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THE GENERIC FINITENESS OF THE *m*-CANONICAL MAP FOR 3-FOLDS OF GENERAL TYPE

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Abstract

Let X be a projective minimal threefold of general type with only $\mathbb{Q}$-factorial terminal singularities. We study the generic finiteness of the m-canonial map for such 3-folds. Suppose $P_g(X) \geq 2$ and $q(X) \geq 3$. We prove that the m-canonical map is generically finite for $m \geq 3$, which is a supplement to Kollár's result. Suppose $P_g(X) \geq 5$. We prove that the 3-canonical map is generically finite, which improves Meng Chen's result.

0. Introduction

Throughout the ground field is always supposed to be algebraically closed of characteristic zero. Let X be a projective minimal threefold of general type with only $\mathbb{Q}$-factorial terminal singularities. For all integer $m > 0$, one may define the so-called m-canonical map ϕ_m, which is nothing but the rational map corresponding to the complete linear system $|mK_X|$. Many authors have studied the generic finiteness of ϕ_m in quite different ways.

In 1986, J. Kollár presented the following theorem in his paper.

Theorem 0.1 (Theorem (6.2) of [6]). *Let X be a smooth projective 3-fold of general type with $q(X) \geq 4$. Then ϕ_k is generically finite for $k \geq 3$.*

Meanwhile, he pointed out that the bound is the best possible. During our study of generic finiteness of m-canonical map for threefolds, we find we get a better bound if we suppose $P_g(X) \geq 2$. We also improve a result of Meng Chen.

Theorem 0.2 (Theorem 3.9 of [1]). *Let X be a projective minimal Gorenstein threefold of general type. Then ϕ_3 is generically finite whenever $P_g(X) \geq 39$.*

The following is our main theorem.

Main Theorem. *Let X be a projective minimal threefold of general type with only $\mathbb{Q}$-factorial terminal singularities. If ϕ_m is not generically finite whenever $m \geq 3$, then*

(1) $P_g(X) \leq 1$ *whenever* $m \geq 6$;
(2) *either* $P_g(X) \leq 1$ *or* $q(X) \leq 2$ *if* $m = 3$ *or* 4;
(3) *either* $P_g(X) \leq 1$ *or* $q(X) \leq 1$ *if* $m = 5$.

As a direct corollary, the following is a supplement to Kollár's result.

Corollary 1. Let X be a projective minimal threefold of general type with only $\mathbb{Q}$-factorial terminal singularities. If $P_g(X) \geq 2$ and $q(X) \geq 3$, then ϕ_m is generically finite whenever $m \geq 3$.

Corollary 2 improves Theorem 0.2.

Corollary 2. Let X be a projective minimal threefold of general type with only $\mathbb{Q}$-factorial terminal singularities. ϕ_3 is generically finite whenever $P_g(X) \geq 5$ or $P_g(X) = 4$ and $q(X) \geq 2$.

As an application of our method, we will present more detailed results of the m-canonical map for 3-folds of general type.

1. Preliminaries

(1.1) Kawamata-Ramanujam-Viehweg vanishing theorem. We always use the vanishing theorem in the following form.

Vanishing Theorem ([7] or [9]). *Let X be a smooth complete variety, $D \in$ $\mathrm{Div}(X) \otimes \mathbb{Q}$. Assume the following two conditions*:
(i) *D is nef and big*;
(ii) *the fractional part of D has supports with only normal crossings. Then $H^i(X, \mathcal{O}_X(K_X + \ulcorner D \urcorner)) = 0$ for all $i > 0$.*

Most of our notations are standard within algebraic geometry except the following which we are in favor of: $\sim_{\mathrm{lin}}$ means *linear equivalence* while $\sim_{\mathrm{num}}$ means *numerical equivalence* and $=_{\mathbb{Q}}$ means $\mathbb{Q}$-numerical equivalence.

