Osaka Mathematical Journal
Vol. 6, No. 2, December, 1954.

# *The Fundamental Solution of the Parabolic Equation in a Differentiable Manifold, II*

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**§0. Introduction (and supplements to the previous paper).** Recently we have shown the existence of the fundamental solution of parabolic differential equations in a differentiable manifold (under some assumptions) in a previous paper[1] which will be quoted here as [FS]. We have set no boundary condition in [FS], while we shall here show the existence of the fundamental solution of parabolic differential equations with some boundary conditions in a compact subdomain of a differentiable manifold.

We shall first add the following supplements 1°) and 2°) to [FS], as we shall quote not only the results obtained in the paper but also the procedures used in it:

1°) CORRECTIONS. Throughout the paper [FS]

for $\exp\{M_1(t-s)^{\frac{1}{2}}\}$, read $\exp\{M_1(t-s)\}$;
for $\exp\{2M_1(t-s)^{\frac{1}{2}}\}$, read $\exp\{2M_1(t-s)\}$.

In the inequality (3.4),

for $(t-s)^{-(\frac{m}{2}+1)}$, read $(t-s)^{-\frac{m+1}{2}}$

2°) The proof of Theorem 4 in [FS, §4] is available only for the case: $t_0=\infty$. Instead of completing the proof, we are enough to establish a slightly ameliorated theorem as follows:

**Theorem 4.** i) *The function* $u(t, x; s, y)$ *is non-negative, and* $\int_M u(t, x; s, y)d_a y \leq \exp\{\lambda(t-s)\}$ *where* $\lambda=\sup_{t,x} c(t, x)$; ii) *if especially* $c(t, x)\equiv 0$, *then* $\int_M u(t, x; s, y)d_a y=1$.

We see that $|\lambda| \leq K(<\infty)$ by virtue of the assumption II) in [FS, p. 76]. To prove this theorem, we consider the functions

$$f_s(t, x)=\int_M u(t, x; s, y)f(y)d_a y \tag{0.1}$$

and

1) S. Itô: The fundamental solution of the parabolic equation in a differentiable manifold, Osaka Math. J. 5 (1953) 75-92.

$$g_s^{(\tau,n)}(t,x)=f_s(t,x)\exp\left\{-\left(\frac{t-s}{\tau-s}\right)^n\right\} \tag{0.2}$$

where $f(x)$ is an arbitrary function continuous on $\boldsymbol{M}$, with a compact support $\subset \boldsymbol{M}$ and satisfying $0\leqq f(x)\leqq 1$, and $n$ is a natural number $\geqq 2$ and $s<\tau<t_0$. Then $g_s^{(\tau,n)}(t,x)$ is continuous in $[s,t_0)\times\boldsymbol{M}$ and

$$g_s^{(\tau,n)}(s,x)\equiv f(x)\,, \quad \text{consequently} \quad 0\leqq g_s^{(\tau,n)}(s,x)\leqq 1\,. \tag{0.3}$$

By virtue of [FS, (3.10)] and the correction 1°) stated just above, we have

$$|f_s(t,x)|\leqq M\exp\{M(t-s)\} \tag{0.4}$$

for a suitable constant $M>0$.

**Lemma A.** *If $c(t,x)\leqq 0$, then the function $g_s^{(\tau,n)}(t,x)$ takes neither positive maximum nor negative minimum at any point in $(s,t_0)\times\boldsymbol{M}$.*

The proof may be achieved by the well known method and so will be omitted.

**Lemma B.** *If $c(t,x)\leqq 0$, then $u(t,x\,;s,y)\geqq 0$ and $\int_M u(t,x\,;s,y)d_a y\leqq 1$.*

Proof. By virtue of the continuity of $u(t,x\,;\,s,y)$ (see [FS, Theorem 1]), it is sufficient to prove that $0\leqq f_s(t,x)\leqq 1$ for any function $f(x)$ satisfying the above stated conditions (see (0.1)).

Suppose that $f_{s_1}(t_1,x_1)>1$ for some $t_1>s_1$ and $x_1$. Then, if we take $\tau$ and $\tau'$ such that $t_1<\tau<\tau'<t_0$ and sufficiently large $n$, we have

$$g_{s_1}^{(\tau,n)}(t_1,x_1)>1$$

and

$$|g_{s_1}^{(\tau,n)}(t_1,x_1)|>|g_{s_1}^{(\tau,n)}(t,x)| \quad \text{for any } t>\tau' \text{ and } x\in\boldsymbol{M}$$

by virtue of (0.2) and (0.4). From this fact and (0.3), it follows that $g_{s_1}^{(\tau,n)}(t,x)$ takes the positive maximum at some point in $(s,t_0)\times\boldsymbol{M}$; this contradicts Lemma A. Hence we have $f_s(t,x)\leqq 1$.

Similar argument shows that, if $f_{s_1}(t_1,x_1)<0$ for some $t_1<s_1$ and $x_1$, there exist $\tau$ and $n$ such that $g_{s_1}^{(\tau,n)}(t,x)$ takes the negative minimum at some point in $(s,t_0)\times\boldsymbol{M}$ contradictly to Lemma A. Hence we get $f_s(t,x)\geqq 0$, q.e.d.

Proof of Theorem 4. Let $u(t,x\,;s,y)$ be the fundamental solution of the equation $Lf=0$. Then we may easily prove that the function

$$u_\lambda(t,x\,;\,s,y)=e^{-\lambda(t-s)}u(t,x\,;\,s,y)$$

is the fundamental solution of the equation $(L-\lambda)f=0$. Since $c(t,x)-\lambda\leqq 0$, we have

$$u_\lambda(t,x;s,y)\geqq 0 \quad\text{and}\quad \int_M u_\lambda(t,x;s,y)d_ay\leqq 1$$

by Lemma B, and hence

$$u(t,x;s,y)\geqq 0 \quad\text{and}\quad \int_M u(t,x;s,y)d_ay\leqq e^{\lambda(t-s)}.$$

Finally, if $c(t,x)\equiv 0$, we may apply Theorem 2 in [FS] to the function $f(t,x)\equiv 1$ and we get

$$\int_M u(t,x;s,y)d_ay=1, \qquad \text{q.e.d.}$$

§ 1. **Fundametal notions and main results.** We shall say, by definition, that a function $f(x)$ defined on a subset $E$ of the Euclidean $m$-space $R^m$ satisfies the *generalized Lipschitz condition* in $E$ if, for any $x\in E$, there exist positive numbers $N$, $\delta$ and $\gamma$ (each of them may depend on $x$) such that $|f(x)-f(y)|\leqq N\sum_i|x^i-y^i|^\gamma$ whenever $y\in E$ and $|x^i-y^i|\leqq\delta (i=1,\dots,m)$, where $(x^i)$ and $(y^i)$ denote the coordinates of $x$ and $y$ respectively[2].

A function $f(x)$ defined on a domain $G\subset R^m$ is said to be *of $C^{k,L}$-class* if $f(x)$ is of $C^k$-class in the usual sense and each partial derivative of $k$-th order of $f(x)$ satisfies the generalized Lipschitz condition in $G$. A *manifold of $C^{k,L}$-class*, a *hypersurface of $C^{k,L}$-class*, etc. should be understood analogously.

Let $\boldsymbol{M}$ be an $m$-dimensional manifold of $C^{4,L}$-class, and $\boldsymbol{G}$ be a domain in $\boldsymbol{M}$ such that the closure $\bar{\boldsymbol{G}}$ is compact and the boundary $\boldsymbol{B}=\bar{\boldsymbol{G}}-\boldsymbol{G}$ consists of a finite number of hypersurfaces of $m$-1 dimension and of $C^{4,L}$-class.

