Baklouti, A. and Tounsi, K.
Osaka J. Math.
44 (2007), 399–414

ON THE BENSON-RATCLIFF INVARIANT OF COADJOINT ORBITS ON NILPOTENT LIE GROUPS

ALI BAKLOUTI and KHALED TOUNSI

(Received February 2, 2006, revised June 6, 2006)

Abstract

We provide in this paper a counterexample to the Benson-Ratcliff conjecture about a cohomology class invariant on coadjoint orbits on nilpotent Lie groups. We prove that this invariant never vanishes on generic coadjoint orbits for some restrictive classes. As such, it does separate up to invariant factor, unitary representations associated to generic orbits in some cases.

1. Introduction

Let G be a connected Lie group with Lie algebra $\mathfrak{g}$ and let $\mathfrak{g}^*$ be the vector dual space of $\mathfrak{g}$. The left invariant forms on G yield a sub-complex of the de Rham complex $\Omega(G)$ which can be identified with the exterior algebra $\bigwedge(\mathfrak{g}^*)$. We denote by $H^*(\mathfrak{g})$ the cohomology of this complex which agrees with the algebraic notion of the Lie algebra cohomology with trivial real coefficients. Let $\mathcal{O} \subset \mathfrak{g}^*$ be a coadjoint orbit of dimension $2q$. For any $l \in \mathcal{O}$, viewed as an element of $\bigwedge^1(\mathfrak{g}^*)$, the differential form $l \wedge (dl)^q$ is a closed form and lies in $\bigwedge^{2q+1}(\mathfrak{g}^*)$. In [4], C. Benson and G. Ratcliff proved that its cohomology class $[l \wedge (dl)^q] \in H^{2q+1}(\mathfrak{g})$ is independent of the choice of $l \in \mathcal{O}$. When G is exponential and simply connected, it is well known that there is a topological homeomorphism between the space of coadjoint orbits $\mathfrak{g}^*/\mathrm{Ad}^*$ and the unitary dual $\hat{G}$ of G. That is, every unitary and irreducible representation π is uniquely associated with a coadjoint orbit $\mathcal{O}_\pi$ via the Kirillov theory. With the above in mind, it comes out that the invariant in question can be defined on $\hat{G}$. For a given representation $\pi \in \hat{G}$, set

$$i(\pi) = i(\mathcal{O}_\pi) = [l \wedge (dl)^q] \in H^{2q+1}(\mathfrak{g}), \quad l \in \mathcal{O}_\pi.$$

It appears so natural to seek the features of such invariant, especially if it can be used to distinguish between representations whose orbits have the same dimension and what kind of properties of the representation in question it detects. In [4], C. Benson and G. Ratcliff compute the invariant for numerous examples in the context of connected simply connected Lie groups, namely the case of infinite dimensional representations of the Heisenberg group and some other examples in higher step nilpotent Lie groups.

2000 Mathematics Subject Classification. 22E27, 22D10.

They remarked, however, that such invariant fails to separate unitary representations in general and they substantiated the following conjecture:

Conjecture 1.1 (Benson and Ratcliff [4]). *Let G be a connected simply connected nilpotent Lie group with one dimensional center. Let $l \in \mathfrak{g}^*$ be a linear form dual to a basis element of the center of $\mathfrak{g}$, then $i(\pi_l) \neq 0$.*

In the same context, the authors gave an affirmative answer to the above conjecture for square integrable representations modulo the center of G. The present work is a continuation of the articles [4] and [5]. We prove in a first time that the above conjecture fails to hold. We shall produce a counterexample, and even more show that the invariant may vanish in general settings on the whole Pedersen-Pukanszky maximal layer, whose image via the Kirillov mapping obviously constitutes a dense subset of $\hat{G}$ with respect to the relative topology. In a second step, we prove in the context of arbitrary nilpotent Lie groups a general criterion for the invariant to be non trivial which consists in looking at other simpler cohomology class making use of the results of [5]. We consequently show that this invariant never vanishes on the Pedersen-Pukanszky maximal layer for nilpotent Lie groups for which coadjoint orbits are at most two dimensional and for some Lie groups admitting a normal subgroup which polarizes all generic linear forms as well. It is somehow noticeable that even for these classes, the invariant may vanish on the set of maximal dimension coadjoint orbits.

The non-vanishing cohomology invariant fails as mentioned earlier to separate unitary and irreducible representations whose orbits lie in the same stratum. As such, the invariant does separate trivial orbits (unitary characters). That is, for $\pi = \chi_l$, $\mathcal{O}_\pi = \{l\}$ and $i(\pi) = [l]$. We pay attention in the last section of the paper to the possibility whether the definition of the invariant could be slightly shifted in order to guarantee such separation. The task basically consists in multiplying the cohomology class $[l \wedge (dl)^q]$ by some G-invariant rational non singular function depending only on l (so constant on G-orbits). When restricted to a single orbit, it appears then clear that the cohomology class of the subsequent invariant coincides with the original, up to a scalar factor. We prove that such operation is feasible in the case of one codimensional maximal coadjoint orbits. Seemingly, this process stands to be pretty tough to realize in general contexts, and this is due to a pair of reasons. Firstly, the invariant does strongly depend on the features of the associated orbit which may be difficult to accurately describe in high dimensional Lie groups. Secondly, the structure of the ambiant Lie algebra greatly intervenes in the cohomology calculus which sometimes contributes to utterly lose the control on some variables related to the orbit in question within a corresponding cross-section, which typically happens in Example **5.5**.

We study in the last section some examples of exponential non-nilpotent Lie groups, we basically remark that the invariant vanishes on infinite dimensional representations in the case where $\dim \mathfrak{g} \leq 3$ which is rather unexpected.

2. Background

2.1. The general setting and notations. We begin this section by reviewing some useful facts and notations for a nilpotent Lie group. This material is quite standard. We refer the reader to [6] for details. Throughout and unless specific mention, $\mathfrak{g}$ will be a n-dimensional real nilpotent Lie algebra, G will be the associated connected and simply connected nilpotent Lie group. The exponential map

$$\exp\colon \mathfrak{g} \to G$$

is a global C^∞-diffeomorphism of $\mathfrak{g}$ into G. Let $\mathfrak{g}^*$ be the dual vector space of $\mathfrak{g}$. The Lie algebra $\mathfrak{g}$ acts on $\mathfrak{g}$ by the adjoint representation $\mathrm{ad}_\mathfrak{g}$, that is:

$$\mathrm{ad}_\mathfrak{g}(X)(Y) = \mathrm{ad}(X)(Y) = [X, Y], \quad X, Y \in \mathfrak{g}.$$

The group G acts on $\mathfrak{g}$ by adjoint representation Ad_G i.e.

$$\mathrm{Ad}_G(g)(Y) = \mathrm{Ad}(g)(Y) = e^{\mathrm{ad}(X)}Y, \quad g = \exp X \in G,\ Y \in \mathfrak{g}$$

and on $\mathfrak{g}^*$ by the coadjoint representation Ad^*_G i.e.

$$\mathrm{Ad}^*_G(g)l(X) = g \cdot l(X) = l(\mathrm{Ad}(g^{-1})X), \quad g \in G,\ l \in \mathfrak{g}^*,\ X \in \mathfrak{g}.$$

The coadjoint orbit of l is the set $\mathcal{O}_l = G \cdot l = \{g \cdot l,\ g \in G\}$. The space of coadjoint orbits is noted by $\mathfrak{g}^*/G$.

