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On hypergroups of group right cosets

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In this paper we present certain results for the so-called hypergroups of classes, or more precisely, hypergroups of (group) right cosets. In § 1. we give several definitions. For any hypergroup of right cosets we give in § 2. a representation by permutations which will be used to characterize such hypergroups. By means of some partitions of elements of a hypergroup of right cosets we may define new hypergroups of right cosets which are treated in § 3. Some results on such kind of partitions for cogroups are given in § 4. This investigation is applied to obtain a counter-example for the conjecture of J. E. Eaton that every cogroup is isomorphic to a hypergroup of right cosets. The author expresses many thanks to Prof. K. Shoda for his kind encouragement and valuable remarks.

§ 1.

A set M is called a *hypergroupoid* if a product ab is defined to be a non-empty subset of M for every a and b in M. We define the product ST for any two subsets S and T of a hypergroupoid M as the set-sum of all products st of s in S and t in T. An element e of M satisfying the relation $ae \ni a$ for any a in M is called a *right unit* of M. Similarly we define a *left unit* and a *two-sided unit*. A one-to-one mapping θ of M onto itself is called a *(right) multiplicator* of M if $ab \ni c$ implies $ab^{\theta} \ni c^{\theta}$ and conversely. The totality of multiplicators of M forms an ordinary group which will be denoted by $R(M)$. A subgroup T of $R(M)$ is called a *(right) transferor group* of M if it satisfies the condition: if $ab \ni b'$ and $ac \ni c'$ then there exists a mapping θ in T such that $b^{\theta} = c$ and $b'^{\theta} = c'^{\theta}$. If T is a transferor group of M then any group U between T and $R(M)$ is also a transferor group of M.

Let M and N be two hypergroupoids. A many-to-one mapping θ of

M onto N is called a *homomorphism* [1]) if it satisfies the condition:

1) $ab \ni c$ in M implies $a^\theta b^\theta \ni c^\theta$ in N, and

2) if $a^\theta b^\theta \ni c^\theta$ in N then there exist a_1, b_1 and c_1 in M such that $a_1{}^\theta = a^\theta$, $b_1{}^\theta = b^\theta$, $c_1{}^\theta = c^\theta$ and $a_1 b_1 \ni c_1$.

If a homomorphism θ of M to N is one-to-one, then θ is called an *isomorphism.* An isomorphism onto itself is called an *automorphism.*

Let F be such a family of subsets of a hypergroupoid M that covers M, and S, T be elements of F. If we define a product of S and T as the set of elements of F which have a non-empty intersection with ST, then F forms a hypergroupoid. Particularly if F consists of all the classes of a partition of M, then F is called a *partition hypergroupoid* of M. We shall consider here the rather important notion of *(right) scalar partition hypergroupoid* which consists of all the elements $a, b, \ldots$ of M and the composition of which is given by $a * b = \{a\}b$ where $\{a\}$ is the class containing a.

A hypergroupoid M is called a *hypergroup* [2]) if the following two conditions are satisfied:

1) The multiplication is associative.

2) For any two elements a and b of M there exist x and y such that $xa \ni b$ and $ay \ni b$.

Let G be an ordinary group and H be its subgroup. The partition hypergroupoid with respect to the right coset decomposition of G by H is clearly a hypergroup which is called a *hypergroup of (group right) cosets* [3]) and denoted by G/H. A hypergroup M is called a *D-hypergroup* [4]) if M is isomorphic to a G/H. If H contains no subgroup, except e, which is normal in G, we call the *group pair* (G, H) *irre-*

1) Cf. J. E. Eaton, Associative multiplicative systems, Amer. J. of Math., v. 62 (1940), pp. 222-32.

2) Cf. F. Marty, Sur une généralisation de la notion de groupe, Åttonde Skandinaviska Matematikerkongressen i Stockholm 14-18 Augusti 1934, pp. 45-9. M. Krasner, Sur la primitivité des corps $\mathfrak{P}$-adiques, Mathematica, v. 13 (1937), pp. 72-191. M. Dresher and O. Ore, Theory of Multigroups, Amer. J. of Math., v. 60 (1938), pp. 705-33.

3) It is also called a "hypergroup of classes". Cf. M. Krasner, loc. cit.

