorphic if and only if there are isomorphisms f of S_0 to S_0' and g of S_1 to S_1' such that*

$$(9.3)\quad \varphi'_{g(\alpha)}=f\varphi_\alpha f^{-1}\,,\quad \psi'_{g(\alpha)}=f\psi_\alpha f^{-1}\quad \textit{for all}\quad \alpha\in S_1\,.$$

Proof. First suppose that S is isomorphic to S'. Let f and g be isomorphisms of S_0 and S_1 to S_0' and S_1' respectively, which are caused by the isomorphism of S to S'. Immediately we get

$$(9.4)\quad f(x\alpha)=f(x)\,g(\alpha)\,,\quad f(\alpha x)=g(\alpha)\,f(x)\,.$$

Rewriting them,

$$f(\varphi_\alpha(x))=\varphi'_{g(\alpha)}f(x)\,,\quad f(\psi_\alpha(x))=\psi'_{g(\alpha)}f(x)\quad \text{for all}\quad x\in S_0\,,$$

so that $f\varphi_\alpha=\varphi'_{g(\alpha)}f$, $f\psi_\alpha=\psi'_{g(\alpha)}f$, whence we get (9. 3).

Conversely if there are given isomorphisms f of S_0 to S_0', g of S_1 to S_1', and translations φ_α, ψ_α, $\varphi'_{g(\alpha)}$, $\psi'_{g(\alpha)}$ such that the equalities (9.3) are satisfied, then we get easily (9.4). Hence S and S' are isomorphic under the mapping k of S to S' defined as $k(x)=f(x)$ for $x\in S_0$, $k(\alpha)=g(\alpha)$ for $\alpha\in S_1$.

4. Let $T=\{0, 1, \cdots, t\}$, $t>1$, be an elementary semilattice: $ij=i$ if $i=j$, $ij=0$ if $i\neq j$. As lemma 37 shows, an automorphism of such a semilattice T is a permutation of $\{0, 1, \cdots, t\}$ which fixes 0, and it follows that any subset of T which contains 0 is a subsemilattice of T. Suppose that S is a composition of semigroups S_i, $i\in T$, and S' is a composition of semigroups S_i' $i\in T$, where S_0 and S_0' are isomorphic and all S_i and S_j' are mutually isomorphic. Consider the sequence $\{\bar{S}_i\}$ of the compositions $\bar{S}_i$ $(i=0, 1, \cdots, t)$ which are contained in S and are defined as the following manner: $\bar{S}_0=S_0$, $\bar{S}_t=S$, and $\bar{S}_i$ is the composition of $S_0, S_1, \cdots, S_i$. $\bar{S}_i$ is considered as a composition of $\bar{S}_{i-1}$ and S_i, and we assume that $\bar{S}_i$ is determined by the translations $\varphi_\alpha^{(i-1)}$, $\psi_\alpha^{(i-1)}$ of $\bar{S}_{i-1}$ where $\alpha\in S_i$. Similarly $\bar{S}_i'$, $\varphi_\alpha^{(i-1)'}$ and $\psi_\alpha^{(i-1)'}$ are also defined.

Lemma 41. *Suppose that an isomorphism of S to S' causes an isomorphism of S_0 to S_0' as well as isomorphisms of S_i to some $S'_{i'}$. Such compositions S and S' are isomorphic if and only if there are a permutation p of $\{1, \cdots, t\}$, an isomorphism f_0 of S_0 to S_0'. and isomorphisms $f_{i,p(i)}$ of S_i to $S'_{p(i)}$ such that, for any $1<i\in T$, and any $\alpha\in S_i$,*

$$(9.5)\quad \begin{cases} \varphi'_{f_{i,p(i)}(\alpha)}(x)=f_0\varphi_\alpha^{(i)}f_0^{-1}(x)\,, \\ \psi'_{f_{i,p(i)}(\alpha)}(x)=f_0\psi_\alpha^{(i)}f_0^{-1}(x) \end{cases} \quad \textit{for} \quad x\in S_0,$$

$$(9.6)\quad \begin{cases} \varphi_\alpha^{(i)}(\beta)=f_0^{-1}(f_{j,p(j)}(\beta)\,f_{i,p(i)}(\alpha)) \\ \psi_\alpha^{(i)}(\beta)=f_0^{-1}(f_{i,p(i)}(\alpha)\,f_{j,p(j)}(\beta)) \end{cases} \quad \textit{for} \quad \beta\in S_j,\ 1\leqq j<i\,.$$

