Osaka Mathematical Journal,
Vol. 2, No. 1, March, 1950

On the Quotient Semi-Group of a Noncommutative Semi-Group

By Kentaro MURATA

In this short note we remark, following K. Asano[1]), that a non-commutative semi-group $\mathfrak{g}$ with a certain condition can be embedded into the quotient semi-group G. The necessary and sufficient condition for the existence of the quotient semi-group is the same as the case of a ring. Moreover if $\mathfrak{g}$ is a ring, we can define the addition in G in a natural manner and G is just the quotient ring of $\mathfrak{g}$.

Definition 1. An element λ in a semi-group $\mathfrak{g}$ is called *regular,* if the following two conditions are satisfied: 1) $a\lambda = b\lambda\,(a,\ b \in \mathfrak{g})$ implies $a = b$ and 2) $\lambda a = \lambda b\,(a,\ b \in \mathfrak{g})$ implies $a = b$.

If $\mathfrak{g}$ has the unit, the elements having their inverse elements in $\mathfrak{g}$ are obviously regular.

In the following we assume that a semi-group $\mathfrak{g}$ has regular elements. It is clear that all regular elements in $\mathfrak{g}$ form a sub-semi-group $\mathfrak{g}^*$ of $\mathfrak{g}$.

Definition 2. Let $\mathfrak{m}$ be a sub-semi-group of $\mathfrak{g}^*$. If a semi-group G which contains $\mathfrak{g}$ satisfies the next three conditions, we call G a *left quotient semi-group* of $\mathfrak{g}$ by $\mathfrak{m}$.

(1) G has a unit 1.

(2) Every element α in $\mathfrak{m}$ has an inverse α^{-1} in G.

(3) For every x in G, there exists α in $\mathfrak{m}$ such that αx is contained in $\mathfrak{g}$.

In particular if $\mathfrak{m} = \mathfrak{g}^*$, we call G a left quotient semi-group of $\mathfrak{g}$. According to Definition 2, every element s in G is clearly expressible in the form $s = \alpha^{-1}a$, where $\alpha \in \mathfrak{m}$ and $a \in \mathfrak{g}$. If $\mathfrak{g}$ has a left (or right) unit e, then $e = 1$.[2])

Lemma 1. If for every a in $\mathfrak{g}$ and every α in $\mathfrak{m}$ there exist α' in $\mathfrak{m}$ and a' in $\mathfrak{g}$ such that $\alpha' a = a' \alpha$ then, for any n elements $\lambda_i \in \mathfrak{m}$ $(i = 1, \dots, n)$ there exist n elements $c_i \in \mathfrak{g}$ $(i = 1, \dots, n)$ satisfying the following condition:

---

1) K. Asano, Arithmetische Idealtheorie in nichtkommutativen Ringen, Japan. Journ. Math. **16** (1939); Über die Quotientenbildung von Schiefringen, Journ. Math. Soc. Japan **1** (1949).

2) $e = e1 = e\lambda\lambda^{-1} = \lambda\lambda^{-1} = 1$ $(e = \lambda e = \lambda^{-1}\lambda e = \lambda^{-1}\lambda = 1)$.

$$c_1 \lambda_1 = \ldots\ldots = c_n \lambda_n = \gamma \in \mathfrak{m}\,.$$

Proof. According to the assumption, there exist a_1 in $\mathfrak{g}$ and α_2 in $\mathfrak{m}$ such that $a_1\lambda_1 = \alpha_2 \lambda_2$. Next, for $\alpha_2 \lambda_2$ and λ_3, there exist a_2 in $\mathfrak{g}$ and α_2 in $\mathfrak{m}$ such that $a_1 \lambda_1 = \alpha_2 \lambda_2$. Next, for $\alpha_2 \lambda_2$ and λ_3, there exist a_2 in $\mathfrak{g}$ and α_3 in $\mathfrak{m}$ such that $a_2 \cdot \alpha_2 \lambda_2 = \alpha_3 \lambda_3 \in \mathfrak{m}$. By induction we complete our proof.

Lemma 2. If, under the same assumption as in lemma 1, there exists one element pair x_0, y_0 in $\mathfrak{g}$ such that $x_0 \alpha = y_0 \beta \in \mathfrak{m}\,(\alpha, \beta \in \mathfrak{m})$ and $x_0 a = y_0 b\,(a, b \in \mathfrak{g})$, then every pair x, y in $\mathfrak{g}$ satisfying the condition $x\alpha = y\beta \in \mathfrak{m}$ satisfies $xa = yb$.