2.2. The orbit theory. Let $l \in \mathfrak{g}^*$ and $\mathfrak{g}(l) = \{X \in \mathfrak{g}; l([X, \mathfrak{g}]) = \{0\}\}$ be the stabilizer of $l \in \mathfrak{g}^*$ in $\mathfrak{g}$ which is actually the Lie algebra of the Lie subgroup $G(l) = \{g \in G,\ g \cdot l = l\}$. So, it is clear that $\mathfrak{g}(l)$ is the radical of the skew-symmetric bilinear form B_l defined by

$$B_l(X, Y) = l([X, Y]), \quad X, Y \in \mathfrak{g}. \tag{2.2.1}$$

A subspace $\mathfrak{b}[l]$ of the Lie algebra $\mathfrak{g}$ is called a polarization for $l \in \mathfrak{g}^*$ if it is a maximal dimensional isotropic subalgebra with respect to B_l. So we can consider the unitary character of $B[l] = \exp(\mathfrak{b}[l])$,

$$\chi_l(\exp X) = e^{-2\pi i l(X)}, \quad X \in \mathfrak{b}[l].$$

The unitary dual $\hat{G}$ of G is parameterized via the Kirillov-Bernat orbit method. Let $l \in \mathfrak{g}^*$, we take a real polarization $\mathfrak{b} = \mathfrak{b}[l]$ for l. For such a polarization, define

$$\pi_l = \pi_{l,\mathfrak{b}} = \mathrm{Ind}_B^G\ \chi_l, \quad B = \exp \mathfrak{b}.$$

Then $\pi_{l,\mathfrak{b}}$ is a unitary and irreducible representation of G and its equivalence class $[\pi_{l,\mathfrak{b}}]$ depends only on the coadjoint orbit of l. Moreover, every irreducible representation π is equivalent to an induced representation $\pi_{l,\mathfrak{b}}$ for some $l \in \mathfrak{g}^*$ and a polarization $\mathfrak{b}$ in l with the character χ_l. The following mapping, called the Kirillov-Bernat mapping

$$K : \mathfrak{g}^*/G \to \hat{G}$$
$$G \cdot l \mapsto [\pi_{l,\mathfrak{b}}]$$

is a homeomorphism (see [6]).

2.3. The Pedersen-Pukanszky stratification. Let

$$(\mathcal{S})\colon \{0\} = \mathfrak{g}_0 \subset \mathfrak{g}_1 \subset \cdots \subset \mathfrak{g}_n = \mathfrak{g}$$

be a Jordan-Hölder sequence of the nilpotent Lie algebra $\mathfrak{g}$, i.e., a flag of ideals of $\mathfrak{g}$ such that $\dim \mathfrak{g}_j = j$, $j = 0, \ldots, n$. We extract from $(\mathcal{S})$ a Jordan-Hölder basis $\mathcal{B} = \{X_1, \ldots, X_n\}$ by taking $X_j \in \mathfrak{g}_j \setminus \mathfrak{g}_{j-1}$, $j = 1, \ldots, n$. Let $\mathcal{B}^* = \{X_1^*, \ldots, X_n^*\}$ the dual basis of $\mathfrak{g}^*$ dual to the basis $\{X_1, \ldots, X_n\}$ which is a Jordan-Hölder basis for the coadjoint action of G on $\mathfrak{g}^*$. Let $l \in \mathfrak{g}^*$, an index $j \in \{1, \ldots, n\}$ is said to be a jump index for l if

$$\mathfrak{g}(l) + \mathfrak{g}_j \neq \mathfrak{g}(l) + \mathfrak{g}_{j-1}.$$

We let

$$e(l) = \{j : j \text{ is a jump index for } l\}, \quad \tilde{e}(l) = \{1, \ldots, n\} \setminus e(l)$$

and

$$\mathcal{E} = \{e(l) : l \in \mathfrak{g}^*\}$$

The set $e(l)$ contains exactly $\dim(\mathcal{O}_l)$ indices, which is necessarily an even number. For each $e \in \mathcal{E}$, the set

$$\Omega_e = \{l \in \mathfrak{g}^* : e(l) = e\}$$

is the layer in $\mathfrak{g}^*$ corresponding to e and obviously contains $\mathcal{O}_l$ for $l \in \mathfrak{g}^*$. Note that each layer Ω_e is a semi algebraic set in $\mathfrak{g}^*$ and there exists a strict total ordering $\prec$ on $\mathcal{E}$ defined as follows. For $e, e' \in \mathcal{E}$ we have $e \prec e'$ if either

1. $e = \{j_1 < j_2 < \cdots < j_d\}$, $e' = \{j_1' < j_2' < \cdots < j_{d'}'\}$ where $j_1 = j_1', \ldots, j_{k-1} = j_{k-1}'$ and $j_k' < j_k$ for some $k \leq \min(d, d')$, or
2. $e \subsetneq e'$.

Note that, in view of the second condition, the empty set $e = \varnothing$ is the minimal element in $\mathcal{E}$. The layer $\Omega_\varnothing$ corresponds to the one dimensional representations in $\hat{G}$. The layer Ω_e given by the maximal element $e \in \mathcal{E}$ contains the generic orbits and forms a Zariski open set in $\mathfrak{g}^*$. This layer will be noted by $\Omega_{\max}$ and called the Pedersen-Pukanszky maximal layer. More generally, one has that for $e \in \mathcal{E}$, the layer Ω_e is the intersection of a Zariski open set with $\bigcup_{e' \geq e} \Omega_{e'}$, which is Zariski closed. In fact, there is a G-invariant polynomial function

$$P_e : \mathfrak{g}^* \to \mathbb{R}$$

for each $e \in \mathcal{E}$ with the property that

$$\Omega_e = \{l \in \mathfrak{g}^* : P_e(l) \neq 0 \text{ and } P_{e'}(l) = 0 \text{ for } e' \prec e\}.$$

These are defined explicitly as $P_\varnothing = 1$ and

$$P_e(l) = Pf(M_e(l)), \quad \text{where} \quad M_e(l) = (l[X_i, X_j])_{i,j \in e}$$

for $\varnothing \prec e$. That is, $P_e(l)$ is the Pfaffian of the skew-symmetric matrix $M_e(l)$. For $e \in \mathcal{E}$, let

$$V_{\tilde{e}} = \mathbb{R} - \operatorname{span}\{X_j^* : j \in \tilde{e}\} \subset \mathfrak{g}^*.$$

The set

$$\mathcal{W}_e = \Omega_e \cap V_{\tilde{e}}$$

is a cross-section to the coadjoint orbits in Ω_e which means that $\mathcal{O}_l$ meets $\mathcal{W}_e$ in a unique and single point called the *fundamental element* of the associated representation π_l.