(1.2) Set up for canonical maps. Let X be a projective minimal threefold of general type with only $\mathbb{Q}$-factorial terminal singularities. Suppose $P_g(X) \geq 2$, we study the canonical map ϕ_1 which is usually a rational map. Take the birational modification $\pi: X' \to X$, according to Hironaka [5], such that
(1) X' is smooth;
(2) $|K_{X'}|$ defines a morphism;
(3) the fractional part of $\pi^*(K_X)$ has supports with only normal crossings. Denote by g the composition of $\phi_1 \circ \pi$. So $g: X' \to B \subseteq \mathbb{P}^{P_g(X)-1}$ is a morphism. Let $g: X' \xrightarrow{f} B' \xrightarrow{s} B$ be the Stein factorization of g. We can write $K_{X'} \sim_{\mathrm{lin}} \pi^*(K_X) + E$

and $K_{X'} \sim_{\text{lin}} M_1 + Z_1$, where M_1 is the movable part of $|K_{X'}|$. E is an effective $\mathbb{Q}$-divisor which is a $\mathbb{Q}$-linear combination of distinct exceptional divisors. We can also write $\pi^*(K_X) \sim_{\text{lin}} M_1 + E'$, where $E' = Z_1 - E$ is actually an effective $\mathbb{Q}$-divisor.

If $\dim \phi_1(X) = 2$, we see that a general fiber of f is a smooth projective curve of genus $g \geq 2$. We say X is canonically fibered by curves of genus g.

If $\dim \phi_1(X) = 1$, we see that a general fiber S of f is a smooth projective surface of general type. We say that X is canonically fibered by surfaces with invariants $\left(c_1^2, P_g\right) = \left(K_{S_0}^2, P_g(S)\right)$. Denote by $\sigma : S \to S_0$ to be the contraction onto the minimal model.

2. Proof of Main Theorem

Let the notation be as in (1.2) throughout this section. We study ϕ_m according to the value $d := \dim \phi_1(X)$ and $b := g(B)$. Obviously $1 \leq d \leq 3$.

Theorem 2.1. *Let X be a projective minimal threefold of general type with only $\mathbb{Q}$-factorial terminal singularities. If ϕ_m is not generically finite whenever $m \geq 6$, then $P_g(X) \leq 1$.*

Proof. We suppose $P_g(X) \geq 2$ and try to prove ϕ_m is generically finite for all integers $m \geq 6$.

The case $d = 2$. Denote by S_1 the general member of $|M_1|$. So S_1 is a smooth projective surface of general type. We have

$$\left|K_{X'} + \ulcorner (m-2)\pi^*(K_X) \urcorner + S_1\right| \subseteq |mK_{X'}|\,.$$

Using (1.1), we have

$$\left|K_{X'} + \ulcorner (m-2)\pi^*(K_X) \urcorner + S_1\right|\Big|_{S_1} \supseteq \left|K_{S_1} + \ulcorner (m-2)L \urcorner\right|$$

where $L := \pi^*(K_X)|_{S_1}$. According to [10], we can reduce to the problem on surface S_1 since

$$K_{X'} + \ulcorner (m-2)\pi^*(K_X) \urcorner$$

is effective. Since $d = 2$, we have $h^0((m-2)L) \geq 3$. We know $|K_{S_1} + \ulcorner (m-2)L \urcorner|$ gives a generically finite map by [2]. So does ϕ_m.

The case $d = 1$ and $b > 0$. Because $b > 0$, the movable part of $|K_X|$ is already base point free on X and $M_1 \sim_{\text{num}} aS$ with $a \geq 2$. So one always have $\pi^*(K_X)|_S = \sigma^*(K_{S_0})$. Obviously we have

$$\left|K_{X'} + \ulcorner (m-2)\pi^*(K_X) \urcorner + M_1\right| \subseteq |mK_{X'}|$$

and

$$\left|K_{X'} + \ulcorner(m-2)\pi^*(K_X)\urcorner + M_1\right|\big|_S = \left|K_S + \ulcorner(m-2)L'\urcorner|_S\right|$$

where $L' = \pi^*(K_X)$ by (1.1). According [8], we can reduce to the system $|K_S + \ulcorner(m-2)L'\urcorner|_S|$ on S since

$$K_{X'} + \ulcorner(m-2)\pi^*(K_X)\urcorner$$

is effective and $a \geq 2$. While

$$\left|K_S + \ulcorner(m-2)L'\urcorner\big|_S\right| \supseteq \left|K_S + \ulcorner(m-2)L'|_S\urcorner\right|$$

by Lemma 2.2 in [3], we see

$$\left|K_S + \ulcorner(m-2)L'|_S\urcorner\right| = \left|K_S + (m-2)\sigma^*(K_{S_0})\right|.$$

The right system defines a generically finite map on S by [12]. So does ϕ_m.