Under a *canonical coordinate around* $x\in\boldsymbol{M}$, we understand any local coordinate which maps a neighbourhood of $x$ onto the interier of the unit sphere in $R^m$ and especially transforms $x$ to the centre of the sphere. For each $x\in\boldsymbol{M}$ and any fixed canonical coordinate around $x$, we denote by $U_\varepsilon(x)$ the neighbourhood of $x$ of the form $\{y\in\boldsymbol{M};\ \sum(y^i-x^i)^2<\varepsilon\}$ where $0<\varepsilon\leqq 1$.

We understand the partial derivatives of a function $f(x)$ (defined on $\bar{\boldsymbol{G}}$) at $\xi\in\boldsymbol{B}$ as follows: $\partial f(\xi)/\partial x^i=\alpha_i$ $(\xi\in\boldsymbol{B})$, $i=1,\dots,m$, means that

$$f(x)=f(\xi)+\alpha_i(x^i-\xi^i)+o\left(\textstyle\sum_i|x^i-\xi^i|\right) \quad\text{for any}\quad x\in U(\xi)\cap\bar{\boldsymbol{G}}$$

where $U(\xi)$ is a coordinate neighbourhood of $\xi$.

2) Cf. Footnote 1) in [FS].

We fix $s_0$ and $t_0$ such that $-\infty < s_0 < t_0 < \infty$ and consider the parabolic differential operator $L$:

$$L \equiv L_{tx} = A_{tx} - \frac{\partial}{\partial t}, \qquad (x \in \bar{G},\ s_0 < t < t_0) \tag{1.1}$$

where

$$A \equiv A_{tx} = a^{ij}(t, x)\frac{\partial^2}{\partial x^i \partial x^j} + b^i(t, x)\frac{\partial}{\partial x^i} + c(t, x) \tag{1.2}$$

and $\| a^{ij}(t, x) \|$ is a strictly positive-definite symmetric matrix for each $\langle t, x\rangle \in (s_0, t_0) \times \bar{G}$; $a^{ij}(t, x)$ and $b^i(t, x)$ are transformed between any two local coordinates by means of (1.3) and (1.4) in [FS]. We assume that

(A.1) *the functions*

$$\frac{\partial a^{ij}(t, x)}{\partial t}, \frac{\partial^3 a^{ij}(t, x)}{\partial x^h \partial x^k \partial x^l}, \frac{\partial b^i(t, x)}{\partial x^k} \quad (i, j, h, k, l = 1, \dots, m)$$
$$\text{and} \quad c(t, x)$$

*satisfy the generalized Lipschitz condition in* $[s_0, t_0] \times \bar{G}$.

We define the *partial derivative* $\partial f(\xi)/\partial \boldsymbol{n}_{t\xi}$ *to the outer transversal direction* $\boldsymbol{n}_{t\xi}$ as follows: when $\boldsymbol{B}$ is represented by $\psi(x) \equiv \psi(x^1, \dots, x^m) = 0$ with respect to a local coordinate around $\xi$ and $\psi(x) > 0$ in $\boldsymbol{G}$, we set

$$\frac{\partial f(\xi)}{\partial \boldsymbol{n}_{t\xi}} = -\frac{\partial f(\xi)}{\partial x^i} \cdot \frac{\partial \psi(\xi)}{\partial x^j} a^{ij}(t, \xi); \tag{1.3}$$

this notion is independent of the special choice of the local coordinate around $\xi$ by virtue of the transformation rule for $a^{ij}(t, x)$ (see [FS. (1.3)]). If we take a local coordinate with respect to which $a^{ij}(t, \xi) = \delta^{ij}$ i.e. $a^{ij}(t, \xi)\frac{\partial^2}{\partial x^i \partial x^j}$ = *Laplacian* at the point $\langle t, \xi\rangle$ (fixed), then $\partial f(\xi)/\partial \boldsymbol{n}_{t\xi}$ means the partial derivative to the outer normal direction to $\boldsymbol{B}$. We consider the boundary condition:

$$\alpha(t, \xi) f(\xi) + \{1 - \alpha(t, \xi)\}\frac{\partial f(\xi)}{\partial \boldsymbol{n}_{t\xi}} = 0 \quad (\xi \in \boldsymbol{B}) \tag{$B_{\alpha(t)}$}$$

for each $t$, where $\alpha(t, \xi)$ is a function on $[s_0, t_0] \times \boldsymbol{B}$, of $C^1$-class in $t$ and of $C^{2,\gamma}$-class in $\xi$ and $0 \leqq \alpha(t, \xi) \leqq 1$. We shall say that a function $f(t, x)$ on $(s_0, t_0) \times \bar{G}$ satisfies the boundary condition $(B_\alpha)$ if it satisfies $(B_{\alpha(t)})$ for any $t \in (s_0, t_0)$.

We define the metric tensor $a_{ij}(x)$, as stated in [FS, p. 79], and consider the measure $d_a x = \sqrt{a(x)}dx^1 \cdots dx^m$ ( $a(x) = \det \| a_{ij}(x) \|$) and define the adjoint operator $L^*$ resp. $A^*$ of $L$ resp. $A$ with respect to this measure. If $\boldsymbol{M}$ is an orientable Riemannian manifold with a metric tensor $g_{ij}(x)$ *a priori*, then it is natural to take the measure $d_g x = \sqrt{g(x)}dx^1 \cdots dx^m$ $(g(x) = \det \| g_{ij}(x) \|)$ in place of $d_a x$; in this case, it is sufficient only to replace $a(x)$ by $g(x)$ throughout the course of the present paper, while $a_{ij}(x)$ should not be replaced by $g_{ij}(x)$.

We assume further that:

(A. 2) *the following relations hold on the set*

$$\{\langle t, \xi \rangle;\ \alpha(t, \xi) \neq 1\}\ (\subset [s_0, t_0] \times \boldsymbol{B}):$$

$$\frac{\partial a^{ij}(t, \xi)}{\partial \boldsymbol{n}_{t\xi}} = 0 \text{ [3]} \quad (i, j = 1, \dots, m) \quad \textit{and} \tag{1.4}$$

$$b^i(t, \xi) = \frac{1}{\sqrt{a(\xi)}} \cdot \frac{\partial}{\partial x^j}[\sqrt{a(\xi)}a^{ij}(t, \xi)] \ (i = 1, \dots, m). \tag{1.5}$$

Under the above stated conditions (A. 1) and (A. 2), we shall consider the parabolic differential equations $Lf = 0$ and $L^*f^* = 0$ in the domain $\boldsymbol{G}$ with the boundary condition $(B_\alpha)$.

By definition, a function $u(t, x; s, y)$, $s_0 < s < t < t_0$; $x, y \in \bar{\boldsymbol{G}}$, is called a *fundamental solution of the parabolic equation* $Lf = 0$ *with the boundary condition* $(B_\alpha)$ if, for any $s$ and any function $f(x)$ which is continuous in $\bar{\boldsymbol{G}}$ and satisfies the condition $(B_{\alpha(s)})$, the function

$$f(t, x) = \int_G u(t, x; s, y) f(y) d_a y \quad (t < s) \tag{1.6}$$

satisfies the conditions[4]:

$$\begin{cases} f(t, x) \text{ is of } C^1\text{–class in } t \text{ and of } C^2\text{–class in } x, \text{ and satisfies the} \\ \text{equation } Lf = 0 \text{ as well as the boundary condition } (B_\alpha) \end{cases} \tag{1.7}$$

and

$$\lim_{t \downarrow s} f(t, x) = f(x) \qquad \text{uniformly on } \bar{\boldsymbol{G}}. \tag{1.8}$$

A function $u^*(s, y; t, x)$, $s_0 < s < t < t_0$; $x, y \in \bar{\boldsymbol{G}}$, is called a *fundamental solution of the adjoint equation* $L^*f^* = 0$ (of the equation $Lf = 0$) *with the boundary condition* $(B_\alpha)$ if, for any $t$ and any continuous function

3) It is true that $\partial a^{ij}/\partial \boldsymbol{n}_t$ depends on the local coodinate, but the condition (1.4) is independent of it, because, if $\| a^{ij} \|$ is changed into $\| \bar{a}^{ij} \|$ by means of the coodinate transformation $(x^i) \to (\bar{x}^i)$, then we get $\dfrac{\partial \bar{a}^{ij}}{\partial \boldsymbol{n}_t} = \dfrac{\partial \bar{x}^i}{\partial x^k} \cdot \dfrac{\partial \bar{x}^j}{\partial x^l} \cdot \dfrac{\partial a^{kl}}{\partial \boldsymbol{n}_t}$ by virtue of [FS, (1.3)].

