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A CONCENTRATION PHENOMENON AROUND A SHRINKING HOLE FOR SOLUTIONS OF MEAN FIELD EQUATIONS

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1. Introduction

Let Ω be a bounded smooth domain in $\mathbf{R}^2$. In this paper, we consider the following mean field equation in statistical mechanics of point vortices; see [6, 7, 15]:

$$\text{(P)}\qquad \begin{aligned} -\Delta u &= \rho\frac{e^u}{\int_\Omega e^u} \quad \text{in} \quad \Omega, \qquad \rho > 0\\ u &= 0 \quad \text{on} \quad \partial\Omega. \end{aligned}$$

We note that the problem (P) for $\rho < 0$ is treated in [14]; see also [6, 7]. Analogous problems under Neumann boundary conditions are considered in relation to stationary problems of the Keller-Segel system of chemotaxis in [28]. Analogous problems on two-dimensional manifolds are also considered in relation to the prescribed Gauss curvature problem or Chern-Simons-Higgs gauge theory; see [12, 17, 26, 29] and references therein.

It should be also remarked that the following non-linear eigenvalue problem called the Gel'fand problem (see, for example, [3, 32]) also relates to our problem (P):

$$\text{(G)}\qquad \begin{aligned} -\Delta u &= \lambda e^u \quad \text{in} \quad \Omega, \qquad \lambda > 0\\ u &= 0 \quad \text{on} \quad \partial\Omega. \end{aligned}$$

Indeed, every solution of (G) corresponds to the solution of (P) for $\rho = \int_\Omega \lambda \exp u\, dx$.

(P) is the Euler-Lagrange equation of the following functional:

$$J_\rho(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \rho \log \int_\Omega e^u \quad \text{for} \quad u \in H_0^1(\Omega).$$

Caglioti et al. show the following facts on (P):

Facts 1.1 ([6]; see also [7]).
(1) From the Moser-Trudinger inequality [23],

$$\inf_{u \in H_0^1(\Omega)} J_\rho(u) > -\infty \quad \text{for} \quad 0 < \rho \leq 8\pi.$$

Moreover, the problem (P) for $0 < \rho < 8\pi$ has a solution that minimizes J_ρ.
(2) The disks admit no solution of (P) for every $\rho \geq 8\pi$. More generally, let Ω be a strictly star-shaped domain, that is, there exists a constant $\alpha_0 > 0$ such that $(x \cdot \nu)(\int_{\partial\Omega} d\sigma)^{-1} \geq \alpha_0 \quad \text{on} \quad \partial\Omega$, where ν is the exterior unit outer normal vector field to $\partial\Omega$ and $d\sigma$ is the arclength measure on $\partial\Omega$. Then (P) admits no solutions if $\rho \geq 4/\alpha_0$ from the Pohožaev identity [27]. We note that $\alpha_0 = 1/(2\pi)$ when Ω is a disk.
(3) Each annulus admits the unique radial solution for every $\rho \in \mathbf{R}$.

It should be remarked that parts of Fact 1.1 are already known as results on (G). Indeed, Bandle [3, Theorem 4.16] and Suzuki and Nagasaki [35, Lemma 3] obtained similar conclusions to Fact 1.1 (2) for (G) from the Pohožaev identity (see also [3, p. 201]). The existence of radial solutions of (G) on annuli was proved by Nagasaki and Suzuki [24] (see also [30, 32, 34]) and independently by Lin [19]. Their studies on the solutions are sufficient to obtain Fact 1.1 (3) for $\rho > 0$. We note that they also studied, in different ways, the existence of non-radial solutions of (G) on annuli. It should be also remarked that, in the course of the study of (G), Suzuki proved the unique existence of solutions of (P) when Ω is simply connected and $0 < \rho < 8\pi$ [33] (see also [32, p. 263]).