Proof. By the assumption, the mapping $i\to i'$ determines a permutation of $\{1, \cdots, t\}$. The formulas (9.5) are obtained by rewriting the images of $x\alpha$ and αx, $x\in S_0$, $\alpha\in S_i$, and the formulas (9.6) are similarly obtained from the images of $\alpha\beta$ and $\beta\alpha$ where $\alpha\in S_i$, $\beta\in S_j$. This lemma is proved as easily as the previous lemma.

5. Now, again, let T be a finite semilattice of order $\geqq 2$. Consider two systems $\{S_\tau\}$, $\{S_\tau'\}$ of s-indecomposable semigroups where $\tau\in T$. Let S denote the composition of S_τ, $\tau\in T$, and let S' denote the composition of S_τ', $\tau\in T$. The main problem is to find a necessary and sufficient condition for S and S' to be isomorphic. The method adopted here is induction with to the order of T.

At first, let us treat the case where T is of oroder 2: $S=S_0 \cup S_1$, $S'=S_0' \cup S_1'$. Since S_0, S_1, S_0', and S_1' are s-indecomposable, these s-decompositions are all greatest because of Theorem 7 in [3]. If S is isomorphic to S', then S_0 and S_1 are isomorphic to S_0' and S_1' respectively under the isomorphism of S to S'. (cf. Theorem 30) In this case, Lemma 40 is applicable and hence the condition (9. 3) is necessary and sufficient.

Next we must treat the case where the order of T is r under the assumption that the case, where T is of order$<r$, is solved. Let $M=\{\sigma_i\}$ be the set of all minimal elements of T. In the set M, we define an equivalence relation $\sigma_1 \sim \sigma_2$ meaning that

(9. 7) there is an automorphism p of T such that $\sigma_1=p(\sigma_2)$,

(9. 8) S_{σ_1} and S_{σ_2} are isomorphic.

By this relation M is decomposed into the sum of classes of the equivalent elements: $M=\sum M_i$.
By Lemma 37, each element of M_i is mapped to an element of the same class M_i under an automorphism of T. We can consider the two cases: one case where one class M_i at least contains only one element, and the other case where every class M_i contains more than one element. The former will be called Case I, and the latter Case II.

In case I, suppose that M_1 contains of only one element σ_1. Consider the cut of T from σ_1 in which T_0 denotes the upper class of T, then the s-decompositions of S and S' are defined as in the following manner.

(9. 9) $S=\bar{S}_0 \cup S_{\sigma_1}, \quad S'=\bar{S}_0' \cup S'_{\sigma_1}$

where $\bar{S}_0=\sum_{\tau\in T_0} S_\tau$, $\bar{S}_0'=\sum_{\tau\in T_0} S_\tau'$, and T_0 is of order$<r$. By Theorem 30 and the above remark, it follows that any isomorphism of S to S' maps $\bar{S}_0$ to $\bar{S}_0'$ and maps S_{σ_1} to S'_{σ_1}. Accordingly Lemma 40 is applied to this case.