3. A counterexample to the Benson-Ratcliff conjecture

In this section, we produce a counterexample to the Benson-Ratcliff Conjecture 1.1. We shall even go much further. The content of the counterexample below can be clarified by carrying out quite accurate computations making use of explicit bases. This basically leads to the fact that the invariant $i(\pi)$ may vanish on the whole Pedersen-Pukanszky maximal layer. Let G be the nilpotent Lie group with Lie algebra $\mathfrak{g} = \mathbb{R} - \operatorname{span}\{Z, Y_1, Y_2, A, X_1, X_2\}$ with non zero Lie brackets:

$$[Y_1, X_1] = [Y_2, X_2] = Z, \quad [A, X_1] = Y_1, \quad [A, X_2] = Y_2 \quad \text{and} \quad [X_1, X_2] = A.$$

It is clear that the center $\mathfrak{z}(\mathfrak{g})$ is one dimensional and spanned by Z. We designate by $\{Z^*, Y_1^*, Y_2^*, A^*, X_1^*, X_2^*\}$ the basis of $\mathfrak{g}^*$ dual to the basis $\{Z, Y_1, Y_2, A, X_1, X_2\}$. The maximal jump indices set is given by $e = \{2, 3, 5, 6\}$. So, the Pfaffian $P_e(l)$ is such that

$$P_e(l)^2 = \det M_e(l) = l_1^4, \quad l_1 = l(Z).$$

The Pedersen-Pukanszky maximal layer is then defined as the set

$$\Omega_{\max} = \{l \in \mathfrak{g}^* : l_1 \neq 0\}.$$

Proposition 3.1. *The invariant $i(\pi_l)$ vanishes on $\Omega_{\max}$.*

Proof. Let $V_{\tilde{e}} = \mathbb{R}Z^* \oplus \mathbb{R}A^*$, then as in **2.3** the set $\mathcal{W} = V_{\tilde{e}} \cap \Omega_{\max}$ is a cross-section for the generic coadjoint orbits in $\mathfrak{g}^*$. If $l \in \Omega_{\max}$, the fundamental element of the representation π_l still denoted by l, may be picked as $l = l_1 Z^* + l_4 A^* \in \Omega_{\max}$. So, $\mathfrak{g}(l) = \langle Z, A \rangle$ and $\dim(\mathcal{O}_l) = 4$. Hence:

$$\begin{aligned} dl &= l_1 dZ^* + l_4 dA^* \\ &= -l_1(Y_1^* \wedge X_1^* + Y_2^* \wedge X_2^*) - l_4 X_1^* \wedge X_2^*. \end{aligned}$$

So, an easy computation shows that $(dl)^2 = -2l_1^2 Y_1^* \wedge Y_2^* \wedge X_1^* \wedge X_2^*$ and then

$$\begin{aligned} l \wedge (dl)^2 &= (l_1 Z^* + l_4 A^*) \wedge \left(-2l_1^2 Y_1^* \wedge Y_2^* \wedge X_1^* \wedge X_2^*\right) \\ &= -2l_1^3 Z^* \wedge Y_1^* \wedge Y_2^* \wedge X_1^* \wedge X_2^* - 2l_1^2 l_4 A^* \wedge Y_1^* \wedge Y_2^* \wedge X_1^* \wedge X_2^*. \end{aligned}$$

Let $\beta(l)$ be the 4-differential form defined on $\mathfrak{g}$ by

$$\beta(l) = 2l_1^2 Z^* \wedge Y_1^* \wedge A^* \wedge (l_1 Y_2^* + l_4 X_1^*).$$

Then one easily checks

$$\begin{aligned} d\beta(l) &= 2l_1^2 \big(d(Z^* \wedge Y_1^* \wedge A^*) \wedge (l_1 Y_2^* + l_4 X_1^*) - Z^* \wedge Y_1^* \wedge A^* \wedge (l_1 dY_2^* + l_4 dX_1^*)\big) \\ &= 2l_1^2 \big((dZ^*) \wedge Y_1^* \wedge A^* - Z^* \wedge ((dY_1^*) \wedge A^* - Y_1^* \wedge (dA^*))\big) \wedge (l_1 Y_2^* + l_4 X_1^*) \\ &\quad - 2l_1^2 Z^* \wedge Y_1^* \wedge A^* \wedge (l_1 dY_2^* + l_4 dX_1^*) \\ &= 2l_1^2 \big(-(Y_1^* \wedge X_1^* + Y_2^* \wedge X_2^*) \wedge Y_1^* \wedge A^* \\ &\qquad + Z^* \wedge ((A^* \wedge X_1^*) \wedge A^* - Y_1^* \wedge (X_1^* \wedge X_2^*))\big) \wedge (l_1 Y_2^* + l_4 X_1^*) \\ &\quad + 2l_1^2 Z^* \wedge Y_1^* \wedge A^* \wedge (l_1 A^* \wedge X_2^*) \\ &= -2l_1^2 l_4 Y_2^* \wedge X_2^* \wedge Y_1^* \wedge A^* \wedge X_1^* - 2l_1^3 Z^* \wedge Y_1^* \wedge X_1^* \wedge X_2^* \wedge Y_2^* \\ &= l \wedge (dl)^2, \end{aligned}$$

which proves that $l \wedge (dl)^2$ is an exact differential form and therefore $i(\pi_l) = 0$. □

REMARK. 1) The particular case where $l_4 = 0$ gives a counterexample for the Benson-Ratcliff conjecture.

2) For lower layers and beyond the minimal stratum of characters, the invariant may be non zero as can be observed in the example above. Pick for instance $l = l_2 Y_1^*$, $l_2 \neq 0$ then $\mathfrak{g}(l) = \langle Z, Y_1, Y_2, X_2 \rangle$ and $\dim(\mathcal{O}_l) = 2$. The invariant $i(\pi_l)$ is:

$$i(\pi_l) = [l \wedge dl] = -l_2^2 [Y_1^* \wedge A^* \wedge X_1^*].$$

Let

$$\begin{aligned}\lambda = {} & Z^* \wedge (\lambda_{1,2} Y_1^* + \lambda_{1,3} Y_2^* + \lambda_{1,4} A^* + \lambda_{1,5} X_1^* + \lambda_{1,6} X_2^*)\\ & + Y_1^* \wedge (\lambda_{2,3} Y_2^* + \lambda_{2,4} A^* + \lambda_{2,5} X_1^* + \lambda_{2,6} X_2^*)\\ & + Y_2^* \wedge (\lambda_{3,4} A^* + \lambda_{3,5} X_1^* + \lambda_{3,6} X_2^*) + A^* \wedge (\lambda_{4,5} X_1^* + \lambda_{4,6} X_2^*) + \lambda_{5,6} X_1^* \wedge X_2^*\end{aligned}$$