We note that, on general domains other than disks and annuli, it is not clear whether a solution of (P) for $\rho \geq 8\pi$ exists. Caglioti et al. proved the existence of a minimizer of $J_{8\pi}(\cdot)$, that is, a solution of (P) for $\rho = 8\pi$ when Ω is sufficiently *thin* by analyzing the dual functional to $J_{8\pi}(\cdot)$ [6, p. 523]. In this case, supposing additionally that Ω is strictly star-shaped and admits the unique solution of (P) for $\rho = 8\pi$, they also proved the existence of a sequence $\rho_n \longrightarrow 8\pi + 0$ such that (P) for ρ_n has at least two solutions [7, Theorem 7.1]. On the other hand, when Ω is simply connected and satisfies some additional conditions, we know the existence of the Weston branch of large solutions (λ, u_λ) of (G) for sufficiently small λ [36], which blows up at one point in Ω as $\lambda \longrightarrow 0$. We note that Moseley [22] and subsequently Suzuki [31] (see also [32, Section 3.4]) reduced some sufficient conditions on Ω to construct the branch. Suzuki and Nagasaki proved that the Weston branch satisfies

$$\int_\Omega \lambda e^{u_\lambda} dx = 8\pi + C\lambda + o(\lambda) \quad \text{as} \quad \lambda \longrightarrow 0,$$

where C is a constant determined by a conformal mapping $B_1(0)$ onto Ω [35, Appendix I] (see also [32, Proposition 4.36]). This formula indicates that, on the domains satisfying $C > 0$, the solutions of (P) for $\rho > 8\pi$ and sufficiently close to 8π

exist. Moreover, Mizoguchi and Suzuki proved that the Weston branch and the trivial solution $(\lambda, u) = (0, 0)$ of (G) are connected under the additional conditions on Ω [21, Theorem 13]. This result indicates additionally the existence of solutions of (P) for $\rho = 8\pi$ as well as for $\rho > 8\pi$ and sufficiently close to 8π on the appropriate domains, an example of which is given in [21, pp. 207–208]. We note that this example is also *thin* in some sense. It should be remarked that Nagasaki and Suzuki [25] (see also [32, Section 3.3]) proved that, when a family of solutions $\{(\lambda_n, u_n)\}$ of (G) on a general domain (not necessarily a simply connected one) satisfies $\lambda_n \longrightarrow 0$ and $\int_\Omega \lambda_n \exp u_n dx \longrightarrow \Sigma_0$ as $n \longrightarrow \infty$, the limit Σ_0 must be $8\pi m$ for some $m \in \{0, \infty\} \cup \mathbf{N}$. They also proved that, when $m \in \mathbf{N}$, the solution u_n of (G) blows up at distinct m points in Ω as $n \longrightarrow \infty$ and obtained several necessary conditions of the limiting function of u_n. We note that this result resembles the later results of Brezis and Merle [5] and Li and Shafrir [18], which we refer as Fact 2.5 in this paper. Recently, Baraket and Pacard [4] considered the converse problem to this result of Nagasaki and Suzuki [25]. Baraket and Pacard gave, for each $m \in \mathbf{N}$, a sufficient condition of limiting functions that enables us to construct a one-parameter family of solutions $\{(\lambda, u_\lambda)\}$ of (G) satisfying that $\int_\Omega \lambda \exp u_\lambda dx \longrightarrow 8\pi m$ and u_λ converges to such a limiting function as $\lambda \longrightarrow 0$. This result also suggests the existence of solutions of (P) at least near $\rho = 8\pi m$ on the appropriate domains for each $m \in \mathbf{N}$.

Recently, a new proof of the existence of a solution of (P) for $\rho > 8\pi$ appeared. Ding et al. proved the following fact by the minimax variational method:

Fact 1.2 ([12]). On every smooth bounded domain whose complement contains a bounded region, that is, on every smooth bounded domain with a hole, the mean field equation (P) has a solution for all $\rho \in (8\pi, 16\pi)$.

The purpose of this paper is to investigate the behavior of this solution as the hole of the domain shrinks to a point. To simplify the presentation, assuming that $0 \in \Omega$, we study the behavior of solutions of (P) on $\Omega_\varepsilon = \Omega \setminus \overline{B_\varepsilon(0)}$ as $\varepsilon \longrightarrow 0$, where $B_\varepsilon(0) = \{x \in \mathbf{R}^2 : |x| < \varepsilon\}$. We refer (P) for Ω_ε as (P_ε) and the functional $J_\rho(\cdot)$ on $H_0^1(\Omega_\varepsilon)$ for (P_ε) as $J_\rho^\varepsilon(\cdot)$.