4), 5) Cf. [FS, Definition 2]. The conditions corresponding to (1.9) and (1.9*) in [FS] follow from (1.7) and (1.7*) respectively in the case where $\bar{\boldsymbol{G}}$ is compact.

$f(x)$ on $G$, the function

$$f^*(s,y)=\int_G u^*(s,y\,;\,t,x)f(x)d_a x \quad (s<t) \tag{1.6*}$$

satisfies the conditions[5]:

$$\text{(1.7*)}\ \begin{cases} f^*(s,y) \text{ is of } C^1\text{-class in } s \text{ and of } C^2\text{-class in } y, \text{ and satisfies} \\ \text{the equation } L^*f^*=0 \text{ as well as the boundary condition } (B_\alpha) \end{cases}$$

and

$$\lim_{s\uparrow t} f^*(s,y)=f(y) \tag{1.8*}$$

pointwisely in $G$ and also strongly in $L^1(G)$.

The purpose of the present paper is to prove the following theorems, which are literally the same as those in [FS][6] except the statements concerning the boundary condition.

**Theorem 1.** *There exists a function $u(t,x,s,y)$ of $C^1$-class in $t$ and $s(s_0<s<t<t_0)$ and of $C^2$-class in $x$ and $y$ $(x,y\in\bar{G})$, with the following properties:*

i) *$u(t,x\,;\,s,y)$ is a fundamental solution of the equation $Lf=0$ with the boundary condition $(B_\alpha)$,*

ii) *$u^*(s,y\,;\,t,x)=u(t,x\,;\,s,y)$ is a fundamental solution of the adjoint equation $L^*f^*=0$ with the boundary condition $(B_\alpha)$,*

iii) *$L_{tx}u(t,x\,;\,s,y)=0$, $L^*_{sy}u(t,x\,;\,s,y)=0$ and $u(t,x\,;\,s,y)$ satisfies the boundary condition $(B_\alpha)$ as a function of $<t,x>$ and also as a function of $<s,y>$,*

iv) *$\int_G u(t,x,\tau,z)u(\tau,z\,;\,s,y)d_a z=u(t,x\,;\,s,y)$, $s<\tau<t$.*

**Theorem 2.** *Let $u(t,x\,;\,s,y)$ and $u^*(s,y\,;\,t,x)$ be the functions stated in Theorem* 1.

i) *If a function $f(t,x)$ on $(s,t_0)\times\bar{G}$ satisfies* (1.7) *and* (1.8) *where $f(x)$ is continuous in $\bar{G}$ and satisfies $(B_\alpha)$, then it is expressible by* (1.6).

ii) *If a function $f^*(s,y)$ on $(s_0,t)\times\bar{G}$ satisfies* (1.7*) *and* (1.8*) *where $f(x)$ is a continuous function on $\bar{G}$, then it is expressible by* (1.6*).

**Theorem 3.** *If a function $v(t,x\,;\,s,y)$ is continuous in the region: $s_0<s<t<t_0\,;\,x,y\in\bar{G}$, and fatisfies the condition* i) *or* ii) *in Theorem* 1, *then it is identical with $u(t,x\,;\,s,y)$ stated in Theorem* 1.

**Theorem 4.** i) *$u(t,x\,;\,s,y)\geqq 0$ and $\int_G u(t,x\,;\,s,y)d_a y\leqq e^{\lambda(t-s)}$ where*

6) As for Theorem 4, see the supplement to [FS] in §0 of the present paper.

$\lambda = \sup_{t,x} c(t, x)$ ; ii) *if* $c(t, x) \equiv 0$ *in the differential operator* $A_{tx}$ *and if* $\alpha(t, \xi) \equiv 1$ *in the boundary condition* $(B_\alpha)$, *then* $\int_G u(t, x\,; s, y) d_a y = 1$.

We shall show, in another paper[7], the existence of the fundamental solution of the parabolic differential equation with a boundary condition considered in a domain whose closure is not compact.

§ 2. **Preliminaries.** The following lemma may be proved by means of Lebesgue's convergence theorem, and will be useful throughout the present paper:

**Lemma 1.** *Let* $(X, \mu)$ *be a measure space, and assume that*

i) $f(t, \chi)$ *is measurable in* $\chi \in X$ *for each* $t \in (t_1, t_2)$,
ii) $f(t, \chi)$ *is differentiable in* $t$ *for a.a.* $\chi \in X$ *and*
iii) *there exists a measurable function* $\varphi(\chi)$ *such that*

$$\left|\frac{\partial f(t, \chi)}{\partial t}\right| \leq \varphi(\chi) \text{ in } (t_1, t_2) \text{ and } \int_X \varphi(\chi) d\mu(\chi) < \infty .$$

*Then*

$$\frac{d}{dt}\int_X f(t, \chi) d\mu(\chi) = \int_X \frac{\partial f(t, \chi)}{\partial t} d\mu(\chi) .$$

Now let $\boldsymbol{G}, \boldsymbol{B}$ and $A_{tx}$ be as stated in § 1 and $z$ be any fixed point in $\boldsymbol{B}$. Then, for any canonical coordinate (see § 1) around $z$, $\boldsymbol{B} \cap U_1(z)$ is represented by means of $\psi(x^1, \dots, x^m) = 0$ where $\psi$ is a function of $C^{4,L}$-class. Hence, considering a suitable coordinate transformation in $U_1(z)$, we may show that

**Lemma 2.** *There exists a canonical coordinate* $(x^i)$ *around* $z$ *such that* $\boldsymbol{B} \cup U_1(z)$ *is expressible by* $x^1 = 0$ *and that* $x^1 > 0$ *in* $\boldsymbol{G} \cap U_1(z)$.

Next we shall prove that

**Lemma 3.** *Let* $(x^i)$ *be a canonical coordinate as stated in Lemma* 2, *and consider the coordinate transformation:* $(x^i) \rightarrow (x_t^i)$, *for each* $t (s_0 \leqq t \leqq t_0)$, *defined by*

$$(2.1) \qquad \begin{cases} x_t^1 = \varphi^1(t, x) \equiv \gamma x^1 \\ x_t^j = \varphi^j(t, x) \equiv \gamma \left\{ -\dfrac{a^{1j}(t, \xi_x)}{a^{11}(t, \xi_x)} x^1 + x^j \right\}, \quad j = 2, \dots, m, \end{cases}$$

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7) See the author's paper: Fundamental solutions of parabolic differential equations and eigenfunction expensions for elliptic differential equations, forthcoming to Nagoya Mathematical Journal.