be an element in $\bigwedge^2(\mathfrak{g}^*)$. Then:

$$\begin{aligned}d\lambda = {} & -\lambda_{1,3} Y_1^* \wedge X_1^* \wedge Y_2^* - \lambda_{1,4} Y_1^* \wedge X_1^* \wedge A^* - \lambda_{1,4} Y_2^* \wedge X_2^* \wedge A^*\\ & + \lambda_{1,2} Z^* \wedge A^* \wedge X_1^* + \lambda_{1,3} Z^* \wedge A^* \wedge X_2^* + \lambda_{1,4} Z^* \wedge X_1^* \wedge X_2^*\\ & - \lambda_{2,3} A^* \wedge X_1^* \wedge Y_2^* - \lambda_{2,6} A^* \wedge X_1^* \wedge X_2^* + \lambda_{2,3} Y_1^* \wedge A^* \wedge X_2^*\\ & + \lambda_{2,4} Y_1^* \wedge X_1^* \wedge X_2^* - \lambda_{3,5} A^* \wedge X_2^* \wedge X_1^* + \lambda_{3,4} Y_2^* \wedge X_1^* \wedge X_2^*\\ & + \lambda_{1,6} X_1^* \wedge Y_1^* \wedge X_2^* + \lambda_{1,2} X_2^* \wedge Y_2^* \wedge Y_1^* + \lambda_{1,5} X_2^* \wedge Y_2^* \wedge X_1^*.\end{aligned}$$

Suppose now that $d\lambda = -l_2^2 Y_1^* \wedge A^* \wedge X_1^*$, then one can easily check that we simultaneously have $-l_2^2 = \lambda_{1,4} = 0$ which is absurd and then $i(\pi_l) \neq 0$.

4. Nilpotent Lie groups on which the invariant restricted to $\Omega_{\max}$ never vanishes

This section aims to examine some restrictive classes of nilpotent Lie groups for which the invariant $i(\pi_l)$ is non zero for every l in a dense subset of $\mathfrak{g}^*$. Our first result deals with the case where the dimension of coadjoint orbits is at most two.

Theorem 4.1. *Let G be a connected and simply connected nilpotent Lie group with Lie algebra $\mathfrak{g}$. Assume that coadjoint orbits in $\mathfrak{g}^*$ are at most two dimensional. Then $i(\pi_l) \neq 0$ for every $l \in \Omega_{\max}$. The result holds on $\mathfrak{g}^* \setminus \{0\}$ if in addition G is two step.*

We also study the case of nilpotent Lie groups admitting an ideal which polarizes generic elements of the dual vector space. The result of Theorem 4.1 may fail in such setting as shown in the counterexample provided in Section 3. A nilpotent Lie group G is said to be SNPC (or to meet the special normal polarization condition) if there exists in its Lie algebra $\mathfrak{g}$ an Abelian ideal $\mathfrak{c}$ such that $[\mathfrak{g}, \mathfrak{c}]$ is one dimensional and that the centralizer $\mathfrak{h}$ of $\mathfrak{c}$ is Abelian. Likewise, G is said to be special if it is of the form $\mathbb{R}^n \rtimes \mathbb{R}$ being non commutative. At the level of Lie algebras, there exists a co-dimensional one ideal $\mathfrak{h}$ which polarizes all generic orbits and so obviously such ideal must be Abelian. It turns out that special nilpotent Lie groups are SNPC, especially the n-step threadlike group and Heisenberg groups are also SNPC, (see [1] and [2]). It evidently follows that in the case of special Lie groups, generic orbits are at most two dimensional. The following consequence obviously stems from Theorem 4.1:

Corollary 4.2. *Let G be a connected simply connected nilpotent special Lie group. Then $i(\pi_l) \neq 0$ for every $l \in \Omega_{\max}$.*

We shall show later that such result may fail for lower layers. We now provide a quite similar result as above for SNPC nilpotent Lie groups. The upshot is the following:

Theorem 4.3. *Let G be a connected and simply connected SNPC nilpotent Lie group. Assume that $\mathfrak{h}$ admits an Abelian supplementary subspace. Then $i(\pi_l) \neq 0$ for every $l \in \Omega_{\max}$.*

We proceed now to the proof of our results. We shall provide a general criterion for the invariant $i(\pi_l)$, $l \in \mathfrak{g}^*$ to be trivial. As shall be remarked later, such criterion has a great practical features and will be used to get pretty general results in some classes of nilpotent Lie groups. It consists in looking at a new cohomology class depending only on the fundamental element of the representation π_l. We start with a connected simply connected nilpotent Lie group G with Lie algebra $\mathfrak{g}$, let

$$(\mathcal{S}) \colon \{0\} = \mathfrak{g}_0 \subset \mathfrak{g}_1 \subset \cdots \subset \mathfrak{g}_n = \mathfrak{g}$$

be a Jordan-Hölder sequence of $\mathfrak{g}$ from which we extract a Jordan-Hölder basis $\{X_1, \ldots, X_n\}$ of $\mathfrak{g}$. Let l be in $\mathfrak{g}^*$, $\mathcal{O}_l$ the coadjoint orbit through l and $2d = \dim(\mathcal{O}_l)$. Denote as before by

$$e(l) = \{1 < i_1 < \cdots < i_{2d} \leq n\}$$

the set of jump indices of l. For any $s \in \tilde{e}(l)$, there exist some real numbers $\lambda_{s,t}(l)$, $t < s$ in such a way that the vector:

$$Y_s = X_s + \sum_{t<s} \lambda_{s,t}(l) X_t$$

belongs to $\mathfrak{g}(l) \cap \mathfrak{g}_s$. We can then extract from $(\mathcal{S})$ a Jordan-Hölder basis $\{Y_1, \ldots, Y_n\}$ passing through $\mathfrak{g}(l) = \mathbb{R} - \mathrm{span}\{Y_s, s \in \tilde{e}(l)\}$ and such that $Y_i = X_i$ for $i \in e(l)$. Let $W(l)$ be the $(2d+1)$-form defined by:

$$W(l) = l \wedge Y_{i_1}^* \wedge \cdots \wedge Y_{i_{2d}}^*, \tag{1}$$

where $\{Y_1^*, \dots, Y_n^*\}$ is a basis of $\mathfrak{g}^*$ dual to the basis $\{Y_1, \dots, Y_n\}$. Remark that $W(l)$ does not depend on the values of l on the vectors Y_i, $i \in e(l)$. In the case where $e(l) = \emptyset$, we get obviously that $\mathfrak{g}(l) = \mathfrak{g}$ and $W(l) = l$ is a one differential form so that $i(\pi_l) = [l]$. The following lemma plays an important role in the sequel.