Here we recall the minimax method used in [12] for the case (P_ε). Let D_ρ^ε be a family of continuous functions $h : B_1(0) = \{(r, \theta) : 0 \le r < 1, \theta \in [0, 2\pi)\} \longrightarrow H_0^1(\Omega_\varepsilon)$ satisfying

$$\lim_{r \to 1} J_\rho^\varepsilon(h(r, \theta)) \longrightarrow -\infty \tag{1.1}$$

and

$$\lim_{r \to 1} m_{\Omega_\varepsilon}(h(r, \cdot)) \quad \text{is a continuous curve enclosing} \quad B_\varepsilon(0), \tag{1.2}$$

where

$$m_{\Omega_\varepsilon}(u) = \int_{\Omega_\varepsilon} x \frac{e^{u(x)}}{\int_{\Omega_\varepsilon} e^{u(x)}}.$$

Ding et al. proved that for every $\rho \in (8\pi, 16\pi)$ the minimax value

$$\alpha_\rho^\varepsilon = \inf_{h \in D_\rho^\varepsilon} \sup_{u \in h(B_1(0))} J_\rho^\varepsilon(u)$$

is achieved by a critical point of J_ρ^ε in $H_0^1(\Omega_\varepsilon)$, which is a solution of (P_ε).

In the following, we assume each element of $H_0^1(\Omega_\varepsilon)$ to be an element of $H_0^1(\Omega)$ by extending it by 0 on $B_\varepsilon(0)$. Our result is stated as follows:

Main Theorem. *Fix $\rho \in (8\pi, 16\pi)$, a sequence $\varepsilon_n \downarrow 0$ as $n \longrightarrow \infty$, and a solution u_n of $(\mathrm{P}_{\varepsilon_n})$ that attains the minimax value $\alpha_\rho^{\varepsilon_n}$. Then,*

$$\frac{e^{u_n}}{\int_\Omega e^{u_n}} \longrightarrow \delta_0 \quad \textit{weakly} * \textit{ in} \quad M(\overline{\Omega}) \quad \textit{as} \quad n \longrightarrow \infty,$$

where $M(\overline{\Omega}) = C(\overline{\Omega})^*$ denotes the space of signed Radon measures over the compact space $\overline{\Omega}$ and δ_0 denotes the Dirac measure supported at the origin $0 \in \Omega$.

We note that Lewandowski [16] obtained a concentration phenomenon similar to our Main Theorem in the following higher dimensional problem with the critical Sobolev exponent:

$$(\mathrm{P}') \qquad \begin{aligned} &-\Delta u = u^{(N+2)/(N-2)}, \quad u > 0 \quad \text{in} \quad \Omega \subset \mathbf{R}^N \quad \text{for} \quad N \geq 5, \\ &u = 0 \quad \text{on} \quad \partial\Omega. \end{aligned}$$

Assuming that Ω is a smooth bounded star-shaped domain and $0 \in \Omega$, Lewandowski considered (P') also on the domain $\Omega_\varepsilon = \Omega \backslash \overline{B_\varepsilon(0)}$. We note that Coron [9] proved that Ω_ε admits a solution of (P') for sufficiently small ε; see also [2] for the more general existence result for (P'). Let u_ε be a solution of (P') on Ω_ε satisfying the appropriate conditions. Then Lewandowski proved that $|\nabla u_\varepsilon|^2 \longrightarrow (S_N)^{N/2}\delta_0$ as $\varepsilon \longrightarrow 0$, where S_N is the best constant in the Sobolev inequality, that is, $S_N = \inf\{\int_{\mathbf{R}^N} |\nabla u|^2 : u \in H^1(\mathbf{R}^N), \|u\|_{L^{2N/(N-2)}(\mathbf{R}^N)} = 1\}$.

In contrast to our results, Lewandowski proved more on the behavior of u_ε as $\varepsilon \longrightarrow 0$. Indeed, let $w_\varepsilon(x)$ be a blow-up around an appropriate point $a_\varepsilon \in \mathbf{R}^N$, that is, $w_\varepsilon(x) = t_\varepsilon^{(N-2)/2} u_\varepsilon(t_\varepsilon(x + a_\varepsilon))$ for appropriately chosen $t_\varepsilon \in (0, 1)$ satisfying $t_\varepsilon \longrightarrow 0$. Then $w_\varepsilon(x)$ converges to a solution of (P') for $\Omega = \mathbf{R}^N$ in an appropriate topology.

Thus, also for our problem, it is natural to ask more precise behavior of u_n itself. It seems interesting to study the behavior of u_n by the blow-up analysis for (P) developed by Li and Shafrir [18], though the author now thinks that it seems difficult.