*where* $\xi_x = \langle 0, x^2, \dots, x^m \rangle (\in \boldsymbol{B})$ *for* $x = \langle x^1, \dots, x^m \rangle \in U_1(z)$ *and* $\gamma$ *is a suitable positive constant. Then there exists* $\delta = \delta_z > 0$ *such that* i) $U_\delta(z) \subset U_1^t(z) \subset U_1(z)$ *and* $U_{\delta/3}(z) \subset U_{1/3}^t(z)$ *for any* $t$, ii) $\boldsymbol{B}$ *is represented by* $x_t^1 = 0$ *in* $U_1^t(z)$ *and* iii) *if* $a^{ij}(t,x)$ *is changed into* $a_\varphi^{ij}(t,x)$ *by means of this transformation* $(i,j=1,\dots,m)$, *then*

$$a_\varphi^{1j}(t,\xi) = a_\varphi^{j1}(t,\xi) = 0 \quad \text{and} \quad a_{1j}^\varphi(t,\xi) = a_{j1}^\varphi(t,\xi) = 0\,, \quad j=2,\dots,m, \tag{2.2}$$

*for any* $\xi \in \boldsymbol{B} \frown U_1^t(z)$, *where* $U_\varepsilon^t(x) = \{y \in \boldsymbol{M}\,;\ \sum_i (y_t^i - x_t^i)^2 < \varepsilon\}$ *and* $\| a_{ij}^\varphi(t,x) \| = \| a_\varphi^{ij}(t,x) \|^{-1}$. *The mapping* $\varphi_t(x) = \langle \varphi^1(t,x), \dots, \varphi^m(t,x) \rangle$ *of* $U_\delta(z)$ *into* $U_1^t(z)$*is one-to-one and of* $C^{3,L}$*–class in* $x$, *and* $a_\varphi^{ij}(t,x)$, $i,j = 1,\dots,m$, *are of* $C^1$*–class in* $t$ *and of* $C^{2,L}$*–class in* $x$.

Proof. We notice that $a^{11}(t,x) > 0$ in $U_1(z)$, and consider the coordinate transformation (2.1) around $z$. Then $x_t^1 = 0$ if and only if $x^1 = 0$, and we have for any $\xi = \langle 0, x^2, \dots, x^m \rangle \in \boldsymbol{B} \frown U_1(z)$

$$\begin{cases} \left(\dfrac{\partial x_t^1}{\partial x^k}\right)_{x=\xi} = \gamma \delta_k^1 \\ \left(\dfrac{\partial x_t^j}{\partial x^k}\right)_{x=\xi} = \gamma \left\{ -\dfrac{a^{1j}(t,\xi)}{a^{11}(t,\xi)} \delta_k^1 + \delta_k^j \right\} \quad (\delta_k^j: \text{Kronecker's delta}) \end{cases} \tag{2.3}$$

for $1 \leqq j \leqq m$ and $2 \leqq j \leqq m$. Hence the Jacobian

$$\frac{\partial(x_t^1, \dots, x_t^m)}{\partial(x^1, \dots, x^m)}$$

is bounded away from zero in $U_{\varepsilon_1}(z)$ for suitable $\varepsilon_1$ $(0 < \varepsilon_1 < 1)$ which may be chosen independently of $t$ by virtue of the continuity of $a^{ij}(t,x)$ on the compact set $[s_0, t_0] \times \overline{U_\varepsilon(z)}$ for any $\varepsilon$ $(0 < \varepsilon < 1)$, and hence the transformation (2.1) is well defined in $U_{\varepsilon_1}(z)$. Considering the continuity of $a^{ij}(t,x)$ on $[s_0, t_0] \times \overline{U_{\varepsilon_1}(z)}$ again, we may determine $\gamma$ and $\delta > 0$ so that $U_\delta(z) \subset U_1^t(z) \subset U_1(z)$ and $U_{\delta/3}(z) \subset U_{1/3}^t(z)$ for any $t$. By virtue of the transformation rule for $a^{ij}$ (see [FS, (1.3)]), we have, for any $\langle t, \xi \rangle \in [s_0, t_0] \times (\boldsymbol{B} \frown U_1^t(z))$ and for $j \geqq 2$,

$$\begin{aligned} a_\varphi^{1j}(t,\xi) &= \left(\frac{\partial x_t^1}{\partial x^k}\right)_{x=\xi} \cdot \left(\frac{\partial x_t^j}{\partial x^l}\right)_{x=\xi} a^{kl}(t,\xi) \\ &= -\gamma^2 \frac{a^{1j}(t,\xi)}{a^{11}(t,\xi)} a^{11}(t,\xi) + \gamma^2 a^{1j}(t,\xi) = 0 \quad (\text{see } (2.3))\,, \end{aligned}$$

and consequently we get (2.2). The last part of Lemma 3 is also evident by means of the above arguments.

**§3. Local construction of a quasi-parametrix.** Let $\boldsymbol{G}$, $\boldsymbol{B}$ and $A_{tx}$ be as before, let $z$ be any fixed point in $\boldsymbol{B}$, and let $(x^i)$ and $(x_t^i)$ $(s_0 \leqq t \leqq t_0)$ be canonical coordinates around $z$ as stated in Lemma 3. Then we have

$$\frac{\partial f(\xi)}{\partial \boldsymbol{n}_t} = -\frac{\partial f(\xi)}{\partial x_t^i} a_\varphi^{i1}(t, \xi) = -a_\varphi^{11}(t, \xi)\frac{\partial f(\xi)}{\partial x_t^1} \qquad (\xi \in \boldsymbol{B}) \tag{3.1}$$

for any function $f(x)$ of $C^1$-class, and hence the assumption (1.4) implies that

$$\frac{\partial a_{ij}^\varphi(t, \xi)}{\partial x_t^1} = 0 \quad \text{on } \{\langle t, \xi\rangle ; \alpha(t, \xi) \neq 1\}, \tag{3.2}$$

Now we put for $s_0 \leqq s < t \leqq t_0$ and $X, Y \in R^m$

$$\begin{cases} V_0(A_{ij}; t, X; s, Y) = (t-s)^{-\frac{m}{2}} \exp\left[-\dfrac{A_{ij}(X^i - Y^i)(X^j - Y^j)}{4(t-s)}\right] \\ V_0(A_{ij}) = \displaystyle\int_{R^m} \exp\left[-\dfrac{A_{ij}Y^iY^j}{4}\right] dY^1 \dots dY^m, \end{cases} \tag{3.3}$$

and define for $s_0 \leqq s < t \leqq t_0$ and $x, y \in U_\delta(z) \frown \bar{\boldsymbol{G}}$ ($\delta = \delta_z$ as stated in Lemma 3)

$$\begin{cases} V(t, x; s, y) = V_0(a_{ij}^\varphi(t, x); t, \varphi_t(x); s, \varphi_s(y)) \quad \text{(see Lemma 3)} \\ \bar{V}(t, x; s, y) = V_0(a_{ij}^\varphi(t, x); t, \varphi_t(x); s, \bar{\varphi}_s(y)) \\ V(t, x) = V_0(a_{ij}^\varphi(t, x)) \end{cases} \tag{3.4}$$

where $\bar{\varphi}_s(y) = \langle -\varphi^1(s, y), \varphi^2(s, y), \dots, \varphi^m(s, y)\rangle$. Further we put

$$\begin{cases} p(t, x; s, y) \\ \quad = \dfrac{2(t-s)\cdot\alpha(t, \xi_{tx})}{2(t-s)\alpha(t, \xi_{tx}) + \varphi^1(s, y)[1 - \alpha(t, \xi_{tx}) \exp\{-|\varphi^1(t, x)|^2\}]} \\ q(t, x; s, y) \\ \quad = \dfrac{\varphi^1(s, y)[1 - \alpha(t, \xi_{tx}) \exp\{1 - |\varphi^1(t, x)|^2\}]}{2(t-s)\alpha(t, \xi_{tx}) + \varphi^1(s, y)[1 - \alpha(t, \xi_{tx}) \exp\{-|\varphi^1(t, x)|^2\}]} \end{cases} \tag{3.5}$$

where $\xi_{tx}$ is the point $(\in \boldsymbol{B})$ defined by the equations:

$$\varphi^1(t, \xi_{tx}) = 0, \quad \varphi^j(t, \xi_{tx}) = \varphi^j(t, x) \quad \text{for } j \geqq 2;$$

such $\xi_{tx}$ is uniquely determined for any $x \in U_\delta(z)$ and any $t$ by virtue of Lemma 3.