Lemma 4.4. *Let G be a connected simply connected nilpotent Lie group with Lie algebra $\mathfrak{g}$. Let $l \in \mathfrak{g}^*$ and let $W(l)$ be the $(2d+1)$-form defined as in (1). Then for every $l \in \Omega_e$, we have that $i(\pi_l) = (-1)^d d!\, P_e(l)[W(l)]$.*

Proof. For $l \in \Omega_e$, let $l_{i,j} = l([Y_i, Y_j])$ $(i, j \in \{1, \dots, n\})$. So $M_e(l) = [l_{i_s,i_t}]_{1 \le s,t \le 2d}$ and from [7], one has:

$$P_e(l) = \frac{1}{2^d}\frac{1}{d!}\sum_{\sigma \in S_{2d}} \operatorname{sign}(\sigma) l_{i_{\sigma(1)},i_{\sigma(2)}} l_{i_{\sigma(3)},i_{\sigma(4)}} \cdots l_{i_{\sigma(2d-1)},i_{\sigma(2d)}}.$$

On the other hand, we have

$$dl = -\sum_{1 \le s < t \le 2d} l_{i_s,i_t} Y_{i_s}^* \wedge Y_{i_t}^* = -\frac{1}{2}\sum_{s \ne t} l_{i_s,i_t} Y_{i_s}^* \wedge Y_{i_t}^*$$

and therefore

$$\begin{aligned}(dl)^d &= (-1)^d \frac{1}{2^d} \sum_{\sigma \in S_{2d}} l_{i_{\sigma(1)},i_{\sigma(2)}} l_{i_{\sigma(3)},i_{\sigma(4)}} \cdots l_{i_{\sigma(2d-1)},i_{\sigma(2d)}} Y_{i_{\sigma(1)}}^* \wedge Y_{i_{\sigma(2)}}^* \wedge \cdots \wedge Y_{i_{\sigma(2d-1)}}^* \wedge Y_{i_{\sigma(2d)}}^* \\ &= (-1)^d \frac{1}{2^d}\left(\sum_{\sigma \in S_{2d}} \operatorname{sign}(\sigma) l_{i_{\sigma(1)},i_{\sigma(2)}} l_{i_{\sigma(3)},i_{\sigma(4)}} \cdots l_{i_{\sigma(2d-1)},i_{\sigma(2d)}}\right) Y_{i_1}^* \wedge \cdots \wedge Y_{i_{2d}}^* \\ &= (-1)^d d!\, P_e(l) Y_{i_1}^* \wedge \cdots \wedge Y_{i_{2d}}^*\end{aligned}$$

from above. □

REMARK. Let $K(l)$ be the normal subgroup of $G(l)$ whose Lie algebra is given by $\mathfrak{k}(l) = \ker(l/\mathfrak{g}(l))$. For $X \in \mathfrak{g}$, the substitution operator

$$i(X)\colon \bigwedge^{k}(\mathfrak{g}^*) \to \bigwedge^{k-1}(\mathfrak{g}^*)$$

is given by $i(X)(\beta)(Y_1, \dots, Y_{k-1}) = \beta(X, Y_1, \dots, Y_{k-1})$, for $Y_i \in \mathfrak{g}$, $i = 1, \dots, k-1$. Let now consider the sub-complexes of $K(l)$-basic and $\mathfrak{k}(l)$-basic of $\bigwedge(\mathfrak{g}^*)$ defined by:

$$\begin{aligned}\left(\bigwedge(\mathfrak{g}^*)\right)_{K(l)} &= \left\{\beta \in \bigwedge(\mathfrak{g}^*) \colon i(X)\beta = 0,\ \forall X \in \mathfrak{k}(l) \text{ and } \operatorname{Ad}_s^* \beta = \beta,\ \forall s \in K(l)\right\}, \\ \left(\bigwedge(\mathfrak{g}^*)\right)_{\mathfrak{k}(l)} &= \left\{\beta \in \bigwedge(\mathfrak{g}^*) \colon i(X)\beta = 0,\ \forall X \in \mathfrak{k}(l) \text{ and } \operatorname{ad}_X^* \beta = 0,\ \forall X \in \mathfrak{k}(l)\right\}.\end{aligned}$$

These complexes yield the relative cohomology theories $H^*(\mathfrak{g}, K(l))$ and $H^*(\mathfrak{g}, \mathfrak{k}(l))$. The fundamental upshot proved in [5] is that the isomorphic spaces $H^*(G, K(l))$ and $H^*(\mathfrak{g}, \mathfrak{k}(l))$ are one dimensional. Moreover, the invariant $i(\mathcal{O}_l)$ vanishes if and only if the map $H^{2q+1}(\mathfrak{g}, K(l)) \to H^{2q+1}(\mathfrak{g})$ induced by the inclusion $\bigwedge(\mathfrak{g}^*)_{K(l)} \hookrightarrow \bigwedge(\mathfrak{g}^*)$ is the zero map. It is not hard to check that the form $W(l)$ belongs to the space $H^*(\mathfrak{g}, \mathfrak{k}(l))$ which allows to see again that $i(\pi_l)$ is trivial if and only if the cohomology class $[W(l)]$ regarded as an element of $H^{2q+1}(\mathfrak{g})$ is trivial.

Proof of Theorem 4.1. As in the context of Theorem 4.1, we fix a Jordan-Hölder sequence of $\mathfrak{g}$

$$(\mathcal{S})\colon \{0\} = \mathfrak{g}_0 \subset \mathfrak{g}_1 \subset \cdots \subset \mathfrak{g}_n = \mathfrak{g}$$

such that $\mathfrak{g}_p = \mathfrak{z}(\mathfrak{g})$ the center of $\mathfrak{g}$. Let l be in $\Omega_{\max}$, $\mathcal{O}_l$ the coadjoint orbit through l and $e(l) = \{1 < i_1 < i_2 \leq n\}$ the set of jump indices of l. Then obviously $i_1 = p+1$ and if we extract from $\mathcal{S}$ a Jordan-Hölder basis $\{X_1, \ldots, X_n\}$ of $\mathfrak{g}$ passing through $\mathfrak{g}(l)$, then $l([X_{p+1}, X_{i_2}]) \neq 0$. Remind the 3-form $W(l)$ defined in equation (1) by $W(l) = l \wedge X^*_{p+1} \wedge X^*_{i_2}$ which lies in the space $\bigwedge^3(\mathfrak{g}^*)_{\mathfrak{k}(l)}$, where $\{X^*_1, \ldots, X^*_n\}$ is a basis of $\mathfrak{g}^*$ dual to the basis $\{X_1, \ldots, X_n\}$. It is then sufficient to prove that $W(l)$ does not consist of a coboundary form in the space $\bigwedge^3(\mathfrak{g}^*)$. If the contrary happens, there exists $\beta(l) \in \bigwedge^3(\mathfrak{g}^*)$ fulfilling $W(l) = d\beta(l)$, it comes out that

$$W(l)([X_{p+1}, X_{i_2}], X_{p+1}, X_{i_2}) = l([X_{p+1}, X_{i_2}]) \neq 0,$$

but on the other hand and due to the fact that $[X_{p+1}, \mathfrak{g}] \subset \mathfrak{z}(\mathfrak{g})$, we get

$$d\beta(l)([X_{p+1}, X_{i_2}], X_{p+1}, X_{i_2}) = -\beta(l)([X_{p+1}, X_{i_2}], [X_{p+1}, X_{i_2}]) = 0,$$

which is absurd. Finally $[W(l)] \neq 0$ in $H^3(\mathfrak{g})$ and by Lemma 4.4 the invariant $i(\pi_l)$ is not trivial as claimed.