2. Proof of Main Theorem

The key of the proof of Main Theorem is the following estimate on the minimax value α_ρ^ε:

Proposition 2.1. *For every* $\rho \in (8\pi, 16\pi)$,

$$\alpha_\rho^\varepsilon \longrightarrow -\infty \quad as \quad \varepsilon \longrightarrow 0.$$

Assuming this proposition, which we prove in Section 3, we prove Main Theorem in this section.

Set

$$\mu_n(x) = \frac{e^{u_n(x)}}{\int_\Omega e^{u_n(x)} dx}.$$

We regard $\{\mu_n\}$ as a bounded set in $M(\overline{\Omega})$. Thus, choosing a subsequence if necessary, we may assume that

$$\mu_n \longrightarrow \mu_\infty \quad \text{weakly} * \text{ in} \quad M(\overline{\Omega}) \quad \text{as} \quad n \longrightarrow \infty$$

for some $\mu_\infty \in M(\overline{\Omega})$. In the rest of this section, we prove μ_∞ is always δ_0, which implies that $\mu_n \longrightarrow \delta_0$ without choosing a subsequence, that is, we obtain Main Theorem.

We prove $\mu_\infty = \delta_0$ by the following three steps:

STEP 1. $\mu_\infty = \delta_{x_\infty}$ for some $x_\infty \in \overline{\Omega}$.

STEP 2. $x_\infty \notin \partial\Omega$.

STEP 3. $x_\infty \notin \Omega \backslash \{0\}$, that is, $x_\infty = 0$.

We start from recalling the improved Moser-Trudinger inequality:

Fact 2.2 ([12, Lemma 2.2]; see also [1, Théorème 4] and [8, Theorem 2.1]). Let S_1 and S_2 be two subsets of $\overline{\Omega}$ satisfying $\mathrm{dist}(S_1, S_2) \geq \delta_0 > 0$ and let γ_0 be a number satisfying $\gamma_0 \in (0, 1/2)$. Then for any $\varepsilon > 0$, there exists a constant $c = c(\varepsilon, \delta_0, \gamma_0) > 0$ such that

$$\int_\Omega e^u \leq c \exp\left\{ \frac{1}{32\pi - \varepsilon} \int_\Omega |\nabla u|^2 + c \right\} \tag{2.1}$$

holds for all $u \in H_0^1(\Omega)$ satisfying

$$\frac{\int_{S_1} e^u}{\int_\Omega e^u} \geq \gamma_0 \quad \text{and} \quad \frac{\int_{S_2} e^u}{\int_\Omega e^u} \geq \gamma_0.$$

From Fact 2.2, we obtain the following lemma:

Lemma 2.3. *Suppose that a sequence* $\{u_n\} \subset H_0^1(\Omega)$ *satisfies*

$$J_\rho(u_n) \longrightarrow -\infty \quad \text{and} \quad m_\Omega(u_n) \left(= \int_\Omega x\mu_n = \int_\Omega x \frac{e^{u_n}}{\int_\Omega e^{u_n}} \right) \longrightarrow x_\infty \quad \text{as} \quad n \longrightarrow \infty$$

for some $\rho \in (8\pi, 16\pi)$ *and some* $x_\infty \in \mathbf{R}^2$. *Then* $x_\infty \in \overline{\Omega}$ *and*

$$\mu_n \longrightarrow \delta_{x_\infty} \quad \text{weakly} * \text{ in} \quad M(\overline{\Omega}).$$

Although we are able to prove this lemma easily by similar argument to the proof of [12, Lemma 2.3], we give a proof of Lemma 2.3 in Appendix for convenience.

Proof of Step 1. It is obvious that $J_\rho(u_n) \le J_\rho^{\varepsilon_n}(u_n)$ because $u_n \equiv 0$ in $B_{\varepsilon_n}(0)$. Thus $J_\rho(u_n)(\le J_\rho^{\varepsilon_n}(u_n) = \alpha_\rho^{\varepsilon_n}) \longrightarrow -\infty$ as $n \longrightarrow \infty$ from Proposition 2.1. On the other hand, there exists a subsequence of $m_\Omega(u_n)$ that converges because Ω is bounded. Using Lemma 2.3, we obtain the conclusion because $\mu_n \longrightarrow \mu_\infty$. □

To make the next step, we recall the following fact from [10]:

Fact 2.4 ([10, p. 51 (8')]; see also [20, p. 628].). Let $N \subset \mathbf{R}^2$ be a neighborhood of $\partial\Omega$ (not $\partial\Omega_\varepsilon$) and set $\omega_0 = \overline{\Omega} \cap N$. Then, there exist positive constants ε, γ, and C depending on $\partial\Omega$ and ω_0 satisfying the following properties: $\omega = \{x \in \overline{\Omega};\, \mathrm{dist}(x, \partial\Omega) < \varepsilon\}$ is a subset of ω_0 and, for all $x \in \omega$, there exists a measurable set I_x such that

(1) $\mathrm{meas}(I_x) \ge \gamma$,

(2) $I_x \subset \{y \in \omega_0 : \mathrm{dist}(y, \partial\Omega) \ge \varepsilon/2\}$,

(3) $u(x) \le Cu(\xi)$ for all $\xi \in I_x$,

where u is any $C^2(\omega_0)$ function satisfying

$$-\Delta u = f(u) \quad \text{and} \quad u > 0 \quad \text{in} \quad \omega_0 \cap \Omega\,(\subset \Omega),$$
$$u = 0 \quad \text{on} \quad \omega_0 \cap \partial\Omega\,(= \partial\Omega)$$

for some locally Lipschitz function $f : \mathbf{R} \longrightarrow \mathbf{R}$.

We note that Fact 2.4 is proved by the moving plane method established in [13].

Proof of Step 2. Fix a neighborhood $N \subset \mathbf{R}^2$ of $\partial\Omega$ such that $\Omega_{\varepsilon_n} \cap N$ is independent of n. Applying Fact 2.4 to this N, we obtain $\omega \subset \overline{\Omega}$ satisfying the several properties stated in Fact 2.4. We prove below that $\sup_n \|u_n\|_{L^\infty(\omega)} < \infty$, which prevents $x_\infty \in \partial\Omega$ since $\int_\Omega e^{u_n} \longrightarrow \infty$ as $n \longrightarrow \infty$ from Proposition 2.1 and Fact 1.1 (1).

Let $\omega_1 = \bigcup_{x\in\omega} I_x \subset \Omega$. Then we obtain that

$$0 \le u_n(x) \le \frac{C}{\gamma}\int_{I_x} u_n(y)dy \le \frac{C}{\gamma}\|u_n\|_{L^1(\omega_1)} \le \frac{C}{\gamma}\|u_n\|_{L^1(\Omega)} \quad \text{for every } x \in \omega,$$

that is,

$$\sup_n \|u_n\|_{L^\infty(\omega)} \le \frac{C}{\gamma} \sup_n \|u_n\|_{L^1(\Omega)}.$$

It is rather standard to estimate $\sup_n \|u_n\|_{L^1(\Omega)}$. Indeed, let

$$\psi_n(x) = \int_{\Omega_{\varepsilon_n}} \left(\frac{1}{2\pi} \log |x-y|^{-1} - \frac{1}{2\pi} \log[\mathrm{diam}(\Omega)]^{-1} \right) \rho\mu_n(y)dy.$$

It is obvious that

$$\begin{aligned} -\Delta\psi_n &= \rho\mu_n = \rho \frac{e^{u_n}}{\int_{\Omega_{\varepsilon_n}} e^{u_n}} \quad \text{in} \quad \Omega_{\varepsilon_n}, \\ \psi_n &\ge 0 \quad \text{on} \quad \partial\Omega_{\varepsilon_n}. \end{aligned}$$

We note that $\psi_n - u_n$ is harmonic in Ω_ε and non-negative on $\partial\Omega_{\varepsilon_n}$. Applying the maximum principle of harmonic functions to $\psi_n - u_n$, we obtain $\psi_n - u_n \ge 0$, that is, $\psi_n \ge u_n (\ge 0)$ in Ω_{ε_n}. Using the Young inequality for convolutions, we obtain

$$\begin{aligned} \|u_n\|_{L^1(\Omega)} = \|u_n\|_{L^1(\Omega_{\varepsilon_n})} &\le \|\psi_n\|_{L^1(\Omega_{\varepsilon_n})} \\ &\le \frac{\rho}{2\pi} \| \log|\cdot|^{-1} \|_{L^1(B_{\mathrm{diam}(\Omega)}(0))} \cdot \|\mu_n\|_{L^1(\Omega_{\varepsilon_n})} + C' \\ &\le C'' < \infty \end{aligned}$$

for some constants C' and C'' independent of n because $\|\mu_n\|_{L^1(\Omega_{\varepsilon_n})} \equiv 1$. □