Proof. Putting $\theta = x_0 \alpha = y_0 \beta$ and $\varphi = x\alpha = y\beta$, we take c and δ in $\mathfrak{g}$ and $\mathfrak{m}$ respectively as $c\theta = \delta\varphi \in \mathfrak{m}$. Then $c\theta = c x_0 \alpha = c y_0 \beta = \delta\varphi = \delta x \alpha = \delta y \beta$, $c x_0 = \delta x$, $c y_0 = \delta y$, hence $\delta x a = c x_0 a = c y_0 b = \delta y b$, that is, $xa = yb$.

Theorem 1. In order that there exists a left quotient semi-group of $\mathfrak{g}$ by $\mathfrak{m}$, it is necessary and sufficient that for every $a \in \mathfrak{g}$ and every $\lambda \in \mathfrak{m}$ there exist $a' \in \mathfrak{g}$ and $\lambda' \in \mathfrak{m}$ satisfying $\lambda' a = a' \lambda$. And such a left quotient semi-group is uniquely determined by $\mathfrak{m}$ and $\mathfrak{g}$ apart from its isomorphism.

Proof. Let G be a left quotient semi-group of $\mathfrak{g}$ by $\mathfrak{m}$. Then for $a\lambda^{-1}\,(a \in \mathfrak{g}, \lambda \in \mathfrak{m})$ in G, there exists $\lambda' \in \mathfrak{m}$ such that $\lambda' \cdot a \lambda^{-1} = a' \in \mathfrak{g}$, namely $\lambda' a = a' \lambda$. Hence the condition is necessary. We show now the condition is sufficient. First, we assume $\mathfrak{g}$ has the unit 1. Let G be the set of all symbols (α, a), $\alpha \in \mathfrak{m}$, $a \in \mathfrak{g}$. We can introduce the equality of the elements in G as follows: (α, a) is equal to (β, b) if and only if $xa = yb$ for every x and y satisfying $x\alpha = y\beta \in \mathfrak{m}$, $x, y \in \mathfrak{g}$. Then, according to Lemma 2, in order that $(\alpha, a) = (\beta, b)$ it is sufficient that there exists at least one element pair x, y in $\mathfrak{g}$ satisfying $xa = yb$ and $x\alpha = y\beta \in \mathfrak{m}$. As we can then readily prove, the above-defined equality fulfils the equivalence relation. In particular $(\lambda\alpha, \lambda a) = (\alpha, a)\,(\lambda \in \mathfrak{m})$. Now we define the multiplication of the elements in G as follows:

$$(\alpha, a)(\beta, b) = (\beta' \alpha, a' b), \quad \beta' a = a' \beta, \ \beta' \in \mathfrak{m}, \ a' \in \mathfrak{g}\,.$$

The product is independent of the choice of $a' \in \mathfrak{g}$ and $\beta' \in \mathfrak{m}$. For if $\beta'' a = a'' \beta$, $\beta'' \in \mathfrak{m}$, $a'' \in \mathfrak{g}$, then taking u and δ such that $u\beta' = \delta\beta'' \in \mathfrak{m}$ $(u \in \mathfrak{g}, \delta \in \mathfrak{m})$, we get $u\beta'\alpha = \delta\beta''\alpha \in \mathfrak{m}$, and

$$u a' \beta = u \beta' a = \delta \beta'' a = \delta a'' \beta.$$

Hence $u a' = \delta a''$, $u a' b = \delta a'' b$, that is, $(\beta' \alpha, a' b) = (\beta'' \alpha, a'' b)$. In particular, if the product of a and β is commutative, then $(\alpha, a)(\beta, b) = (\beta\alpha, ab)$. Further, if $(\alpha, a) = (\alpha_1, a_1)$ and $(\beta, b) = (\beta_1, b_1)$ then $(\alpha, a)(\beta, b) = (\alpha_1, a_1)(\beta_1, b_1)$. For if we choose c, c_1, a' in $\mathfrak{g}$ and β' in $\mathfrak{m}$ satisfying $\gamma = c\beta = c_1\beta_1 \in \mathfrak{m}$, $\beta' a = a'\gamma$, then $(\alpha, a)(\beta, b) = (\beta'\alpha, a'cb) = (\beta'\alpha, a'c_1b_1) = (\alpha, a)(\beta_1, b_1)$. Similarly $(\alpha, a)(\beta_1, b_1) = (\alpha_1, a_1)(\beta_1, b_1)$. One can easily verify the associative law of the multiplication introduced above. Hence G is a semi-group. The mapping $a \rightarrow (1, a)$ gives an isomorphism of $\mathfrak{g}$ into G. Identifying a and $(1, a)$, we can see that G contains $\mathfrak{g}$, $1 = (1, 1)$ is the unit of G, every element $\alpha = (1, \alpha)$ in $\mathfrak{m}$ has an inverse $(\alpha, 1)$ in G, and

$$\alpha(\alpha, a) = (1, \alpha)(\alpha, a) = (\alpha, \alpha a) = (1, a) = a.$$