In Case II, let $M_i=\{\sigma_1, \cdots, \sigma_t\}$, $t>1$, and consider a decomposition of T which is given as

$$T=T_0 \cup T_1 \cup \cdots \cup T_t \quad \text{where} \quad T_i=\{\sigma_i\}, \quad i=1, \cdots, t,$$

and T_0 is the set of elements of T beside $\sigma_1, \cdots, \sigma_t$. The factor semilattice of T given by this decomposition is an elementary semilattice. Then

(9. 10) $S=\bar{S}_0 \cup S_{\sigma_1} \cup \cdots \cup S_{\sigma_t}, \quad S'=\bar{S}_0' \cup S'_{\sigma_1} \cup \cdots \cup S'_{\sigma_t}$

where $\bar{S}_0 = \sum_{\tau \in T_0} S_\tau$, $\bar{S}_0' = \sum_{\tau \in T_0} S_\tau'$, and T_0 is of order $< r$.
It is easily seen that any isomorphism of S to S' makes $\bar{S}_0$ correspond to $\bar{S}_0'$ and makes S_{σ_i} correspond to some S'_{σ_j}, that is, a permutation of $\{1, \cdots, t\}$ is caused. Hence the result of Lemma 41 is applied to this case.

Summarizing the above description,

Theorem 31. *Let T be a finite semilattice and let S and S' be compositions of s-indecomposable semigroups $\{S_\tau\}$, $\tau \in T$, and $\{S_\tau'\}$, $\tau \in T$, respectively. The necessary and sufficient condition for S and S' to be isomorphic is by induction on the order of T stated that*

in Case I, *we obtain the decompositions* (9.9) *of S and S' such that $\bar{S}_0$ and $\bar{S}_0'$ are isomorphic, S_{σ_1} and $S'_{\sigma_{p(i)}}$ are isomorphic, and* (9.3) *is satisfied,*

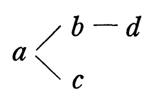
in Case II, *we obtain the decompositions* (9.10) *of S and S' where $\bar{S}_0$ and $\bar{S}_0'$ are isomorphic, and there is a permutation p of $\{1, \cdots, t\}$ such that S_{σ_i} and $S'_{\sigma_{p(i)}}$ are isomorphic, and further* (9.5) *and* (9.6) *are satisfied.*

§ 10. Examples of Computations.

Example 1. Let S_0 be a semilattice denoted by 124_4 in [9]

$$\begin{array}{|cccc|} a & a & a & a \\ a & b & a & b \\ a & a & c & a \\ a & b & a & d \end{array}$$

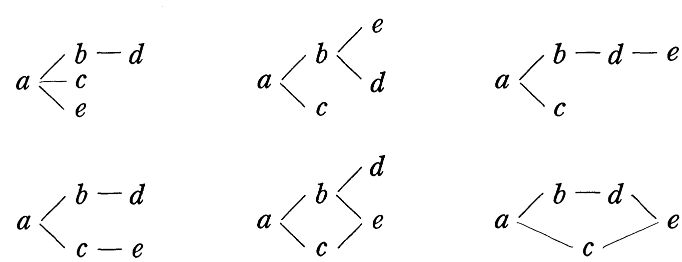
124_4



Find all the compositions of S_0 and $\{e\}$. By Thiorem 17, *its translations are obtained as following*

aaaa, *abab*, *abad*, *aaca*, *abcb*, *abcd*

These determine semilattices respectively:



which are isomorphic to 1149_5, 1151_5, 1156_5, 1152_5, 1150_5 and 1153_5 respectively. Since 124_4 has no automorphism except the identical mapping, these 6 semilattices are not isomorphic mutually (Cf. [9]).

Example 2. *Find all the semilattices of order* 5 *according to Theorems* 22 *ond* 23. All the possible cases of the order of I and S^* which appear in Theorem 22 are as follows.