4) It is so-called "hypergroup$_D$", Cf. M. Krasner, Sur la théorie de la ramification des idéaux de corps non-galoisiens de nombres algebriques, (Thèse Paris) (1938).

ducible.[5] The irreducible group pair is not uniquely determind by M as may be shown easily by examples. Two group pairs (G_1, H_1) and (G_2, H_2) for a D-hypergroup is called *equivalent* if G_1 and H_1 are isomorphic to G_2 and H_2 resp. in the same mapping.

A hypergroup is called a *(right) cogroup*[6] if it satisfies the following conditions:

1) There exists a left unit e such that ea contains only one element a for any a in C: $ea = a$.

2) If $ab \ni c$ there exists b' such that $bb' \ni e$ and $cb' \ni a$.

3) If ab contains k[7]) elements then ac also contains k elements for any c in C.

4) If the intersection of ac and bc is not empty, then $ae \ni b$.

We can easily prove that any D-hypergroup is a cogroup. An element a is called *e-conjugate* to b if $a \in be$: the e-conjugation is evidently an equivalence relation. A left unit e of a cogroup is a two-sided unit; it is uniquely determined and $ab \ni e$ if and only if $ba \ni c$. Then b is called an *inverse* of a. If a is e-conjugate to a' then $ab \ni e$ follows from $a'b \ni e$. Conversely, if $ab \ni e$ and $a'b \ni e$ then a is e-conjugate to a'.

§ 2.

Lemma 1. Let M and N be two hypergroupoids, and M be homomorphic to N by a homomorphism θ. Let M_1 be the partition hypergroupoid of M with respect to the partition given by the equivalence which is defined in M by the homomorphism θ. Then M_1 is isomorphic to N.

This can be proved in the usual way.

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5) The group pair is called "representation" by M. Krasner. Cf. M. Krasner, La caracterisation des hypergroupes de classes et la probléme de Schreier dan les hypergroupes, C. R. Acad. Sci. Paris, v. 212 (1941), pp. 948-50: Errata, ibid. v. 218 (1944), pp. 483-4: Rectification à ma note précédente et quelques nouvelles contributions à la théorie des hypergroupes. ibid. v. 218 (1944), pp. 542-4. I could not see these papers, but saw only reviews by R. Hull and D. C. Muldoch.

6) Cf. J. E. Eaton, Theory of cogroups, Duke Math. J., v. 6 (1940), pp. 101-7. J. E. Eaton has discussed his cogroups in finite case only, but here we drop this restriction and add one axiom which is the theorem 3 of the paper cited above.

7) Here k may be an infinite cardinal number.

Theorem 1. A hypergroupoid M with a right unit e is a D-hypergroup if and only if $R(M)$ is a transferor group of M.

Theorem 2. Every irrdeucible group pair of a D-hypergroup M is equivalent to a group pair (T, T_e) where T is a transferor group of M and T_e is the subst of T consisting of all the mappings in T which make the unit e of M invariant. Conversely, (T, T_e) is an irreducible group pair of M for every transferor group T of M.

Proof. Let M be a D-hypergroup and (G, H) be one of its irreducible group pairs. By m_a we denote the element of M corresponding to the coset Ha for every a in G. Then the mapping

$$\rho(x): \quad m_a \longrightarrow m_{ax}$$

is a multiplicator and all such multiplicators forms a group T. Let e be the unit of M. We denote by T_m the totality of elements of T which map e to m. Then the group pair (G, H) is clearly equivalent to (T, T_e). If $m_a m_b \ni m_c$ and $m_a m_b' \ni m_c'$ then $HaHb \supset Hc$ and $HaHb' \supset Hc'$; therefore $Hahb = Hc$ and $Hah'b' = Hc'$ for some h and h' in H. Let $y = b^{-1}h^{-1}h'b'$. Then $Hby = Hb'$ and $Hcy = Hc'$, i.e., $m_b{}^{\rho(y)} = m_{b'}$ and $m_c{}^{\rho(y)} = m_{c'}$. This implies that T is a transferor group of M.

Conversely, let M be a hypergroupoid with a right unit e. We assume the existence of a transferor group T of M. Now we prove that T is homomorphic to M. We make ρ map to e^ρ, then the mapping is "onto". By the assumption $a \varepsilon ae$, we get $a^\sigma \varepsilon ae^\sigma$ for any a in M. Thus $e^{\rho\sigma} \varepsilon e^\rho e^\sigma$. Conversely let $e^\rho e^\sigma \ni e^\tau$. Since $e^\rho e \ni e^\rho$, there exists π in T such that $e^\pi = e^\sigma$ and $e^{\rho\pi} = e^\tau$. Therefore, by lemma 1, T/T_e is isomorphic to M. Evidently T_e contains no subgroup, except e, which is normal in T.