The case $d = 1$ and $b = 0$. According to [6], we have

$$\mathcal{O}(1) \hookrightarrow f_*\omega^2_{X'}$$

and denote by

$$\varepsilon := f_*\omega^2_{X'/\mathbb{P}^1} \hookrightarrow f_*\omega^6_{X'}.$$

The local sections of $f_*\omega^2_{X'/\mathbb{P}^1}$ give the bicanonical map of the fiber S and they extend to global sections of ε because ε is generated by global sections. On the other hand, $H^0(\mathbb{P}^1, \varepsilon)$ can distinguish different fibers of f because $\deg(\varepsilon) > 0$. So $H^0(\mathbb{P}^1, \varepsilon)$ gives a generically finite map on X' and so does ϕ_6, which means ϕ_m is generically finite whenever $m \geq 6$. □

Theorem 2.2. *Let X be a projective minimal threefold of general type with only $\mathbb{Q}$-factorial terminal singularities. If ϕ_m is not generically finite where $m = 3$ or 4, then either $P_g(X) \leq 1$ or $q(X) \leq 2$.*

Proof. We assume $P_g(X) \geq 2$. Since $|3K_{X'}| \subseteq |4K_{X'}|$, we study ϕ_3 according to d and b.

The case $d = 2$. Choose a 1-dimensional subsystem $\Lambda \subseteq |K_{X'}|$ while taking a birational modification $\pi_1 \colon X' \to X$ such that the pencil Λ defines a morphism $g_1 \colon X' \to \mathbb{P}^1$. We can even take further modification to π_1 so that $\pi_1^*(K_X)$ has supports with only normal crossings. Taking the stein factorization $p \colon X' \to W'$. We note that this fibration is different from the one which was defined in (1.2). Denote $b_1 := g(W')$. Let M be the movable part of the pencil. We obviously have $M \leq K_{X'}$. We can write $M \sim_{\text{lin}} \sum_{i=1}^{a} F_i$ where $a \geq 1$ and F_i is a fiber of p for all i.

Suppose $b_1 > 0$. We consider the system

$$|2K_{X'} + M| \subseteq |3K_{X'}|.$$

Now M contains at least 2 components F_1 and F_2. By (1.1), we have a surjective map

$$H^0(X', K_{X'} + M) \to H^0(F_1, 2K_{F_1}) \oplus H^0(F_2, 2K_{F_2}).$$

This means $\phi_{|2K_{X'}+M|}$ can distinguish F_1 and F_2 and the restriction to F_i is at least a bicanonical map. Then ϕ_3 is generically finite.

Suppose $b_1 = 0$. Now $M \sim_{\text{lin}} F$. Still by (1.1) and since $b_1 = 0$, we consider the following system

$$\left| K_F + \ulcorner \pi^*(K_X) \urcorner |_F \right|.$$

Assume $P_g(F) \geq 2$ and $|K_F|$ is composed of pencils otherwise $q(F) \leq 1$ or ϕ_3 is generically finite. If $q(F) \leq 1$, then $q(X) \leq 1$ by virtue of Corollary 2.3 in [4]. If $|K_F|$ is composed of pencils, then $q(F) \leq 2$ according to [11]. So $q(X) \leq 2$.

The case $d = 1$ and $b > 0$. Now we have

$$\left| K_{X'} + \ulcorner \pi^*(K_X) \urcorner + M_1 \right| \subseteq |3K_{X'}|.$$

One can replace m with 3 in the corresponding proof of Theorem 2.1 and derive that ϕ_3 is generically finite.