Applying (3.1), (3.2), Lemma 1 and Lemma 3 to (3.3), (3.4) and (3.5), and making use of the fact that $\partial f/\partial \boldsymbol{n}_{t\xi}$ is independent of the local coordinate, we obtain

$$\frac{\partial V(t, \xi)}{\partial \boldsymbol{n}_{t\xi}} = 0 \tag{3.6}$$

and

$$\begin{aligned} \frac{\partial V(t, \xi ; s, y)}{\partial \boldsymbol{n}_{t\xi}} &= -a_{\varphi}^{11}(t, \xi) \cdot \left\{ -a_{11}^{\varphi}(t, \xi) \cdot \frac{-\varphi^1(s, y)}{2(t-s)} V(t, \xi ; s, y) \right\} \\ &= \frac{-\varphi^1(s, y)}{2(t-s)} V(t, \xi ; s, y) \end{aligned} \tag{3.7}$$

for $\langle t, \xi \rangle$ such that $\xi \in \boldsymbol{B} \frown U_\delta(z)$ and $\alpha(t, \xi) \neq 1$, and we get also

$$\frac{\partial p(t, \xi ; s, y)}{\partial \boldsymbol{n}_{t\xi}} = \frac{\partial q(t, \xi ; s, y)}{\partial \boldsymbol{n}_{t\xi}} = 0 \tag{3.8}$$

and

$$V(t, \xi ; s, y) = \bar{V}(t, \xi ; s, y) \tag{3.9}$$

for any $\xi \in \boldsymbol{B} \frown U_\delta(z)$. We define

$$\begin{aligned} W_z(t, x ; s, y) &= p(t, x; s, y) J_s(y) \frac{V(t, x ; s, y) - \bar{V}(t, x ; s, y)}{V(t, x)} \\ &\quad + q(t, x ; s, y) J_s(y) \frac{V(t, x ; s, y) + \bar{V}(t, x ; s, y)}{V(t, x)} \end{aligned} \tag{3.10}$$

where

$$J_s(y) = \frac{\partial[\varphi^1(s, y), \dots, \varphi^m(s, y)]}{\partial[y^1, \dots, y^m]} \quad \text{(Jacobian).} \tag{3.11}$$

Then we may prove from (3. 6—9) and by simple calculation that

$$\alpha(t, \xi) W_z(t, \xi ; s, y) + \{1 - \alpha(t, \xi)\} \frac{\partial W_z(t, \xi ; s, y)}{\partial \boldsymbol{n}_{tx}} = 0 \quad \text{for} \quad \xi \in U_\delta(z) \frown \boldsymbol{B}, \tag{3.12}$$

that is, $W_z(t, x ; s, y)$ satisfies the boundary condition $(B_\alpha)$ as a function of $\langle t, x \rangle \in [s_0, t_0] \times U_\delta(x)$. Since

$$\begin{aligned} &\int_{U_\delta(z) \cap \boldsymbol{G}} V(t, x ; s, y) J_s(y) dy + \int_{U_\delta(z) \cap \boldsymbol{G}} \bar{V}(t, x ; s, y) J_s(y) dy \\ &\leqq \int_{R^m} V_0(a_{ij}^{\varphi}(t, x) ; t, \varphi_t(x) ; s, Y) dY = V(t, x) \\ &\qquad (dy = dy^1 \cdots dy^m, \ dY = dY^1 \cdots dY^m) \end{aligned} \tag{3.13}$$

and since the denominators and numerators in the right-hand side of (3. 5) are positive for any $x, y \in U_\delta(z) \frown \boldsymbol{G}$, we get

(3. 14) $\quad \int_{U_\delta(z)\cap G} |W_z(t, x; s, y)| dy \leqq 1$ for any $x \in U_\delta(z) \frown \bar{G}$.

Now we have the following

**Lemma 4.** *If $f(x)$ is continuous in $\bar{G}$ and vanishes outside $U_\delta(z)$, then*

$$\lim_{t \downarrow s} \int_G \frac{V(t, x; s, y) + \bar{V}(t, x; s, y)}{V(t, x)} f(y) J_s(y) dy = f(x) \tag{3.15}$$

*uniformly in $U_\delta(z) \frown \bar{G}$.*

PROOF. By virtue of (3. 3) and the uniform continuity of $\varphi^j(t, x)$ on $[s_0, t_0] \times \overline{U_\delta(z)}$, we may show that

$$\lim_{t \downarrow s} \int_{R^m} \frac{V_0(a_{ij}^\varphi(t, x); t, \varphi_t(x); s, Y)}{V_0(a_{ij}^\varphi(t, x))} F(Y) dY = F(\varphi_s(x)) \quad \text{uniformly in } U_\delta(z) \frown \bar{G}$$

for any continuous function $F(Y)$ with a compact support; and hence, if especially $F(\bar{Y}) = F(Y)$ where $\bar{Y} = \langle -Y^1, Y^2, \dots, Y^m \rangle$ for $Y = \langle Y^1, Y^2, \dots, Y^m \rangle$, then

$$\lim_{t \downarrow s} \int_{R^m(Y^1>0)} \frac{V_0(a_{ij}^\varphi(t, x); t, \varphi_t(x); s, Y) + V_0(a_{ij}^\varphi(t, x); t, \varphi_t(x); s, \bar{Y})}{V_0(a_{ij}^\varphi(t, x))} F(Y) dY$$
$$= F(\varphi_s(x)) \quad \text{uniformly in} \quad U_\delta(z) \frown \bar{G}.$$

Putting

$$F(Y) = F(\bar{Y}) = \begin{cases} f(\varphi_s^{-1}(Y)) & \text{if} \quad \sum_i (Y^j)^2 < 1 \\ 0 & \text{if not} \end{cases}$$

in the above relation, and considering (3. 4) and (3. 11), we obtain (3. 15).

**Lemma 5.** *If $f(x)$ is such a function as stated in Lemma 3 and if $\boldsymbol{D}$ is an open set containing $\boldsymbol{B}^{(s)} = \{\xi \in \boldsymbol{B}; \alpha(s, \xi) = 1\}$, where $s$ is any fixed real number $(s_0 < s < t_0)$, then*

$$\lim_{t \downarrow s} \int_G W_z(t, x; s, y) f(y) dy = f(x) \quad \textit{uniformly in} \quad U_\delta(z) \frown \bar{G} - \boldsymbol{D}.$$

PROOF. Let $\varepsilon$ be an arbitrary positive number. Then, by virtue of Lemma 4, there exists $\Delta_1 > 0$ such that

$$\left| \int_{U_\delta(z)\cap \bar{G}} \frac{V(t, x; s, y) + \bar{V}(t, x; s, y)}{V(t, x)} J_s(y) f(y) dy - f(x) \right| < \frac{\varepsilon}{4} \tag{3.16}$$

for any $x \in U_\delta(z) \frown \bar{G}$ whenever $s < t < s + \Delta_1$. On the other hand, by virtue of (3.13) and (3.14), there exists $\eta_1>0$ such that

$$(3.17)\quad \left|\int_{U_\delta(z)\cap\{\varphi^1(s,y)<\eta_1\}\cap\bar{G}} \frac{V(t,x,;s,y)+\bar{V}(t,x;s,y)}{V(t,x)} J_s(y)f(y)dy\right|<\frac{\varepsilon}{4}$$

and that

$$(3.18)\quad \left|\int_{U_\delta(z)\cap\{\varphi^1(s,y)<\eta_1\}\cap\bar{G}} W_z(t,x;s,y)f(y)dy\right|<\frac{\varepsilon}{4}.$$