§ 3.

A partition $M = \sum\{a\}$ of a D-hypergroup M is called a *(right) copartition* if the scalar partition hypergroupoid of M with respect to the partition is also a D-hypergroup.

As an immediate consequence of the theorem 1, we prove

Theorem 3. A partition $M = \sum \{a\}$ of a D-hypergroup M is a copartition if and only if for each pair a and b with $\{a\} = \{b\}$ there exists a multiplicator θ of the scalar partition hypergroupoid M^* of M with respect to the partition, such that $e^\theta = e$ and $a^\theta = b$.

Proof. Let $M = \sum \{a\}$ be a copartition. If $\{a\} = \{b\}$ then $a * e = \{a\} e = \{b\} e = b * e \ni b$. Since $a * e \ni a$, by the theorem 1., there exists θ in $R(M^*)$ such that $e^\theta = e$ and $a^\theta = b$.

Conversely, let the condition of the theorem be satisfied. If $a * b \ni c$ and $a * b' \ni c'$ then there exist a_1 and a_2 such that $\{a_1\} = \{a_2\} = \{a\}$, $a_1 b \ni c$ and $a_2 b' \ni c'$. From $a_1 e \ni a_1$ and $a_2 e \ni a_2$, there exist multiplicators ρ and σ of M such that $b^\rho = e$, $c^\rho = a_1$, $e^\sigma = b'$ and $a_2^\sigma = c'$. By the assumption, there is a multiplicator θ of M^* such that $e^\theta = e$ and $a_1^\theta = a_2$. Hence $b^{\rho\theta\sigma} = b'$ and $c^{\rho\theta\sigma} = c'$.

Theorem 4. Let H and K be two subgroups of group G such that $HK = KH$. Then HK/H is a scalar partition hypergroupoid of $K/K \frown H$ with respect to a copartition.

Conversely, if M^* be a scalar partition hypergroupoid of a D-hypergroup M with respect to a copartition, then $R(M^*) = R_e(M^*) R(M)$ and $R_e(M) = R_e(M^*) \frown R(M)$ in the symmetric group on M, where $R_e(M^*)$ and $R_e(M)$ are the totalities of elements in $R(M^*)$ and $R(M)$ resp. which make unit e invariant.

Proof. It is sufficient to prove that $(HaH \frown K)(Hb \frown K) = HaHb \frown K$. The left side of the expression is evidently included in the right side. Let $h_1 a h_2 b = k$ be an arbitrary element in the right side, and $h_3 b = k'$ be an element in $Hb \frown K$. Then $kk'^{-1} = h_1 a h_2 h_3^{-1}$ is in $HaH \frown K$.

To prove the other part of the theorem it is sufficient to prove that $R(M^*) = R_e(M^*) R(M)$. Clearly, by the assumption, $R(M^*) > R(M)$, hence $R(M^*) > R_e(M^*) R(M)$. Let $\theta \in R(M^*)$. Then there exists ρ in $R(M)$ such that $e^\rho = e^\theta$. Now $e^{\theta\rho^{-1}} = e$ and $\theta\rho^{-1} \in R_e(M^*)$, hence $R(M^*) < R_e(M^*) R(M)$.

Theorem 5. The partition join of two copartitions of a D-hypergroup is also a copartition.

Proof. Let M_1 and M_2 be two scalar partition hypergroupoids with respect to the given two copartitions resp. And let T be the join group of $R(M_1)$ and $R(M_2)$ in the symmetric group. Then, by the theorem 3, we may easily prove that T is a transferor group of the scalar partition hypergroupoid with respect to the join copartition.

Theorem 6. Let A be a subgroup of the automorphism group of a D-hypergroup M and $M = \sum \{a\}$ be a partition of M to transitive systems with respect to A. Then $M = \sum \{a\}$ is a copartition.

This follows easily from the theorem 3. In fact, every automorphism in A is a multiplicator of the scalar partition hypergroupoid of M with respect to the partition.