The case $d = 1$ and $b = 0$. In this case we have

$$\pi^*(K_X) =_{\mathbb{Q}} aS + E'$$

where $a = P_g(X) - 1 \geq 1$ and

$$\left| K_{X'} + \ulcorner 2\pi^*(K_X) - \frac{E'}{a} \urcorner \Big|_S \right| = \left| K_S + \ulcorner \left(2 - \frac{1}{a}\right) \pi^*(K_X) \urcorner \Big|_S \right|.$$

If ϕ_3 is not generically finite, nor is the map defined by the right system above. We suppose $P_g(S) \geq 2$. Then $|K_S|$ is compose of pencils and $q(S) \leq 2$ according to [11]. Thus $q(X) \leq 2$ by [4]. □

Theorem 2.3. *Let X be a projective minimal threefold of general type with only $\mathbb{Q}$-factorial terminal singularities. If ϕ_5 is not generically finite, then either $P_g(X) \leq 1$ or $q(X) \leq 1$.*

Proof. We suppose $P_g(X) \geq 2$.

The case $d = 2$. One can replace m with 5 in the corresponding proof of Theorem 2.1 and derive that ϕ_5 is generically finite.

The case $d = 1$ and $b > 0$. The proof of Theorem 2.2 implies that ϕ_5 is generically finite since $|3K_{X'}| \subseteq |5K_{X'}|$.

The case $d = 1$ and $b = 0$. We can write $\pi^*(K_X) =_Q aS + E'$ where $a = P_g(X) - 1$ and

$$\left| K_{X'} + \ulcorner 4\pi^*(K_X) - \frac{E'}{a} \urcorner \right| \subseteq |5K_{X'}|.$$

For the same reason, we consider the system

$$\left| K_S + \ulcorner \left(4 - \frac{1}{a}\right) \pi^*(K_X) \urcorner \Big|_S \right| = \left| K_{X'} + \ulcorner 4\pi^*(K_X) - \frac{E'}{a} \urcorner \right| \Big|_S$$

on surface S. If $P_g(X) \geq 3$, then we have

$$\mathcal{O}(2) \hookrightarrow f_*\omega_{X'}$$

and

$$f_*\omega^2_{X'/\mathbb{P}^1} \hookrightarrow f_*\omega^4_{X'}.$$

Thus ϕ_4 is generically finite for the same reason given in the proof of Theorem 2.1. So is ϕ_5.

Next we suppose $P_g(X) = 2$ and then $a = 1$. By [6],

$$\mathcal{O}(1) \hookrightarrow f_*\omega_{X'}$$

and

$$f_*\omega^2_{X'/\mathbb{P}^1} \hookrightarrow f_*\omega^6_{X'}.$$

We assume $P_g(S) \geq 2$ and then denote by G the movable part of $|\sigma^*(K_{S_0})|$, we have $6\pi^*(K_X)|_S \geq 2G$ since $|2\sigma^*(K_{S_0})|$ is base-point-free for $P_g(S) > 0$. Furthermore, we suppose $|K_S|$ is composed of pencils otherwise ϕ_5 is generically finite. Then we can write

$$\sigma^*(K_{S_0}) \sim_{\text{num}} bC + Z$$

where C is a general fiber of the canonical map of S and $b \geq P_g(S) - 1 \geq 1$. If $|K_S|$ is composed of irrational pencils, then $b \geq P_g(S) \geq 2$. Denote by M' the movable part of

$$|7K_{X'} + S| \supseteq |K_{X'} + \ulcorner 6\pi^*(K_X) \urcorner + S|.$$

Thus we have $M'|_S \geq 3G$ by Lemma 2.7 in [3]. Now we consider the subsystem

$$|K_{X'} + (7K_{X'} + S) + S| \subseteq |10K_X|.$$

From Theorem 2.1, we know ϕ_7 is generically finite. Then M' is nef and big. By (1.1), we have surjective map

$$H^0(X', K_{X'} + M' + S) \to H^0(S, K_S + M'|_S).$$

Then we see that $M_{10}|_S \geq 4G$ and thus $10\pi^*(K_X)|_S \geq 4G$. Pick up a general member C of $|G|$. Then we can write