Since $1-\alpha(t,\xi_{tx})\exp\{-|\varphi^1(t,x)|^2\}>0$ for any $t$ and any $x\in U_\delta(z)\cap\bar{G}-B^{(s)}$, there exists $\eta_2>0$ such that

$$1-\alpha(t,\xi_{tx})\exp\{-|\varphi^1(t,x)|^2\}\geqq\eta_2 \qquad \text{(see (3.5))}$$

for any $t$ and any $x\in U_\delta(z)\cap\bar{G}-D$. Hence $\varphi^1(s,y)\geqq\eta_1$ implies that

$$|1-q(t,x;s,y)|=|p(t,x;s,y)|\leqq(t-s)/\eta_1\eta_2$$

for any $t\geqq s$ and any $x\in U_\delta(z)\cap\bar{G}-D$, and hence it follows from (3.10) and (3.13) that there exists $\Delta_2>0$ such that

$$(3.18)\quad \left|\int_{U_\delta(z)\cap\{\varphi^1(s,y)\geqq\eta_1\}\cap\bar{G}}\left\{W_z(t,x;s,y)-\right.\right.$$
$$\left.\left.-\frac{V(t,x;s,y)+\bar{V}(t,x;s,y)}{V(t,x)}J_s(y)\right\}f(y)dy\right|<\frac{\varepsilon}{4}$$

for any $x\in U_\delta(z)\cap G-\bar{D}$ whenever $s<t<s+\Delta_2$. Since $f(y)=0$ for $y\in\bar{G}-U_\delta(z)$, it follows from (3.16—19) that

$$|\textstyle\int_G W_z(t,x;s,y)f(y)dy-f(x)|<\varepsilon \quad \text{for any } x\in U_\delta(z)\cap\bar{G}-D$$

whenever $s<t<s+\min\{\Delta_1,\Delta_2\}$. Thus we obtain Lemma 5.

**Lemma 6.** *Assume that $f(x)$ is continuous in $\bar{G}$, vanishes outside $U_\delta(z)$ and satisfies the boundary condition $(B_{\alpha(s)})$. Then*

$$\lim_{t\downarrow s}\textstyle\int_G W_z(t,x;s,y)f(y)dy=f(x) \quad \textit{uniformly in } U_\delta(z)\cap\bar{G}.$$

Proof. Let $\varepsilon$ be an arbitrary positive number, and put

$$D=\{x;x\in\bar{G},|f(x)|<\varepsilon/5\}\cup\{\bar{x};x\in\bar{G},|f(x)|<\varepsilon/5\}$$

where $\bar{x}=\langle -x^1,x^2,\dots,x^m\rangle$ for $x=\langle x^1,x^2,\dots,x^m\rangle$. Then, by virtue of the assumption of this lemma, $D$ is an open set containing $B^{(s)}=\{\xi\in B;\ \alpha(s,\xi)=1\}$ and hence, by Lemma 5, there exists $\Delta>0$ such that

(3.20) $\quad |\int_G W_z(t, x; s, y) f(y) dy - f(x)| < \varepsilon$ for any $x \in U_\delta(z) \cap \bar{G} - D$

whenever $s < t < s + \Delta$. On the other hand, by Lemma 4, there exists $\Delta' > 0$ such that

$$\int_G \frac{V(t, x; s, y) + \bar{V}(t, x; s, y)}{V(t, x)} |f(y)| J_s(y) dy < \frac{2}{5}\varepsilon$$
$$\text{for any } x \in U_\delta(z) \cap \bar{G} \cap D$$

whenever $s < t < s + \Delta'$. Hence, considering the non-negativity of $V(t, x; s, y)$, $\bar{V}(t, x; s, y)$ and $J_s(y)$ (see the proof of Lemma 3) and using the facts: $0 \leqq p(t, x; s, y) \leqq 1$ and $0 \leqq q(t, x; s, y) \leqq 1$, we obtain from (3.10) that

$$\left| \int_G W_z(t, x; s, y) f(y) dy \right| < \frac{4}{5}\varepsilon \quad \text{for any } x \in U_\delta(z) \cap \bar{G} \cap D$$

and accordingly

(3.21) $\quad |\int_G W_z(t, x; s, y) f(y) dy - f(x)| < \varepsilon$ for any $x \in U_\delta(z) \cap \bar{\bar{G}} \cap D$

whenever $s < t < s + \Delta'$. From (3.20) and (3.21) we get

$$|\int_G W_z(t, x; s, y) f(y) dy - f(x)| < \varepsilon \quad \text{for any } x \in U_\delta(z) \cap \bar{\bar{G}}$$

whenever $s < t < s + \min\{\Delta, \Delta'\}$. Thus we obtain Lemma 6.

Next, let $f(\tau, y)$ be a continuous function on $(s, t_0) \times G$ which vanishes outside $U_\delta(z)$ and satisfies the condition: $\int_s^t \int_G |f(\tau, y)| dy d\tau < \infty$, and put

$$f(t, x, \tau) = \int_G W_z(t, x; \tau, y) f(\tau, y)\, dy, \quad t > \tau > s,$$
$$F(t, x) = \int_s^t f(t, x, \tau)\, d\tau .$$

Then we have

**Lemma 7.** i) *$f(t, x, \tau)$ and $F(t, x)$ satisfy the boundary condition $(B_\alpha)$ in $U_\delta(z) \cap B$;* ii) *for any $s'(t_0 > s' > s)$*

$$\lim_{\tau \downarrow s'} \int_G f(\tau, x) W_z(\tau, x; s', y) dx = f(s', y) \text{ in } G \cap U_\delta(z);$$

iii) *if $f(\tau, y)$ satisfies the generalized Lipschitz condition in $(s, t_0) \times \bar{G}$, then*

$$\frac{\partial F(t, x)}{\partial t} = f(t, x) + \int_s^t \int_G \frac{\partial W_z(t, x; \tau, y)}{\partial t} f(\tau, y) dy d\tau ,$$
$$A_{tx} F(t, x) = \int_s^t \int_G A_{tx} W_z(t, x; \tau, y) f(\tau, y) dy d\tau .$$

OUTLINE OF THE PROOF. The proposition i) may be shown by means of (3.12) and Lemma 1, and the proposition ii) may be proved similarly to [FS, Lemma 2]. The proposition iii) is proved as follows. Considering the fact that the mapping $\varphi_t(x)$ is one-to-one and of $C^{2,L}$-class for any $t$ (see Lemma 3), using the same idea as in [FS, Lemmas 1 and 3] and applying Lemma 1 (§1), we may show that

$$\frac{\partial f(t, x, \tau)}{\partial t} = \int_G \frac{\partial W_z(t, x; \tau, y)}{\partial t} f(\tau. y)dy,$$

$$\frac{\partial f(t, x, \tau)}{\partial x^i} = \int_G \frac{\partial W_z(t, x; \tau, y)}{\partial x^i} f(\tau, y)dy,$$

$$\frac{\partial^2 f(t, x, \tau)}{\partial x^i \partial x^j} = \int_G \frac{\partial^2 W(t, x; \tau, y)}{\partial x^i \partial x^j} f(\tau, y)dy.$$

and

$$\lim_{\substack{t > t' > \tau \\ t \to \tau}} f(t, x, t') = f(\tau, x)$$

and that there exist $M > 0$ and $\gamma = \gamma(t, x) > 0$ such that

$$\frac{\partial f(t', s, \tau)}{\partial t'} \leqq M(t-s)^{-(1-\frac{\gamma}{2})} \text{ whenever } s < \tau < t \leqq t';$$

further we have

$$\int_s^t \left| \frac{\partial f(t, x, \tau)}{\partial x^i} \right| d\tau < \infty \text{ and } \int_s^t \left| \frac{\partial^2 f(t, x, \tau)}{\partial x^i \partial x^j} \right| d\tau < \infty.$$

Hence we may prove the proposition iii) by the same manner as in [FS, Lemma 4].

**Lemma 8.** *If $\omega(t, x)$ is a function of $C^1$-class in $t$ and of $C^2$-class in $x$, and vanishes outside $U_\delta(z)$, then there exists a constant $M_0 > 0$ such that*

$$|L_{tx}[\omega(t, x) W_z(t, x; s, y)]| \leqq M_0(t-s)^{-\frac{m+1}{2}} \exp\left\{-\frac{M_0 \sum_i (x^i - y^i)^2}{4(t-s)}\right\}.$$

This may be proved similarly to [FS, Lemma 5].