Theorem 7. Let A be a subgroup of the automorphism group of a group G and $H = GA$ be the holomorph of G for A. Let $H = \sum AhA$ be the double coset decomposition of H by A. Then the partition $G = \sum (AhA \frown G)$ of G coincides with the partition into the transitive systems with respect to A and it is a copartition. The scalar partition hypergroupoid of G with respect to the copartition is isomorphic to H/A.

Proof. Let $G = \sum \{a\}$ be the partition of G into the transitive systems of G with respect to A. In the decomposition $H = \sum AhA$, we can assume that h is in G. If $AhA \frown G \ni g_1, g_2$, then $g_1 = \alpha_1 h \alpha_1'$ and $g_2 = \alpha_2 h \alpha_2'$ where $\alpha_1, \alpha_1', \alpha_2, \alpha_2'$ are in A. If $g' = \alpha_2'^{-1}\alpha_1'\alpha_1 h \alpha_2'$ then $g' g_2^{-1} \varepsilon A$ and $g' = [(\alpha_1')^{-1}\alpha_2']^{-1}\alpha_1 h \alpha_1' [(\alpha_1')^{-1}\alpha_2'] \varepsilon G$. Hence $g' g_2^{-1} \varepsilon A \frown G$, therefore $g_2 = g' = g_1^{[(\alpha_1')^{-1}\alpha_2']}$. Conversely if $G \ni g_1$, g_2 and $g_1^{\alpha} = g_2$ for α in A. then $g_2 = \alpha^{-1} g_1 \alpha \varepsilon A g_1 A$. Hence two partitions $G = \sum (AhA \frown G) = \sum \{a\}$ coincide. The second part of the theorem follows easily from the first part of the proof of theorem 4.

§ 4.

Theorem 8, Let C be a cogroup and $C = \sum \{a\}$ be its partition such that $a\, e < \{a\}$. Then the scalar partition hypergroupoid C^* of C with respect to the partition is a cogroup if and only if the partition satisfies the condition:

1) *$\{e\} = e$ for the unit e of C, and*

2) $\{\{a\}b^{-1}\} = \{a\}\ \{b\}^{-1}$ *where* $\{\{a\}\ b^{-1}\}$ *is the set-sum of* $\{c\}$ *for all* c *which is contained in* $a'b^{-1}$ *with* $a' \varepsilon \{a\}$, *and* b^{-1} *is some inverse of* b, *and* $\{b\}^{-1}$ *is the set of all inverses of all elements in* $\{b\}$.

Proof. By $*$ we denote the multiplication in C, and by $-1*$ the inverse in C^*, that is, $a * b = \{a\}\ b$ and $a^{-1*} * a \ni e$. From the assumption, $\{a\} < \{a\}e < \{a\}$ and hence $\{a\} = \{a\}e = a * e$. If C^* forms a cogroup, then $b^{-1} * e = (b * e)^{-1}$. In fact, $b^{-1} * e = b^{-1*} * e = (b * e)^{-1*} > (b * e)^{-1}$ and let $x \varepsilon b^{-1} * e$, then $x \varepsilon (b * e)^{-1*}$ and there exist $y \varepsilon b * e$ such that $\{y\}x = y * x \ni e$, hence there exist $y' \varepsilon \{y\} = y * e = b * e$ such that $y'x \ni e$ or $y' = x^{-1}$, therefore $x \varepsilon (b * e)^{-1}$, i.e., $b^{-1} * e < (b * e)^{-1}$, hence $b^{-1} * e = (b * e)^{-1}$. Now $\{e\} = e * e = e$, $\{\{a\}b^{-1}\} = (a * b^{-1}) * e = a * (b^{-1} * e) = a * (b * e)^{-1} = \{a\}\ \{b\}^{-1}$ which prove the first part of the theorem.