	(1)	(2)	(3)	(4)
I	5	4	3	2
S^*	1	2	3	4

In the case (1) we get, at once, the elementary semilattice 1146_5. In the cade (2), letting I be $a \begin{smallmatrix} \diagup b \\ - c \\ \diagdown d \end{smallmatrix}$ and e^* be a non-zero element of S^*, we have only $\varphi_e = (abcd)$, so that S is nothing but 1147_5.
In the case (3) : I $\quad a < \begin{smallmatrix} b \\ c \end{smallmatrix}$
If S^* is $0^* - d^* - e^*$, then $\varphi_d = \varphi_e = (abc)$, obtaining 1154_5, and if S^* is $0^* < \begin{smallmatrix} d^* \\ e^* \end{smallmatrix}$, then we have the three isomorphically distinct semilattices

φ_d	*aaa*	*aba*	*aba*
φ_e	*abc*	*abc*	*aac*
Result	1148_5	1150_5	1152_5

In the case (4) : I $\quad a - b$

S^*	122_4 $0^* \begin{smallmatrix} \diagup c^* \\ - d^* \\ \diagdown e^* \end{smallmatrix}$			123_4 $0^* < \begin{smallmatrix} c^* \\ d^* \end{smallmatrix} > e^*$		124_4 $0^* < \begin{smallmatrix} c^* - e^* \\ d^* \end{smallmatrix}$		125_4 $0^* - c^* < \begin{smallmatrix} d^* \\ e^* \end{smallmatrix}$	126_4 $0^* - c^* - d^* - e^*$
φ_c	*ab*	*aa*	*aa*	*ab*	*aa*	*ab*	*ab*	*ab*	*ab*
φ_d	*ab*	*ab*	*aa*	*ab*	*ab*	*ab*	*aa*	*ab*	*ab*
φ_e	*ab*	*ab*	*ab*	*ab*	*ab*	*ab*	*ab*	*ab*	*ab*
Result	1155_5	1151_5	1149_5	1157_5	1153_5	1158_5	1156_5	1159_5	1160_5

Example 3. *Find all compositions of s-indecomposable semigroups* S_0 *and* S_1

$$S_0 \quad \begin{vmatrix} a & b & c & d \\ a & b & c & d \\ a & b & c & d \\ a & b & c & d \end{vmatrix}, \qquad S_1 = \{e\}.$$

Since $\psi_e = (abcd)$, the conditions (1.2), (1.4), (1.5), (1.6) are satisfied. For (1.1) and (1.3), we may consider all idempotent translations of S_0.

Without proof, the following properties are arranged.

(10.1) The right translation semigroup Φ of the right singular semigroup S_0 consists of all the mappings of S_0 into S_0.

(10.2) All the mappings of S_0 into itself are automorphisms of S_0.

Now let us decompose all the right translations into five classes by the relation $\varphi_e \sim \varphi'_e$ meaning that $\varphi_e = f\varphi'_e f^{-1}$ for some automorphism f of S_0. Thus we have five types:

φ_e	*aaaa*	*abaa*	*abab*	*abca*	*abcd*
Result	387	388	389	390	391

Example 4. *Find all the compositions of the two s-indecomposable semigroups* S_0 *and* S_1:

$$\begin{array}{c|ccc} & a & b & c \\ \hline a & a & a & a \\ b & a & a & a \\ c & a & a & b \end{array} \qquad \begin{array}{c|cc} & d & e \\ \hline d & d & d \\ e & d & d \end{array}$$

$$S_0 \qquad S_1$$
$$3_3 \qquad 2_2$$

All the right and left translations of S_0 are

$$aaa, \quad aab, \quad abc.$$

Considering (1.3) and (1.4), all pairs of $\begin{pmatrix}\varphi_d \\ \varphi_e\end{pmatrix}$, or $\begin{pmatrix}\psi_d \\ \psi_e\end{pmatrix}$ are

$$\begin{pmatrix}aaa \\ aaa\end{pmatrix} \quad \begin{pmatrix}aaa \\ aab\end{pmatrix} \quad \begin{pmatrix}abc \\ abc\end{pmatrix}$$

Then

$\psi \backslash \varphi$	$\begin{pmatrix}aaa \\ aaa\end{pmatrix}$	$\begin{pmatrix}aaa \\ aab\end{pmatrix}$	$\begin{pmatrix}abc \\ abc\end{pmatrix}$
$\begin{pmatrix}aaa \\ aaa\end{pmatrix}$	312_5	313_5	none
$\begin{pmatrix}aaa \\ aab\end{pmatrix}$	anti-isomorphic to 313_5	314_5	*none*
$\begin{pmatrix}abc \\ abc\end{pmatrix}$	*none*	*none*	315_5

For pairs written "none", (1.6) is not satisfied.