$$3\pi^*(K_X)|_S - C - H \sim_{\text{num}} \frac{1}{2}\pi^*(K_X)\Big|_S$$

where H is an effective divisor or zero. Since

$$|K_S + \ulcorner 3\pi^*(K_X)\urcorner|_S| \supseteq |K_S + \ulcorner 3\pi^*(K_X)\urcorner|_S - H|,$$

by (1.1) we have a surjective map

$$H^0(S, K_S + \ulcorner 3\pi^*(K_X)\urcorner|_S - H) \to H^0(K_C + D)$$

where

$$D := (\ulcorner 3\pi^*(K_X)\urcorner|_S - H - C)|_C.$$

Whether $|K_S|$ is composed of rational pencils or irrational pencils, we can reduce to the curve C. Since C is nef on S, $\deg D > 0$. Thus $|K_C + D|$ gives a finite map and ϕ_5 is generically finite. Then we can derive that if ϕ_5 is not generically finite then $P_g(S) \leq 1$ and thus $q(S) \leq 1$. By virtue of Corollary 2.3 in [4], we have $q(X) \leq 1$. So we are done. □

3. Generic finiteness of ϕ_m

In this section we will keep the same notation as in (1.2) and denote $d := \dim \phi_1(X)$ and $b := g(B)$.

Corollary 3.1. *Let X a projective minimal threefold of general type with only $\mathbb{Q}$-factorial terminal singularities. If $P_g(X) \geq 2$, then ϕ_m is generically finite for all integers $m \geq 6$.*

Proof. This is a direct result from Theorem 2.1. □

Corollary 3.2. *Let X be a projective minimal threefold of general type with only $\mathbb{Q}$-factorial terminal singularities. Assume $P_g(X) \geq 3$. Then ϕ_m is generically finite when $m = 4$ or 5.*

Proof. The proof of Theorem 2.3 implies the case $m = 5$. As for $m = 4$.

The case $d = 2$. One can still derive it from the proof of Theorem 2.1;

The case $d = 1$ and $b > 0$. The proof of Theorem 2.1 also implies ϕ_4 is generically finite as long as replacing m with 4 there.

The case $d = 1$ and $b = 0$. From proof of Theorem 2.3, we know this corollary is true. □

In the following, we will study ϕ_3 and then present several probabilities.

Corollary 3.3. *Let X be a projective minimal threefold of general type with only $\mathbb{Q}$-factorial terminal singularities. Assume $P_g(X) \geq 5$. Then ϕ_3 is generically finite.*

Proof. The case $d = 2$. Denote by S_1 the general member of $|M_1|$ where M_1 is the movable part of $|K_{X'}|$. We have

$$|K_{X'} + \ulcorner \pi^*(K_X) \urcorner + S_1| \subseteq |3K_{X'}|$$

and

$$|K_{X'} + \ulcorner \pi^*(K_X) \urcorner + S_1|\,|_{S_1} = |K_{S_1} + L|$$

where $L := \ulcorner \pi^*(K_X) \urcorner|_{S_1}$. Since $K_{X'} + \ulcorner \pi^*(K_X) \urcorner$ is effective, we can reduce to the problem on the surface S_1 by [10]. Obviously $h^0(L) \geq P_g(X) - 1 \geq 4$. Then $|K_S + L|$ gives a generically finite map by [2], so does ϕ_3.

The case $d = 1$ and $b > 0$. The proof of Theorem 2.2 implies this is true.

The case $d = 1$ and $b = 0$. Then

$$\mathcal{O}(4) \hookrightarrow f_*\omega_{X'}$$

and

$$f_*\omega^2_{X'/\mathbb{P}^1} \hookrightarrow f_*\omega^3_{X'}.$$

Thus ϕ_3 is generically finite for the same reason given in the proof of Theorem 2.1. □

Corollary 3.4. *Let X be a projective minimal threefold of general type with only $\mathbb{Q}$-factorial terminal singularities. Assume $P_g(X) = 4$ and $d = 2$. Then ϕ_3 is generically finite.*