Hence G is a left quotient semi-group of $\mathfrak{g}$ by $\mathfrak{m}$. Secondly, if $\mathfrak{g}$ has no unit, then we add it to $\mathfrak{g}$, and denote the new by $\mathfrak{g}'$. Then there exists a left quotient semi-group G of $\mathfrak{g}'$ by $\mathfrak{m}$. And it is easily seen that G is a left quotient semi-group of $\mathfrak{g}$ by $\mathfrak{m}$. Finally let G' be an arbitrary left quotient semi-group of $\mathfrak{g}$ by $\mathfrak{m}$, then every element in G' is expressible as the form $\alpha^{-1}a$ where α is in $\mathfrak{m}$ and a in $\mathfrak{g}$. The mapping $\alpha^{-1}a \rightarrow (\alpha, a)$ gives an isomorphism of G' onto G, q. e. d.

Corollary 1. In order that there exists a left quotient semi-group of a semi-group $\mathfrak{g}$, *it is necessary and sufficient that for every regular element* λ *and every element* a *in* $\mathfrak{g}$ *there exist* a' *in* $\mathfrak{g}$ *and a regular element* λ' *satisfying* $a'\lambda = \lambda' a$.

Corollary 2. Let all elements in a semi-group $\mathfrak{g}$ *be regular. In order that there exists a left quotient group of* $\mathfrak{g}$, *it is necessary and sufficient that for any two elements* α, β *in* $\mathfrak{g}$ *there exist another two elements* α', β' *in* $\mathfrak{g}$ *satisfying* $\alpha'\alpha = \beta'\beta$.

Lemma 3. Let G *be a left quotient semi-group of* $\mathfrak{g}$ *by* $\mathfrak{m}$. *Let further* x_i $(i = 1, \dots, n)$ *be any* n *elements in* $\mathfrak{g}$, *then we can choose an element* $\gamma \in \mathfrak{m}$ *such that* $\gamma x_i \in \mathfrak{g}$ $(i = 1, \dots, n)$.

Proof. There exist α_i in $\mathfrak{m}$ such that $\alpha_i x_i \in \mathfrak{g}$ $(i = 1, \dots, n)$. And if we take, according to Lemma 1, $\gamma = c_1\alpha_1 = \dots = c_n\alpha_n \in \mathfrak{m}$, then $\gamma x_i \in \mathfrak{g}$ $(i = 1, \dots, n)$.

Now, we shall consider a quotient system of some algebraic system. Let $\mathfrak{o}$ be a semi-group with regular elements, and $\mathfrak{m}$ a sub-semi-group of $\mathfrak{o}$ consisting of regular elements. Let further $\mathfrak{o}$ has besides the multiplication, binary operations denoted by $\circ$, which satisfy the following conditions:

[1] If a and b $(a, b \in \mathfrak{o})$ are composable, (with respect to $\circ$) then ca and cb are composable, also ac and bc are composable. And $ca \circ cb = c(a \circ b)$, $ac \circ bc = (a \circ b)c$.

[2] If, for some $\gamma \in \mathfrak{m}$, γa and $\gamma b\,(a,\ b \in \mathfrak{o})$ are composable, then a and b are composable, also if $a\gamma$ and $b\gamma$ are composable then so are a and b.

For example, let $\mathfrak{o}$ be a (noncommutative or commutative) ring, and $\circ$ the addition, then the above-mentioned conditions are of course fulfilled.

Theorem 2. Let S be a left quotient semi-group of $\mathfrak{o}$ by $\mathfrak{m}$. If there exists an element $\lambda \in \mathfrak{m}$, such that λx and λy $(x,\ y \in S)$ are contained in $\mathfrak{o}$ and composable, we define the composition of x and y by

$$(*) \qquad x \circ y = \lambda^{-1}(\lambda x \circ \lambda y).$$

Then S becomes an algebraic system of the same kind of $\mathfrak{o}$.