If in addition G is 2-step, then for given $l \in \mathfrak{g}^*$ we have either $\dim(\mathcal{O}_l) = 0$, in which case $l([\mathfrak{g}, \mathfrak{g}]) = \{0\}$ and then $i(\pi_l) = [l]$ is the orbit itself viewed as a cohomology class, or $\dim(\mathcal{O}_l) = 2$. In the last case, we make use of the same above notations and arguments, we have:

$$\begin{aligned}
&d\beta(l)([X_{i_1}, X_{i_2}], X_{i_1}, X_{i_2}) \\
&= -\beta(l)([X_{i_1}, X_{i_2}], [X_{i_1}, X_{i_2}]) + \beta(l)([[X_{i_1}, X_{i_2}], X_{i_2}], X_{i_1}) \\
&\quad - \beta(l)([[X_{i_1}, X_{i_2}], X_{i_1}], X_{i_2}) \\
&= 0,
\end{aligned}$$

which is impossible as again $l([X_{i_1}, X_{i_2}]) \neq 0$ and $[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{z}(\mathfrak{g})$. So we are also done in this case. This achieves the proof of the Theorem. □

Proof of Theorem 4.3. Let Z be a non zero vector in $\mathfrak{g}$ such that $[\mathfrak{g},\mathfrak{c}] = \mathbb{R}Z$. Then obviously Z lies in the center $\mathfrak{z}(\mathfrak{g})$ of $\mathfrak{g}$. We may and do assume that $\mathfrak{z}(\mathfrak{g}) \subset \mathfrak{c}$. It is then proved in [2] that $\mathfrak{h}$ stands to be a common polarizing algebra of $\mathfrak{g}^*$ of generic linear forms. Let $\mathfrak{m}$ be an Abelian supplementary subspace in $\mathfrak{g}$, we pick a Jordan-Hölder basis $\{Z_1,\dots,Z_n\}$ of $\mathfrak{g}$ extracted from the sequence $\mathcal{S}$ as follows:

$$\mathfrak{z}(\mathfrak{g}) = \mathbb{R}-\mathrm{span}\{Z_1,\dots,Z_p\} \quad \text{and} \quad [\mathfrak{g},\mathfrak{c}] = \mathbb{R}Z_1.$$

$$\mathfrak{c} = \mathbb{R}-\mathrm{span}\{Z_1,\dots,Z_p,Z_{p+1},\dots,Z_{p+d}\}.$$

$$\mathfrak{h} = \mathbb{R}-\mathrm{span}\{Z_1,\dots,Z_{p+d},Z_{p+d+1},\dots,Z_m\}.$$

$$\mathfrak{g} = \mathbb{R}-\mathrm{span}\{Z_1,\dots,Z_m,Z_{m+1},\dots,Z_{m+d}\},\ Z_{m+i}\in\mathfrak{m} \quad \text{for} \quad i\in\{1,\dots,d\}.$$

The Pukanszky index set e of generic elements with respect to the above basis is:

$$e = \{p+1 < \cdots < p+d < m+1 < \cdots < m+d = n\}.$$

Moreover, $\Omega_{\max} = \{\xi \in \mathfrak{g}^* : \xi(Z_1) \neq 0\}$ and $\dim(\mathcal{O}_l) = 2d$ for every $l \in \Omega_{\max}$. The $(2d+1)$-form $W(l)$ associated to l can be written as:

$$W(l) = l \wedge Z^*_{p+1} \wedge \cdots \wedge Z^*_{p+d} \wedge Z^*_{m+1} \wedge \cdots \wedge Z^*_{m+d},$$

where the basis above is shifted in such a way that it passes through $\mathfrak{g}(l)$. We are going to use the same means as above. Suppose that there exits $\beta(l) \in \bigwedge(\mathfrak{g}^*)$ such that $W(l) = d\beta(l)$. Remark first that

$$W(l)(Z_1,Z_{p+1},\dots,Z_{p+d},Z_{m+1},\dots,Z_{m+d}) = l_1 \neq 0.$$

On the other hand, and using the fact that $\mathfrak{m}$ is Abelian, we get

$$\begin{aligned}
&d\beta(l)(Z_1,Z_{p+1},\dots,Z_{p+d},Z_{m+1},\dots,Z_{m+d})\\
&= \sum_{i=1}^{d}\sum_{j=1}^{d}(-1)^{p+m+i+j}\\
&\qquad \times \beta(l)([Z_{p+i},Z_{m+j}],Z_1,Z_{p+1},\dots,\check{Z}_{p+i},\dots,Z_{p+d},Z_{m+1},\dots,\check{Z}_{m+j},\dots,Z_{m+d})\\
&= \sum_{i=1}^{d}\sum_{j=1}^{d}(-1)^{p+m+i+j}\\
&\qquad \times \beta(l)(\lambda_{i,j}Z_1,Z_1,Z_{p+1},\dots,\check{Z}_{p+i},\dots,Z_{p+d},Z_{m+1},\dots,\check{Z}_{m+j},\dots,Z_{m+d})
\end{aligned}$$

for some real numbers $\lambda_{i,j}$. Finally $d\beta(l)(Z_1,Z_{p+1},\dots,Z_{p+d},Z_{m+1},\dots,Z_{m+d}) = 0$, which achieves the proof using Lemma 4.4. □

REMARK. 1) The result of the above theorem remains true if we replace the hypothesis $\mathfrak{m}$ Abelian by the fact that $[\mathfrak{m},\mathfrak{m}] \subset \mathbb{R}-\mathrm{span}\{Z,Z_{p+1},\dots,Z_{p+d}\}$.

2) The result of Theorem 4.3 may fail for general SNPC nilpotent Lie groups. Take the example given in Section 3 with $\mathfrak{c} = \mathbb{R} - \operatorname{span}\{Z, Y_1, Y_2\}$, $\mathfrak{h} = \mathbb{R} - \operatorname{span}\{Z, Y_1, Y_2, A\}$ and $\mathfrak{m} = \mathbb{R} - \operatorname{span}\{X_1, X_2\}$. We see that $\mathfrak{g}$ is a SNPC nilpotent Lie algebra for which the invariant vanishes on $\Omega_{\max}$.