Conversely, let the partition $C = \sum \{a\}$ satisfies the conditions above. If $a = e$ in 2), then $\{b^{-1}\} = \{b\}^{-1}$ by 1), and $\{\{a\}b^{-1}\} = \{a\}\ \{b^{-1}\}$ by 2). Since every element in a cogroup is an inverse of some element, we obtain $\{\{a\{b\} = \{a\}\ \{b\}$. We shall prove now that the axioms of cogroups is satisfied in C^*. Since $(a * b) * c = \{\{a\}b\}\ c = \{a\}\ \{b\}\ c = \{a\}\ (b * c) = a * (b * c)$, the multiplication is associative. Let a, b be two elements, then there exist x and y such that $a\,x \ni b$ and $y\,a \ni b$, hence $a * x \ni b$ and $y * a \ni b$. $e * a = \{e\}a = ea = a$. If $a * b \ni c$ or $\{a\}b \ni c$ then $a'\,b \ni c$ for some $a' \varepsilon \{a\}$ and $a' \varepsilon c\,b^{-1}$ for some b^{-1}, hence $a \varepsilon \{a'\} < \{\{c\}b^{-1}\} = \{c\}\ \{b\}^{-1}$, therefore $a \varepsilon \{c\}b' = c * b'$ for some $b' \varepsilon \{b\}^{-1}$ and $b * b' = \{b\}b' \ni e$. $\{a\}$ is a set-sum of certain number of e-conjugate classes of C: $\{a\} = \sum a_i e$. But if $a_i e \neq a_j e$ then $a_i b \frown a_j b$ is empty by the axiom 4 of cogroups. Hence the axiom 3 for C^* follows from the same for C. If $a * c \frown b * c \ni d$, then $a'\,c \frown b'\,c \ni d$ for some $a' \varepsilon \{a\}$ and $b' \varepsilon \{b\}$. Hence $a'e \varepsilon b'$. $a * e = \{a\} > a'e \ni b'$. Therefore $a * e < \{b'\} \ni b$, which completes the proof.

The above restriction $a\,e < \{a\}$ is not essential in the sense that the two partitions $C = \sum \{a\} = \sum \{a\}e$ define the same scalar partition hypergroupoid.

Now let $Z = [a]$ be a cyclic group of order 8. Then the partition

$$(e),\ (a,\ a^4,\ a^7),\ (a^2,\ a^3,\ a^5,\ a^6)$$

of elements of Z to three classes evidently satisfies the conditions 1) and 2) in the theorem 8. Hence the scalar partition hypergroupoid Z^* of Z with respect to the partition forms a cogroup. We shall prove now that $Z*$ is not a D-hypergroup. Obviously, $a * a = (e, a^2, a^5)$, $a * a^2 = (a, a^3, a^6)$, $a * a^3 = (a^2, a^4, a^7)$, $a * a^4 = (e, a^3, a^5)$ and $a * a^5 = (a, a^4, a^6)$. If Z^* were a D-hypergroup, then by the theorem 3 there would exist a multiplicator θ of Z^* such that $e^\theta = e$ and $a^\theta = a^4$. From $a * a \ni a^2$, follows $a * a^4 = a * a^\theta \ni (a^2)^\theta$. But $(a^2)^\theta \neq e = e^\theta$. Hence $(a^2)^\theta = a^3$ or a^5. First, let $(a^2)^\theta = a^3$. Then $a * a^3 = a * (a^2)^\theta \ni (a^3)^\theta$, since $a * a^2 \ni a^3$. But $(a^6)^\theta \neq a^4 = a^\theta$. Hence $(a^6)^\theta = a^2$ or a^7. On the other hand, $a * a \ni a^5$, $a * a^4 = a * a^\theta \ni (a^5)^\theta$, $(a^5)^\theta \neq e = e^\theta$ and $(a^5)^\theta \neq a^3 = (a^2)^\theta$, whence $(a^5)^\theta = a^5$. Since $a * a^5 \ni a^6$, $a * a^5 = a * (a^5)^\theta \ni (a^6)^\theta$, $(a^6)^\theta \neq a^4 = a^\theta$, thus we get $(a^6)^\theta = a$ or a^6 which contradicts to an earlier expression. Next, let $(a^2)^\theta = a^5$. Then, in a similar way, we can prove that $(a^6)^\theta = a$ or a^6 and $(a^6)^\theta = a^2$ or a^7 which is also a contradiction. Therefore Z^* is not a D-hypergroup, while it is a cogroup.

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Added in proof. Let r, s, s', and t be elements of a D-hypergroup. If s and s' are e-conjugate, then $rs \frown \{t\}$ and $rs' \frown \{t\}$ contain the same number of elements. But cogroups have not necessarily this property. In fact, in Z^* defined above, $a * a^2 \smile \{a\} = a$, but $a * a^3 \frown \{a\} = a^4$ and a^7, while $\{a^2\} = \{a^3\}$. (February 11, 1949)