Proof. One can easily derive it from above since $h^0(L) \geq 3$ in this case. □

Corollary 3.5. *Let X be a projective minimal threefold of general type with only $\mathbb{Q}$-factorial terminal singularities. Assume $P_g(X) \geq 2$ and $d = 1$ and $b > 0$. Then ϕ_3 is generically finite.*

Proof. This is just one part of the proof of Theorem 2.2. □

Corollary 3.6. *Let X be a projective minimal threefold of general type with only $\mathbb{Q}$-factorial terminal singularities. Assume $P_g(X) = 3$ and $d = 2$. Then ϕ_3 is generically finite except $q(S_1) = 1$ or 2 and $|L|$ is composed of a rational pencil of genus $g = q(S_1) + 1$ where S_1 is the general member of $|M_1|$ and $L := \ulcorner \pi^*(K_X) \urcorner|_{S_1}$.*

Proof. We only need to consider the system $|K_{S_1} + L|$. One can easily derive this result from Proposition 2.1 and 2.2 in [2]. □

Proposition 3.7. *Let X be a projective minimal threefold of general type with only $\mathbb{Q}$-factorial terminal singularities. Assume $P_g(X) = 4$ and $d = 1$ and $b = 0$. Then ϕ_3 is generically finite if $P_g(S) \geq 2$.*

Proof. One can easily see that we only need to study the system

$$\left| K_S + \ulcorner \frac{5}{3}\pi^*(K_X) \urcorner \Big|_S \right| = \left| K_{X'} + \ulcorner 2\pi^*(K_X) - \frac{E'}{3} \urcorner \right| \Big|_S$$

since

$$\pi^*(K_X) =_{\mathbb{Q}} 3S + E'.$$

Because

$$\mathcal{O}(3) \hookrightarrow f_*\omega_{X'},$$

we have

$$f_*\omega^3_{X'/\mathbb{P}^1} \hookrightarrow f_*\omega^5_{X'}.$$

Suppose $P_g(S) \geq 2$ and denote by G the movable part of $|\sigma^*(K_{S_0})|$. Then we have $5\pi^*(K_X)|_S \geq 3G$ since $|3\sigma^*(K_{S_0})|$ is base point free. Denote by $\overline{M}$ the movable part of

$$|6K_{X'} + S| \supseteq |K_{X'} + \ulcorner 5\pi^*(K_X) \urcorner + S|.$$

We know $\overline{M}|_S \geq 4G$ from Lemma 2.7 in [3]. Denote by $\overline{\overline{M}}$ the movable part of $|2(6K_{X'} + S)|$. Now we consider the subsystem

$$|K_{X'} + 2(6K_{X'} + S) + S| \subseteq |14K_{X'}|.$$

Since ϕ_{12} is generically finite, $\overline{\overline{M}}$ is nef and big. By (1.1), we have a surjective map

$$H^0\left(X', K_{X'} + \overline{\overline{M}} + S\right) \to H^0\left(S, K_S + \overline{\overline{M}}\Big|_S\right).$$

Obviously we have $\overline{\overline{M}}|_S \geq 2\overline{M}|_S$. So $M_{14}|_S \geq 9G$ by Lemma 2.7 in [3]. Thus $14\pi^*(K_X)|_S \geq 9G$. Then we can write

$$\frac{5}{3}\pi^*(K_X)\Big|_S - G - H \sim_{\text{num}} \frac{1}{9}\pi^*(K_X)\Big|_S$$

where H is an effective divisor or zero. Pick up a general member C of $|G|$. Then we have a surjective map

$$H^0\left(S, K_S + \left\ulcorner\frac{5}{3}\pi^*(K_X)\right\urcorner\Big|_S - H\right) \to H^0(C, K_C + D)$$

by (1.1) where $D := (\ulcorner(5/3)\pi^*(K_X)\urcorner|_S - C - H)|_C$. Since C is nef on S, $|K_C + D|$ gives a finite map. Thus ϕ_3 is generically finite. □

Proposition 3.8. *Let X be a projective minimal threefold of general type with only $\mathbb{Q}$-factorial terminal singularities. Assume $P_g(X) = 3$ and $d = 1$ and $b = 0$. Then ϕ_3 is generically finite when $P_g(S) \geq 3$.*

Proof. In this case, we have

$$\pi^*(K_X) =_Q 2S + E'.$$

Then one can reduce to the system $|K_S + \ulcorner(3/2)\pi^*(K_X)\urcorner|_S|$ since

$$\left|K_S + \left\ulcorner\frac{3}{2}\pi^*(K_X)\right\urcorner\Big|_S\right| = \left|K_{X'} + \left\ulcorner 2\pi^*(K_X) - \frac{E'}{2}\right\urcorner\right|\Big|_S$$

by (1.1).