Finally we define a quasi-parametrix $W_z(t, x; s, y)$ around any inner point $z$ of $G$ as follows. We fix a canonical coordinate $(x^i)$ around $z$ satisfying $U_1(z) \subset G$ and put

$$\begin{cases} \delta_z = 1 \\ x_t^i \equiv \varphi^i(t, x) = x^i, \ i = 1, \dots, m, \text{ for any } t \end{cases}$$

(consequently $\varphi_t(x) = \langle x^1, \dots x^m \rangle$ and $a^{\varphi}_{ij}(t, x) = a_{ij}(t, x)$—cf. Lemma 3). Using this local coordinate, we define $V(t, x; s, y)$ and $V(t, x)$ by means of (3.3) and (3.4), and put

$$W_z(t, x; s, y) = \frac{V(t, x; s, y)}{V(t, x)} \qquad (s_0 < s < t < t_0; x, y \in U_1(z)).$$

Then we may easily prove that Lemmas 6, 8 and Lemma 7 ii), iii) hold for $W_z(t, x; s, y)$ defined here. (See Lemmas 2, 4 and 5 in [FS].)

**§4. Gloval construction of a quasi-parametrix and a fundamental solution.** For each $z \in \bar{G}(= G + B)$, we fix canonical coordinates $(x^i)$ and $(x^i_t)$ around $z$ as stated in §2, and put

$$U(z, \varepsilon) = \{x \in M; \sum_i (x^i - z^i)^2 < \varepsilon\} \qquad (\varepsilon > 0).$$

Since $\bar{G}$ is compact, there exists a finite sequence $\{z_1, \dots, z_N\} \subset \bar{G}$ such that

(4.1) $$\bar{G} \subset \bigcup_{\nu=1}^{N} U(z_\nu, \delta_\nu/3) \text{ where } \delta_\nu = \delta_{z_\nu} \text{ (see §2)},$$

and then, since

(4.2) $$z_\nu \in G \text{ implies } U(z_\nu, \delta_\nu) \subset G \text{ (see §2)},$$

we have

(4.3) $$B \subset \bigcup_{z_\nu \in B} U(z_\nu, \delta_\nu/3).$$

Let $\omega(\lambda)$ be a function of $C^{2,L}$-class in $0 \leq \lambda < \infty$ such that $\omega(\lambda) = 1$ or $0$ if $0 \leq \lambda \leq 1/3$ or $\lambda \geq 2/3$ respectively and that $0 \leq \omega(\lambda) \leq 1$ for any $\lambda$, and put for each $\nu$

$$\omega_\nu(t, x) = \begin{cases} \omega(\sum_i [x^i_t - (z_\nu)^i_t]^2) & \text{for } x \in \bar{G} \cap U(z_\nu, \delta_\nu) \\ 0 & \text{for } x \in \bar{G} - U(z_\nu, \delta_\nu). \end{cases}$$

Then $\omega_\nu(t, x)$, $\nu = 1, \dots, N$, are of $C^1$-class in $t$ and of $C^{2,L}$-class in $x \in \bar{G}$, and

(4.4) $$\frac{\partial \omega_\nu(t, \xi)}{\partial \boldsymbol{n}_{t\xi}} = 0 \qquad \text{for any } \langle t, \xi \rangle \in [s_0, t_0] \times B;$$

this may be proved by considering the local coordinate $(x^i_t)$ around $z_\nu$ for each $t$ since the operator $\partial/\partial \boldsymbol{n}_t$ is independent of the special choice of the local coordinate.

Now let $a_\nu(x)$ be the restriction of $a(x) = \det \| a_{ij}(x) \|$ (see §1) to $U(z_\nu, \delta_\nu)$ with the local coordinate $(x^i)$ around $z$ stated above, and put, for $s_0 < s < t < t_0$,

$$W_\nu(t, x ; s, y) = \begin{cases} W_{z_\nu}(t, x ; s, y) \text{ (as stated in § 3) } & \text{if } x, y \in U(z_\nu, \delta_\nu) \cap \bar{G} \\ 0 & \text{if not.} \end{cases}$$

We define a quasi-parametrix:

$$Z(t, x ; s, y) = \frac{\sum_\nu \omega_\nu(t, x)\omega_\nu(s, y) W_\nu(t, x ; s, y)}{\sum_\nu \omega_\nu(t, x)^2 \sqrt{a_\nu(y)}} \quad \begin{pmatrix} s_0 < s < t < t_0 \\ x, y \in \bar{G} \end{pmatrix}$$

Then $Z(t, x ; s, y)$ is of $C^1$-class in $t$ and $s$, and of $C^{2,\Gamma}$-class in $x$ and $y$, and it follows from (3.12), (4.2), (4.3) and (4.4) that

$$\alpha(t, \xi)Z(t, \xi ; s, y) + \{1-\alpha(t, \xi)\} \frac{\partial Z(t, \xi ; s, y)}{\partial \boldsymbol{n}_{t\xi}} = 0 \quad (\xi \in \boldsymbol{B}), \tag{4.5}$$

that is, $Z(t, x ; s, y)$ satisfies the boundary condition $(B_\alpha)$ as a function of $\langle t, x \rangle$. Further, by virtue of Lemmas 6, 7 and 8, we obtain the following three lemmas.

**Lemma 9.** i) *If $f(x)$ is continuous in $\bar{G}$, then*

$$\lim_{t \downarrow s} \int_G Z(t, x ; s, y) f(y) d_a y = f(x) \text{ in } \boldsymbol{G};$$

*if especially $f(x)$ satisfies the boundary condition $(B_{\alpha(s)})$, then the above convergence is uniform in $\bar{G}$.*

ii) *if $f(t, x)$ is continuous in $[s, t_0) \times \bar{G}$, then*

$$\lim \int_G f(t, x) Z(t, x ; s, y) d_a x = f(s, y) \text{ in } \boldsymbol{G}.$$

**Lemma 10.** *If $f(\tau, y)$ is continuous in $(s, t_0) \times \bar{G}$ and satisfies the condition: $\int_s^t \int_G | f(\tau, y) | d_a y d\tau < \infty$, then*

$$f(t, x, \tau) = \int_G Z(t, x ; \tau, y) f(\tau, y) d_a y \qquad (t < \tau < s)$$

*and*

$$F(t, x) = \int_s^t f(t, x, \tau) d\tau$$

*satisfy the boundary condition $(B_\alpha)$; if further $f(\tau, y)$ satisfies the generalized Lipschitz condition in $(s, t_0) \times \bar{G}$, then*

$$\begin{cases} \dfrac{\partial F(t, x)}{\partial t} = f(t, x) + \displaystyle\int_s^t \int_G \frac{\partial Z(t, x ; \tau, y)}{\partial t} f(\tau, y) d_a y d\tau, \\ A_{tx} F(t, x) = \displaystyle\int_s^t \int_G A_{tx} Z(t, x ; \tau, y) f(\tau, y) d_a y d\tau. \end{cases}$$

**Lemma 11.** *$Z(t, x; s, y)$ satisfies all inequalities stated in* [FS, Lemma 8] *for a suitable constant $M > 0$.*

Thus we see that $Z(t, x; s, y)$ has all properties stated in [FS, § 2]. Hence, starting from this quasi-parametrix $Z(t, x; s, y)$, we may construct $u(t, x; s, y)$ in the entirely same way as in [FS, § 3]. We may also construct $u^*(t, x; s, y)$ in the similar manner for the adjoint equation $L^* f^* = 0$ with the same boundary condition $(B_\alpha)$. The functions $u(t, x; s, y)$ and $u^*(t, x; s, y)$ defined here have the properties stated in [FS, § 3] where the manifold $\boldsymbol{M}$ should be replaced by the compact domain $\bar{\boldsymbol{G}}$ and the uniformity of the convergence in [FS, (3.13)] may be proved if and only if $f(x)$ is the limit of a uniformly convergent sequence of functions satisfying the the boundary condition $(B_{\alpha(s)})$[8]. Moreover $u(t, x; s, y)$ and $u^*(t, x; s, y)$ satisfy the boundary condition $(B_\alpha)$ as functions of $< t, x >$ — see Lemma 10 and the procedure of the construction of $u(t, x; s, y)$ (in [FS, § 3]).