Example 5. *Find all compositions S of S_0, S_1, S_2, and S_3 by T*: $0 \begin{array}{l} \nearrow 1-2 \\ \searrow 3 \end{array}$

$$S_0 = \begin{array}{c|cc|} & a & b \\ \hline a & a & b \\ b & a & b \\ \hline \end{array} \qquad S_1 = \{c\}\,, \qquad S_2 = \{d\}\,, \qquad S_3 = \{e\}\,.$$

At first, all the compositions of S_0 and S_1, by 0—1, are

$$\begin{array}{|ccc|} \hline a & b & a \\ a & b & b \\ a & b & c \\ \hline \end{array} \qquad\qquad \begin{array}{|ccc|} \hline a & b & b \\ a & b & b \\ a & b & c \\ \hline \end{array}$$

$$11_3 \qquad\qquad\qquad 12_3$$

Secondly, let us find all compositions of S_0, S_1, aud S_2.

For 11_3,

$$\begin{array}{|ccc|c|} \hline a & b & a & \\ a & b & b & \\ a & b & c & c \\ \hline & & c & d \\ \hline \end{array}$$

while a right translation φ_d of 11_3 which fulfils $\varphi_d(c) = c$, is only (abc). ψ_d is als similar.

Hence we have 112_4.

For 12_3,

$$\begin{array}{|ccc|c|} \hline a & b & b & \\ a & b & b & \\ a & b & c & c \\ \hline & & c & d \\ \hline \end{array}$$

ψ_d of 12_3 is only (abc)

the translations φ_d of 12_3, which fulfil $\varphi_d(c) = c$, are only

$$(abc)\,, \qquad (bbc)\,.$$

Thus we have

$$\begin{array}{|cccc|} \hline a & b & b & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \\ \hline \end{array} \qquad\qquad \begin{array}{|cccc|} \hline a & b & b & b \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \\ \hline \end{array}$$

$$113_4 \qquad\qquad\qquad 114_4$$

At last we shall find S.

$$\begin{array}{c|c|c|} & a\ b\ c\ d & e \\ \hline \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{array}{c} 113 \\ \text{or } 114 \\ \text{or } 112 \end{array} & \varphi_e \\ \hline e & \psi_e & e \\ \hline \end{array}$$

where

$$\varphi_e(x) = a \text{ or } b \qquad \psi_e(x) = a \text{ or } b$$

For 112, the required idempotent translations are

$$\varphi_e = (aaaa), \quad (bbbb)\,; \qquad \psi_e = (abaa), \quad (abbb)$$

Hence

ψ_e \ φ_e	*aaaa*	*bbbb*
abaa	1048_5	*none*
abbb	*none*	*none*

where "none" means "(1. 6) is not fulfilled."

For 113,

ψ_e \ φ_e	*aaaa*	*bbbb*
abba	1052_5	*none*
abbb	*none*	1053_5

For 114,

ψ_e \ φ_e	*aaaa*	*abbb*	*bbbb*
abbb	1049_5	1050_5	1051_5

See [10].

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[10] Errata of the paper [9]

Page	Title	Error	Correction
35	XXI 1049 1050 1051	113	114
35	XXI 1052 1053	114	113

[11] M. Yamada: Compositions of Semigroups, Kōdai Math. Sem. Rep. **8**, 107–111 (1956)

[12] Yamada obtained also the same example in [11]. I obtained it independently of him.