If $|K_S|$ is not composed of pencils, then ϕ_3 is generically finite.

If $|K_S|$ is composed of pencils, then we can write $K_S \sim_{\text{num}} bC + Z''$ where $b \geq P_g(S) - 1 \geq 2$. Since

$$\mathcal{O}(2) \hookrightarrow f_*\omega_{X'}$$

and

$$f_*\omega^3_{X'/\mathbb{P}^1} \hookrightarrow f_*\omega^6_{X'},$$

we have $6\pi^*(K_X)|_S \geq 3G$ where G is the movable part of $|\sigma^*(K_{S_0})|$. By Lemma 2.7 in [3] and (1.1) considering the system $|K_{X'} + \ulcorner 6\pi^*(K_X)\urcorner + S|$, we have $M'|_S \geq 4G$ where M' is the movable part of $|7K_{X'} + S|$. Then considering the subsystem

$$|K_{X'} + (7K_{X'} + S) + S| \subseteq |9K_{X'}|,$$

by (1.1), we have a surjective map

$$H^0(X', K_{X'} + M' + S) \to H^0(S, K_S + M'|_S).$$

Denote by M'' the movable part of $|K_{X'} + (7K_{X'} + S) + S|$. Then $M''|_S \geq 5G$. So $9\pi^*(K_X)|_S \geq 5G$. Then we can write

$$\frac{3}{2}\pi^*(K_X)\bigg|_S - C - H \sim_{\text{num}} \frac{3}{5}\pi^*(K_X)\bigg|_S$$

where H is an effective divisor or zero. Thus we can reduce to the problem on the smooth curve C of $g \geq 2$. Then we are done. □

Proposition 3.9. *Let X be a projective minimal threefold of general type with only $\mathbb{Q}$-factorial terminal singularities. Assume $P_g(X) = 2$ and $d = 1$ and $b = 0$. Then ϕ_3 is generically finite when $P_g(S) \geq 4$.*

Proof. We can write $\pi^*(K_X) =_{\mathbb{Q}} S + E'$ and reduce to the problem on the system $|K_S + \ulcorner\pi^*(K_X)\urcorner|_S|$ on surface S.

If $|K_S|$ is not composed of pencils, then we are done.

If $|K_S|$ is composed of pencils, we can write $\sigma^*(K_{S_0}) \sim_{\text{num}} bC + Z''$ where $b \geq P_g(S) - 1 \geq 3$. Now

$$\mathcal{O}(1) \hookrightarrow f_*\omega_{X'}$$

and

$$f_*\omega^2_{X'/\mathbb{P}^1} \hookrightarrow f_*\omega^6_{X'}.$$

Then we see that $M'|_S \geq 3G$ where M' is the movable part of $|7K_{X'} + S|$ and G the movable part of $\sigma^*(K_{S_0})$. Then consider the subsystem

$$|K_{X'} + (7K_{X'} + S) + S| \subseteq |10K_{X'}|.$$

Denote by M'' the movable part of the left system above. By (1.1) we have a surjective map

$$H^0(X', K_{X'} + M' + S) \to H^0(S, K_S + M'|_S)$$

and then $M''|_S \geq 4G$. Thus $10\pi^*(K_X)|_S \geq 4G$. We can write

$$\pi^*(K_X)|_S - C - H \sim_{\text{num}} \frac{1}{6}\pi^*(K_X)\bigg|_S$$

where H is an effective divisor or zero. Then we can consider the system $|K_C + D|$ on curve C where $D \sim_{\text{num}} (\ulcorner(1/6)\pi^*(K_X)\urcorner|_S)|_C$. So we are done. □

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