## § 5. Proof of Theorems.

**Lemma 12.** *If $f(x)$ and $h(x)$ are functions of $C^2$-class on $\bar{\boldsymbol{G}}$ satisfying the boundary condition $(\boldsymbol{B}_{\alpha(t)})$ ($t$ : fixed), then*

$$\int_G f(x) \cdot A_{tx} h(x) d_a x = \int_G A^*_{tx} f(x) \cdot h(x) d_a x .$$

PROOF. By partial integration, we obtain the Green's formula:

$$\begin{aligned} &\int_G f(x) \cdot A_{tx} h(x) d_a x - \int_G A^*_{tx} f(x) \cdot h(x) d_a x \\ &= \int_{\boldsymbol{B}} \left\{ f(\xi) \frac{\partial h(\xi)}{\partial \boldsymbol{n}_t} - \frac{\partial f(\xi)}{\partial \boldsymbol{n}_t} h(\xi) \right\} \tilde{d}\xi \\ &+ \int_{\boldsymbol{B}} \left\{ \frac{\partial}{\partial x^j} [\sqrt{a(\xi)}\, a^{ij}(t, \xi)] - \right. \\ &\left. - \sqrt{a(\xi)}\, b^i(t, \xi) \right\} \frac{\partial \psi(\xi)}{\partial x^i} f(x) h(x) \tilde{d}\xi \end{aligned}$$

where $\tilde{d}\xi = d\xi^1, \dots, d\xi^{m-1}$ is the hypersurface area on $\boldsymbol{B}$ and $\psi(x)$ is such function that $\psi(x) = 0$ determines $\boldsymbol{B}$ and that $\psi(x) > 0$ in $\boldsymbol{G}$. But the right-hand side equals zero by virtue of the boundary condition $(\boldsymbol{B}_{\alpha(t)})$ and the assumption (1.5). Hence we obtain Lemma 12.

From this lemma we obtain the following (see [FS, Lemma 11])

**Lemma 13.** *If a function $f^*(s, y)$ on $(s_0, t) \times \bar{\boldsymbol{G}}$ satisfies* (1.7*) *and* $(B_\alpha)$, *then*

8) This assumption for $f(x)$ is equivalent to the following one: $f(\xi) = 0$ on $\boldsymbol{B}^{(s)} = \{\xi \in \boldsymbol{B} ;\ \alpha(s, \xi) = 1\}$

$$\int_G f^*(\tau, x)u(\tau, x; s, y)d_a x = f^*(s, y) \text{ for any } \tau \in (s, t).$$

Therefore, we may see that:

PROOF OF THEOREMS 1, 2 AND 3 *may be performed in the same way as the proof of the corresponding theorems in* [FS] (see [FS, pp. 89–90]). It seems not to be necessary to repeat the entirely same argument. *The propositions concerning the boundary condition which are not included in* [FS] *may be easily proved from properties of* $u(t, x; s, y)$ *and* $u^*(t, x; s, y)$ *stated in* § 4 *of the present paper.*

In order to prove Theorem 4, we consider, as in § 0, the functions

$$f_s(t, x) = \int_G u(t, x; s, y) f(y) d_a y \tag{5.1}$$

and

$$g(t, x) \equiv g_s^{(\tau, n)}(t, x) = f_s(t, x) \exp\left\{-\left(\frac{t-s}{\tau-s}\right)^n\right\} \tag{5.2}$$

where $f(x)$ is an arbitrary continuous function on $\boldsymbol{G}$ such that $0 \leq f(x) \leq 1$ and the support of $f(x)$ is a compact set contained in the domain $\boldsymbol{G}$, and $\tau$ and $n$ are as stated in § 0. Then $g_s^{(\tau, n)}(t, x)$ is continuous in $(s, t_0) \times \bar{\boldsymbol{G}}$ and satisfies (0. 3), (0. 4) and the boundary condition $(B_\alpha)$.

**Lemma 14.** *If* $c(t, x) \leq 0$, *then the function* $g(t, x)$ *takes neither positive maximum nor negative minimum at any point in* $(s, t_0) \times \bar{\boldsymbol{G}}$ (*for any fixed* $\tau$, $n$ *and* $s$).

PROOF. It is easily proved by the well known method that $g(t, x)$ takes neither positive maximum nor negative minimum at any point in the open set $(s, t_0) \times \boldsymbol{G}$.

Suppose that:

(5. 3) $g(t, x)$ *takes the positive maximum at* $\langle t_1, \xi_1 \rangle \in (s, t_0) \times \boldsymbol{B}$.

$f_s(t, x)$ satisfies $Lf = 0$ in $(s, t_0) \times \bar{\boldsymbol{G}}$ as may be seen from the properties of $u(t, x; s, y)$, where the partial derivatives at any $\xi \in \boldsymbol{B}$ should be understood as defined in § 1, and $g(t, x)$ satisfies the boundary condition $(B_\alpha)$ as well as $f_s(t, x)$. We adopt a canonical coordinate around $\xi_1$ as stated in Lemma 3. Then we obtain from (5. 3), (3. 1) and $(B_\alpha)$ that $\partial g(t_1, \xi_1)/\partial x_\xi^1 \leq 0$ and that

$$\alpha(t_1, \xi_1) g(t_1, \xi_1) - \{1 - \alpha(t_1, \xi_1)\} a_\varphi^{11}(t_1, \xi_1) \frac{\partial g(t_1, \xi_1)}{\partial x_\xi^1} = 0.$$

Since $g(t_1, \xi_1) > 0$ and $a_\varphi^{11}(t_1, \xi_1) > 0$, it follows that $\alpha(t_1, \xi_1)$ should be zero, consequently $\partial g(t_1, \xi_1)/\partial x_t^1 = 0$, and accordingly $\partial^2 g(t_1, \xi_1)/(\partial x_t^1)^2 \leqq 0$ by virtue of (5.3). Moreover, since $\langle t_1, \xi_1 \rangle$ may be considered as the maximising point of $g(t, \xi)$ restricted to $(s, t_0) \times \boldsymbol{B}$, we have

$$\sum_{i,j \geqq 2} a_\varphi^{ij}(t_1, \xi_1) \frac{\partial^2 g(t_1, \xi_1)}{\partial x_t^i \partial x_t^j} \leqq 0 \text{ and } b_\varphi^i(t_1, \xi_1) \frac{\partial g(t_1, \xi_1)}{\partial x_t^i} = 0$$

where we use the following facts: $a_\varphi^{1j}(t_1, \xi_1) = a_\varphi^{j1}(t_1, \xi_1) = 0$ for $j \geqq 2$ (see Lemma 3) and accordingly $\|a_\varphi^{ij}(t_1, \xi_1)\|_{i,j=2,\dots,m}$ is a positive-definite symmetric matrix. Thus we get $Ag(t_1, \xi_1) \leqq 0$, and hence

$$0 = \frac{\partial g(t_1, \xi_1)}{\partial t} = Ag(t_1, \xi_1) - \frac{n(t_1 - s)^{n-1}}{(\tau - s)^n} g(t_1, \xi_1) < 0\,;$$

that is a contradiction. Hence the function $g(t, x)$ on $(s, t_0) \times \bar{\boldsymbol{G}}$ does not take the positive maximum at any point in $(s, t_0) \times \boldsymbol{B}$. Similarly it does not take the negative minimum at any point in $(s, t_0) \times B$.

PROOF OF THEOREM 4 may be performed by means of the entirely same manner as in § 0 by making use of Lemma 14 in place of Lemma A in § 0. We omit to repeat here the argument in § 0.

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(Received August 17, 1954)