5. Examples and separation of unitary representations

5.1. On threadlike nilpotent Lie groups. We present hereby a sequence of examples which are often referred to as threadlike nilpotent Lie algebras belonging to the class of special nilpotent Lie algebras. For $n \geq 2$, let $\mathfrak{g}_n$ be the $(n+1)$-dimensional real nilpotent Lie algebra with basis $\{X_1, \ldots, X_{n+1}\}$ and non-trivial Lie brackets:

$$[X_{n+1}, X_j] = X_{j-1}, \quad j = 2, \ldots, n.$$

Let $G_n = \exp(\mathfrak{g}_n)$ be the associated connected and simply connected nilpotent Lie group. Note that $\mathfrak{g}_2$ is the Heisenberg Lie algebra, G_n is n-step nilpotent and a semi-direct product of the one parameter group $\exp(\mathbb{R}X_{n+1})$ and the Abelian subgroup $G^0 = \exp(\mathfrak{g}^0)$ where $\mathfrak{g}^0 = \mathbb{R} - \operatorname{span}\{X_1, \ldots, X_n\}$. In addition the center of $\mathfrak{g}$ is one dimensional and $\mathfrak{z}(\mathfrak{g}_n) = \mathbb{R}X_1$. We know already from Corollary 5.2 above that $i(\pi_l) \neq 0$ for every $l \in \Omega_{\max}$. We shall proceed to an explicit computation of the invariant and show that the result may fail on lower layers. In this example one has

$$\mathcal{E} = \{e_1 \succ e_2 \succ \cdots \succ e_n\}$$

where $e_j = \{j+1, n+1\}$ for $j = 1, \ldots, n-1$ and $e_n = \emptyset$. The layer Ω_{e_j} are

$$\Omega_{e_j} = \{l \in \mathfrak{g}^* : l_1 = \cdots = l_{j-1} = 0,\ l_j \neq 0\}$$

for $j = 1, \ldots, n-1$, and

$$\Omega_{e_n} = \Omega_{\emptyset} = \{l \in \mathfrak{g}^* : l_1 = \cdots = l_{n-1} = 0\} = \mathbb{R} - \operatorname{span}\{X_n^*, X_{n+1}^*\}.$$

Let $l \in \mathfrak{g}_n^*$ such that $l_1 = l(X_1) \neq 0$ then obviously $e_1 = e(l) = \{2, n+1\}$ and $\dim(\mathcal{O}_l) = 2$. As in [3], the generic orbit associated to the fundamental element $l = (l_1, 0, l_3, \ldots, l_n, 0)$, $l_1 \neq 0$ has the form:

$$\mathcal{O} = \Bigg\{ \bigg(l_1, x_2, l_3 + \frac{1}{2l_1}x_2^2, l_4 + \frac{l_3}{l_1}x_2 + \frac{1}{6l_1^2}x_2^3, l_5 + \frac{l_4}{l_1}x_2 + \frac{l_3}{2l_1^2}x_2^2 + \frac{1}{24l_1^3}x_2^4, \ldots,$$
$$l_n + \frac{l_{n-1}}{l_1}x_2 + \frac{l_{n-2}}{2l_1^2}x_2^2 + \cdots + \frac{l_3}{(n-3)!\, l_1^{n-3}}x_2^{n-3} + \frac{1}{(n-1)!\, l_1^{n-2}}x_2^{n-1}, x_{n+1} \bigg) :$$
$$x_2, x_{n+1} \in \mathbb{R} \Bigg\}.$$

We have then,

$$dl = l_1 X_2^* \wedge X_{n+1}^* + \sum_{i=3}^{n} l_i X_{i+1}^* \wedge X_{n+1}^*,$$

and therefore

$$l \wedge dl = \left(l_1^2 X_1^* \wedge X_2^* + l_1 \sum_{i=3}^{n} l_i X_i^* \wedge X_2^* + l_1 \sum_{i=3}^{n-1} l_i X_1^* \wedge X_{i+1}^* \right) \wedge X_{n+1}^*$$
$$+ \left(l_3 \sum_{i=4}^{n} l_{i-1} X_3^* \wedge X_i^* + \sum_{4 \le i < j \le n} (l_i l_{j-1} - l_{i-1} l_j) X_i^* \wedge X_j^* \right) \wedge X_{n+1}^*.$$

Since for all $i \geq 2$, $X_i^* \wedge X_n^* \wedge X_{n+1}^* = d(X_{i-1}^* \wedge X_n^*)$, the invariant $i(\pi_l)$ reads

$$i(\pi_l) = l_1^2 [X_1^* \wedge X_2^* \wedge X_{n+1}^*] + l_1 \sum_{i=3}^{n-1} l_i [X_i^* \wedge X_2^* \wedge X_{n+1}^*] + l_1 \sum_{i=3}^{n-1} l_i [X_1^* \wedge X_{i+1}^* \wedge X_{n+1}^*]$$
$$+ l_3 \sum_{i=4}^{n-1} l_{i-1} [X_3^* \wedge X_i^* \wedge X_{n+1}^*] + \sum_{4 \le i < j \le n-1} (l_i l_{j-1} - l_{i-1} l_j) [X_i^* \wedge X_j^* \wedge X_{n+1}^*].$$

We remark hereafter that the invariant can be trivial on lower non minimal layers, so on the set of coadjoint orbits of maximal dimension as well. In fact, consider the group G_3 above. For the index set $e_2 = \{3, 4\}$ the corresponding layer is $\Omega_2 = \{l \in \mathfrak{g}_3^* \colon l_1 = 0,\ l_2 \neq 0\}$. We take $l = l_2 X_2^*$, $l_2 \neq 0$, then:

$$l \wedge dl = l_2^2 X_2^* \wedge X_3^* \wedge X_4^* = d\left(l_2^2 X_3^* \wedge X_1^*\right),$$

which shows that the invariant $i(\pi_l)$ is trivial.

5.2. Case of non-nilpotent Lie groups. We study in the section the behavior of the invariant in the case of exponential non-nilpotent Lie groups. We put the emphasis on the case where $\dim G \leq 3$. The unique exponential solvable two dimensional Lie group is the group $ax+b$ whose Lie algebra admits a basis $\{X, Y\}$ such that $[X, Y] = Y$. This group admits only two infinite dimensional representations π_+ and π_- associated respectively to the linear forms $+Y^*$ and $-Y^*$ whose coadjoint orbits are open sets in $\mathfrak{g}^*$. So, obviously $i(\pi_\pm) = 0$ as being element of $H^3(\mathfrak{g})$ in the two dimensional space $\mathfrak{g}$.