To make the final step, we recall the results of [5] and [18] concerning the solutions of $-\Delta u = V(x)\exp u$. Combining their results, we obtain the following fact:

Fact 2.5 ([5, Theorem 3] and [18, Theorem]). Let Ω be a bounded domain in $\mathbf{R}^2$ and let $\{w_n\} \subset C(\Omega)$ be a sequence of solutions of

$$-\Delta w = \rho e^w \quad \text{in} \quad \mathcal{D}'(\Omega)$$

for some $\rho > 0$ such that $\sup_n \int_\Omega e^{w_n} < \infty$. Then, there exists a subsequence $\{u_{n_k}\}$ satisfying one of the following alternatives:

(1) $\{u_{n_k}\}$ is bounded in $L^\infty_{\mathrm{loc}}(\Omega)$,

(2) $u_{n_k} \longrightarrow -\infty$ uniformly on compact subsets of Ω,

(3) there exists a finite non-empty blow-up set $S = \{a_1, \dots, a_m\} \subset \Omega$ such that, for any $i = 1, \dots, m$, there exists $\{x_{n_k}\} \subset \Omega$ satisfying $x_{n_k} \longrightarrow a_i$, $u_{n_k}(x_{n_k}) \longrightarrow \infty$, and $u_{n_k}(x) \longrightarrow -\infty$ uniformly on compact subsets of $\Omega\backslash S$. Moreover, $\rho\exp(u_{n_k}) \longrightarrow \sum_{i=1}^m 8\pi m_i \delta_{a_i}$ weakly in the sense of measures on Ω, where m_i is a positive integer for all $i = 1, \dots, m$.

It should be remarked that, prior to [5] and [18], an analogous result to Facts 2.5 for the asymptotic behavior of the solutions of (G) as $\lambda \longrightarrow 0$ exists [25], which we mentioned in Section 1.

Proof of Step 3. Suppose $x_\infty \in \Omega\backslash\{0\}$. Then, we are able to take $R > 0$ such that $\Omega_{\varepsilon_n} \supset B_R(x_\infty)$ for sufficient large n. Let $w_n(x) = u_n(x) - \log \int_{\Omega_{\varepsilon_n}} u_n(x)dx$. Then this $\{w_n(x)\}$ satisfies the assumptions of Fact 2.5 on the bounded domain $B_R(x_\infty)$. Since $\rho\exp(w_n) = \rho\mu_n \longrightarrow \rho\delta_{x_\infty}$, only the alternative (3) is able to occur with $S = \{x_\infty\}$ and ρ must be $8\pi m$ for some positive integer m. Nevertheless, $\rho \in (8\pi, 16\pi)$ from the hypothesis. This is a contradiction and we obtain $x_\infty = 0$. □

3. Estimate of the minimax value

To prove Proposition 2.1, it is enough to construct $h_\varepsilon \in D_\rho^\varepsilon$ such that

$$\sup_{u\in h_\varepsilon(B_1(0))} J_\rho^\varepsilon(u) \longrightarrow -\infty \quad \text{as} \quad \varepsilon \longrightarrow 0. \tag{3.1}$$

Fix $s_0 > 0$ and set

$$u_s(t) = \begin{cases} 4\log\dfrac{s_0}{s} & 0 \le t \le s, \\ 4\log\dfrac{s_0}{t} & s \le t \le s_0, \\ 0 & s_0 \le t. \end{cases}$$

We use $u_s(t)$ to construct h_ε. It is obvious that $u_{s,p}(x) = u_s(|x-p|) \in H_0^1(\Omega_\varepsilon) \subset H_0^1(\Omega)$ if $B_{s_0}(p) \subset \Omega_\varepsilon$. Moreover, we are able to obtain the following estimates:

Proposition 3.1. *Suppose $B_{s_0}(p) \subset \Omega_\varepsilon$. Then we obtain*

$$\int_{\Omega_\varepsilon} |\nabla u_{s,p}|^2 = \int_{B_{s_0}(0)\backslash B_s(0)} |\nabla(u_s(|x|)|^2 = 32\pi\log\frac{s_0}{s}, \tag{3.2}$$

$$\int_{\Omega_\varepsilon} e^{u_{s,p}} \ge \int_{B_{s_0}(0)\backslash B_s(0)} e^{u_s(|x|)} = \frac{1}{s^2}\pi s_0^4\left[1-\left(\frac{s}{s_0}\right)^2\right] \tag{3.3}$$

for every $0 < s < s_0$. Especially, we have

$$\frac{e^{u_{s,p}}}{\int_\Omega e^{u_{s,p}}} \longrightarrow \delta_p \quad \textit{weakly} * \quad \textit{in} \quad M(\bar{\Omega}) \quad \textit{as} \quad s \longrightarrow 0, \tag{3.4}$$

$$J_\rho^\varepsilon(u_{s,p}) \le -2(\rho-8\pi)\log\frac{1}{s} + O(1) \longrightarrow -\infty \quad \textit{as} \quad s \longrightarrow 0, \tag{3.5}$$

where $O(1)$ is independent of ε and p.

Since we obtain Proposition 3.1 by elementary calculations, we omit the proof.

We are able to take positive numbers R and $s_0 \le R$ such that $B_{4R}(0)\backslash B_{2R}(0) \subset \Omega_\varepsilon$ for sufficiently small ε and $B_{4R+s_0}(0) \subset \Omega$. Take $s = s(\varepsilon) \longrightarrow 0$ as $\varepsilon \longrightarrow 0$ such that $\varepsilon \le s \le s_0$, which we specify later. We define

$$h_\varepsilon^0(r,\theta)(x) := \begin{cases} u_{s,p(4Rr,\theta)}(x) & 0 \le r \le \dfrac{1}{2}, \\ u_{2(1-r)s,p(4Rr,\theta)}(x) & \dfrac{1}{2} \le r < 1, \end{cases}$$

where $p(r,\theta) = (r\cos\theta, r\sin\theta) \in \mathbf{R}^2$. From (3.4) and (3.5), it is easy to see that $h_\varepsilon^0(r,\theta)(\cdot)$ satisfies (1.1) and (1.2), though $h_\varepsilon^0(r,\theta)(\cdot) \notin H_0^1(\Omega_\varepsilon)$ if r is small, that is, $h_\varepsilon^0(\cdot) \notin D_\rho^\varepsilon$ yet.

We introduce the following logarithmic cut-off function, which is also used in [11]:

$$\eta_\varepsilon(t) := \begin{cases} 0, & 0 \le t \le \varepsilon \\ -\dfrac{2\log(t/\varepsilon)}{\log\varepsilon}, & \varepsilon \le t \le \sqrt{\varepsilon}, \\ 1, & \sqrt{\varepsilon} \le t. \end{cases}$$

Let

$$h_\varepsilon(r,\theta)(x) := \eta_\varepsilon(|x|)h_\varepsilon^0(r,\theta)(x).$$

This h_ε obviously belongs to D_ρ^ε and we are able to prove the following fact:

Proposition 3.2. *For every $\delta > 0$, if we take sufficiently small positive number $\sigma < 1/2$ and set $s = \varepsilon^\sigma (\ge \sqrt{\varepsilon} \ge \varepsilon)$, we obtain*

$$\sup_{(r,\theta)\in B_1(0)} J_\rho^\varepsilon(h_\varepsilon(r,\theta)(x)) \le -2\sigma\{\rho - (1+\delta)8\pi\}\log\frac{1}{\varepsilon} + O(1) \quad \textit{as} \quad \varepsilon \longrightarrow 0.$$

Proof. We note that $h_\varepsilon(r,\theta)(x) \equiv h_\varepsilon^0(r,\theta)(x)$ if $1/2 \le r < 1$. From (3.5), we obtain that

$$\begin{aligned} J_\rho^\varepsilon(h_\varepsilon(r,\theta)) &= J_\rho^\varepsilon(u_{2(1-r)s,p(4Rr,\theta)}) \le -2(\rho-8\pi)\log\frac{1}{2(1-r)s} + O(1) \\ &\le -2(\rho-8\pi)\log\frac{1}{s} + O(1) \quad \text{as} \quad s \longrightarrow 0 \quad \text{if} \quad \frac{1}{2} \le r < 1. \end{aligned} \tag{3.6}$$