Proof. If x and y (where $x,\ y \in S$) are composable in the sense of the above-defined (*), then $x \circ y$ is uniquely determined independently of the choice of λ in $\mathfrak{m}$. For if μx and μy (where $\mu x \in \mathfrak{o}$, $\mu y \in \mathfrak{o}$, $\mu \in \mathfrak{m}$) are composable, then, by taking η and c satisfying $\eta\mu = c\lambda = \gamma \in \mathfrak{m}\ (\eta \in \mathfrak{m},\ c \in \mathfrak{o})$, we get

$$\gamma\lambda^{-1}(\lambda x \circ \lambda y) = c\lambda\lambda^{-1}(\lambda x \circ \lambda y) = c(\lambda x \circ \lambda y)$$
$$= c\lambda x \circ c\lambda y = \eta\mu x \circ \eta\mu y = \eta(\mu x \circ \mu y) = \gamma\mu^{-1}(\mu x \circ \mu y).$$

Namely $\lambda^{-1}(\lambda x \circ \lambda y) = \mu^{-1}(\mu x \circ \mu y)$.

Next, if x and y in S are composoble, then so are zx and $zy\ (z \in S)$ and $zx \circ zy = z(x \circ y)$. By the hypothesis, there exists λ in $\mathfrak{m}$ such that $x \circ y = \lambda^{-1}(\lambda x \circ \lambda y)$. If we choose α, τ and a such that $\alpha z \in \mathfrak{o}$, $\alpha \cdot zx \in \mathfrak{o}$, $\alpha \cdot zy \in \mathfrak{o}\ (\alpha \in \mathfrak{m})$ and $\tau(\alpha z) = a\lambda\ (\tau \in \mathfrak{m},\ a \in \mathfrak{o})$, then $\sigma zx = a\lambda x$, $\sigma zy = a\lambda y\ (\sigma = \tau\alpha)$. Since $a\lambda x = \sigma zx$ and $a\lambda y = \sigma zy$ are composable in $\mathfrak{o}$, zx and zy are composable in S, and

$$z(x \circ y) = z\lambda^{-1}(\lambda x \circ \lambda y) = \sigma^{-1}a(\lambda x \circ \lambda y)$$
$$= \sigma^{-1}(a\lambda x \circ a\lambda y) = \sigma^{-1}(\sigma zx \circ \sigma zy) = zx \circ zy, \quad (\sigma \in \mathfrak{m}).$$

If x and y in S are composable, then xz and yz are composable, and $(x \circ y)z = xz \circ yz$. By the hypothesis, there exists $\lambda \in \mathfrak{m}$ such that $x \circ y = \lambda^{-1}(\lambda x \circ \lambda y)$. If we take μ and α such that $\mu z \in \mathfrak{o}\ (\mu \in \mathfrak{m})$, $\alpha\lambda x\mu^{-1} \in \mathfrak{o}$ and $\alpha\lambda y\mu^{-1} \in \mathfrak{o}\ (\alpha \in \mathfrak{m})$, then $x \circ y = \sigma^{-1}(\sigma x \circ \sigma y)$, $\sigma = \alpha\lambda$. According to the latter part of the condition [2], $\sigma x\mu^{-1}$ and $\sigma y\mu^{-1}$ are composable in $\mathfrak{o}$. Hence

$$(x \circ y)z = \sigma^{-1}(\sigma x \circ \sigma y)z = \sigma^{-1}(\sigma x\mu^{-1}\mu \circ \sigma y\mu^{-1}\mu)z$$
$$= \sigma^{-1}(\sigma x\mu^{-1} \circ \sigma y\mu^{-1})\mu z = \sigma^{-1}(\sigma x\mu^{-1}\mu z \circ \sigma y\mu^{-1}\mu z)$$
$$= \sigma^{-1}(\sigma xz \circ \sigma yz) = xz \circ yz.$$

Every element in $\mathfrak{m}$ has an inverse in S. Then condition [2] is therefore easily obtained from the condition [1]. Thus our theorem is proved.

Let $\mathfrak{o}$ be a noncommutative ring containing non-nilfactors. The non-nilfactor is clearly a regular element in $\mathfrak{o}$ in the sense of Definition 1. Hence we have

Corollary.[3]) *Let* $\mathfrak{m}$ *be a sub-semi-group consisting of non-nilfactors in a given ring* $\mathfrak{o}$. *In order that there exists a left quotient ring* S *of* $\mathfrak{o}$ *by* $\mathfrak{m}$, *it is necessary and sufficient that, for every* $\alpha \in \mathfrak{m}$ *and every* $a \in \mathfrak{o}$, *there exist* $a' \in \mathfrak{o}$ *and* $\alpha' \in \mathfrak{m}$ *satisfying* $a' \alpha = \alpha' a$.

Putting the word *right* in place of *left* in the above-mentioned argument, we can argue similarly as above. But the existence of a left quotient semi-group and a right one are independent, and if there exist both, then they are the same[4]).

(Received June 8, 1949)

---

[3]), [4]) K. Asano, 1. c.