Suppose now that G is three dimensional, then up to isomorphism, one can assume that $\mathfrak{g}$ admits a basis $\{A, X, Y\}$ with non trivial brackets:

$$[A, X] = X - \alpha Y, \quad [A, Y] = \alpha X + Y$$

for some $\alpha \in \mathbb{R}^*$ (see [8]). This group admits two layers, the unitary characters and the layer of two dimensional coadjoint orbits $\mathcal{O}_l$ such that $l(X)^2 + l(Y)^2 \neq 0$. So every non trivial orbit contains a representative linear form $l_\theta = \cos\theta X^* + \sin\theta Y^*$ for some $\theta \in [0, 2\pi[$. Having fixed such a form, a routine computation shows that:

$$\begin{aligned} dl_\theta &= \cos\theta dX^* + \sin\theta dY^* \\ &= -\cos\theta(A^* \wedge X^* + \alpha A^* \wedge Y^*) - \sin\theta(-\alpha A^* \wedge X^* + A^* \wedge Y^*) \\ &= (\alpha\sin\theta - \cos\theta)A^* \wedge X^* - (\alpha\cos\theta + \sin\theta)A^* \wedge Y^*. \end{aligned}$$

So,

$$l_\theta \wedge dl_\theta = \alpha A^* \wedge X^* \wedge Y^* = d\left(-\frac{1}{2}X^* \wedge Y^*\right)$$

which merely entails that the invariant $i(\pi_{l_\theta})$ is trivial for every $\theta \in [0, 2\pi[$. This example shows that the result of Theorem 4.1 may fail for general exponential solvable Lie groups.

5.3. Separation of unitary representations using the invariant. We pay attention in this section to the possibility whether the definition of the invariant could be slightly shifted in order to be used to distinguish non equivalent unitary and irreducible representations. For that purpose, the idea is to multiply the cohomology class $[l \wedge (dl)^q]$ by some G-invariant rational nonsingular function defined on the corresponding cross-section, (or G-invariant $C^{+\infty}$ function in more general contexts) depending only on l in order to guarantee such separation within a fixed stratum. In what follows, we prove that such process is efficiently feasible in the following class of Lie groups.

Proposition 5.1. *Let G be a connected simply connected nilpotent Lie group. Assume that the coadjoint orbits of maximal dimension are one codimensional. Then the invariant separates representations associated to generic orbits up to a G-invariant factor.*

Proof. Denote by $O_{\max}$ the set of coadjoint orbits of maximal dimension, and let $\mathcal{O} \in O_{\max} \subset \mathfrak{g}^*$, the dual space of the Lie algebra $\mathfrak{g}$ of G. For $l \in \mathcal{O}$, $\dim(\mathfrak{g}(l)) = 1$ and then $\mathfrak{g}(l) = \mathfrak{z}(\mathfrak{g})$. It comes out then that representation π_l is square integrable modulo the center of G and that from Theorem (5.1) in [4], $i(\pi_l) \neq 0$. Note in addition that $\mathcal{O} = l + \mathfrak{z}(\mathfrak{g})^\perp$. Let $\{X_i : 1 \leq i \leq m+1\}$ be a Jordan-Hölder basis of $\mathfrak{g}$ such that $\mathfrak{z}(\mathfrak{g}) = \mathbb{R}X_1$. We have then $e(l) = e = \{2, \ldots, m+1\}$, $l(X_1) \neq 0$ and the cross-section of the coadjoint orbits in Ω_e is

$$\mathcal{W} = \{\lambda X_1^* : \lambda \in \mathbb{R}^*\}.$$

The invariant $i(\pi_l)$ is then given by

$$i(\pi_l) = P(l)[W], \quad l \in O_{\max}$$

where P is a G-invariant polynomial on $\mathfrak{g}^*$ which never vanishes on $O_{\max}$ and W is the volume form on $\mathfrak{g}$ defined by

$$W = X_1^* \wedge \cdots \wedge X_{m+1}^*.$$

Denote by p_1 the projection

$$p_1 \colon \mathfrak{g}^* \to \mathbb{R}, \quad l = \sum_{i=1}^{m+1} l_i X_i^* \mapsto l_1.$$

We then define the invariant

$$i'(\pi_l) = \frac{p_1(l)}{P(l)} i(\pi_l) = l_1[W],$$

which is an invariant for generic orbits in $\mathfrak{g}^*$ and obviously separates the representations of $O_{\max}$. □

5.4. Remark. Note that $\dim(G)$ is necessarily odd in the Proposition 5.1 above. The proof of Theorem (5.1) in [4] shows that the invariant $i(\pi)$, $\pi \in \hat{G}$ separates representations associated to generic orbits if and only if $\dim(G) \equiv 1 \bmod 4$ (i.e. $\dim(G) = 2q+1$ with q even).

5.5. Example. We consider finally the threadlike nilpotent Lie group G_4. Fix a unitary representation π_l associated to its fundamental element $l = (l_1, 0, l_3, l_4, 0)$. So as in Subsection 5.1 above, one has that

$$i(\pi_l) = l_1^2[X_1^* \wedge X_2^* \wedge X_5^*] + l_1 l_3[X_1^* \wedge X_4^* \wedge X_5^*] - l_1 l_3[X_2^* \wedge X_3^* \wedge X_5^*].$$

Take now another unitary representation $\pi_{l'}$ associated to its fundamental element $l' = (l_1', 0, l_3', l_4', 0)$ such that $i(\pi_l) = i(\pi_{l'})$. We see then that the invariant may be shifted in such a way to get that $l_1 = l_1'$ and $l_3 = l_3'$. Nevertheless, no control on the variable l_4 is accessible.

ACKNOWLEDGEMENT. The authors are grateful to the referee for useful comments and guidance in improving some proofs in the paper.

References

[1] A. Baklouti and N. Ben Salah: *The L^p-L^q version of Hardy's theorem on nilpotent Lie groups*, Forum Math. **18** (2006), 245–262.

[2] A. Baklouti, N. Ben Salah and K. Smaoui: *Some uncertainty principles on nilpotent Lie groups*; in Banach Algebras and Their Applications, Contemp. Math. **363**, Amer. Math. Soc., Providence, RI, 2004, 39–52.

[3] A. Baklouti, C. Benson and G. Ratcliff: *Moment sets and the unitary dual of a nilpotent Lie group*, J. Lie Theory **11** (2001), 135–154.

[4] C. Benson and G. Ratcliff: *An invariant for unitary representations of nilpotent Lie groups*, Michigan Math. J. **34** (1987), 23–30.

[5] C. Benson and G. Ratcliff: *Quantization and invariant for unitary representation of nilpotent Lie groups*, Illinois J. Math. **32** (1988), 53–64.

[6] L.J. Corwin and F.P. Greenleaf: Representations of Nilpotent Lie Groups and Their Applications. Part 1: Basic Theory and Examples, Cambridge Univ. Press, Cambridge, 1990.

[7] R. Goodman and N. Wallach: Representations and Invariants of the Classical Groups, Encyclopedia of Mathematics and its Applications **68**, Cambridge Univ. Press, Cambridge, 1998.

[8] H. Leptin and J. Ludwig: Unitary Representation Theory of Exponential Lie Groups, de Gruyter Expositions in Mathematics **18**, Walter de Gruyter & Co., Berlin, 1994.

Ali Baklouti
Département de Mathématiques
Faculté des sciences de Sfax
BP 802
3038 Sfax
Tunisie
e-mail: ali.baklouti@fss.rnu.tn

Khaled Tounsi
Département de Mathématiques
Faculté des sciences de Sfax
BP 802
3038 Sfax
Tunisie
e-mail: khaled.tounsi@fss.rnu.tn