For every $r \le 1/2$ and every $\delta > 0$, we obtain

$$\begin{aligned} &\int_{\Omega_\varepsilon} |\nabla h_\varepsilon(r,\theta)|^2 \\ &\le \left(1+\frac{\delta}{2}\right)\int_{\Omega_\varepsilon} |\nabla h_\varepsilon^0(r,\theta)|^2 + C(\delta)\left(\sup_{x\in\Omega_\varepsilon}|h_\varepsilon^0(r,\theta)(x)|\right)^2 \int_{\Omega_\varepsilon} |\nabla(\eta_\varepsilon(|x|))|^2, \end{aligned} \tag{3.7}$$

where $C(\delta)$ is a constant depending only on δ. We note that $h^0_\varepsilon(r,\theta)(x)$ is a translation of $u_s(|x|)$ if $0 \le r \le 1/2$ and $\operatorname{supp} h^0_\varepsilon(1/2,\theta) = B_{s_0}(p(2R,\theta)) \subset \Omega_\varepsilon$. Thus we obtain from (3.2) that

$$\begin{aligned}\int_{\Omega_\varepsilon} |\nabla h^0_\varepsilon(r,\theta)|^2 &\le \int_{\Omega_\varepsilon} \left|(\nabla h^0_\varepsilon)\left(\frac{1}{2},\theta\right)\right|^2 \\ &= \int_{\Omega_\varepsilon} |\nabla u_{s,p(2R,\theta)}|^2 = 32\pi \log\frac{1}{s} + O(1) \quad \text{as} \quad s \longrightarrow 0.\end{aligned} \tag{3.8}$$

It is easy to see that

$$\sup_{x\in\Omega_\varepsilon} |h^0_\varepsilon(r,\theta)(x)| = \sup_t |u_s(t)| = 4\log\frac{s_0}{s} \tag{3.9}$$

and

$$\int_{\Omega_\varepsilon} |\nabla(\eta_\varepsilon(|x|))|^2 = \frac{4\pi}{\log(1/\varepsilon)}. \tag{3.10}$$

Combining (3.7–10) and choosing $s = \varepsilon^\sigma$ for sufficiently small $\sigma \in (0, 1/2)$, we obtain

$$\begin{aligned}\int_{\Omega_\varepsilon} |\nabla h_\varepsilon(r,\theta)|^2 &\le 32\pi\left(1+\frac{\delta}{2}\right)\log\frac{1}{s} + C(\delta)'\frac{\log s}{\log\varepsilon}\log\frac{1}{s} + O(1), \\ &\le 32\pi\sigma(1+\delta)\log\frac{1}{\varepsilon} + O(1) \quad \text{as} \quad \varepsilon \longrightarrow 0,\end{aligned} \tag{3.11}$$

where $C(\delta)'$ is a constant independent of ε.

On the other hand, we obtain from (3.3) that

$$\begin{aligned}\int_{\Omega_\varepsilon} e^{h_\varepsilon(r,\theta)} &\ge \int_{\Omega_\varepsilon} e^{h_\varepsilon(0,\theta)} \ge \int_{B_{s_0}(0)\setminus B_s(0)} e^{u_s(|x|)} \\ &\ge \frac{1}{s^2}\pi s_0^4\left[1-\left(\frac{s}{s_0}\right)^2\right] = \frac{1}{\varepsilon^{2\sigma}}\pi s_0^4\left[1-\left(\frac{\varepsilon^\sigma}{s_0}\right)^2\right].\end{aligned} \tag{3.12}$$

Combining (3.11–12), we obtain

$$J^\varepsilon_\rho(h_\varepsilon(r,\theta)) \le -2\sigma\{\rho-(1+\delta)8\pi\}\log\frac{1}{\varepsilon} + O(1) \quad \text{as} \quad \varepsilon \longrightarrow 0 \quad \text{if} \quad 0 \le r \le \frac{1}{2}. \tag{3.13}$$

Thus we obtain the conclusion from (3.6) and (3.13). □

Proof of Proposition 2.1. As we assumed that $\rho > 8\pi$, we are able to take a sufficiently small $\delta > 0$ such that $\rho-(1+\delta)8\pi > 0$. Then h_ε satisfies required property (3.1). □

Appendix. Proof of Lemma 2.3

It is enough to see that, for every sufficiently small $r > 0$, there exists $x_{r,n} \in \overline{\Omega}$ such that

$$\int_{\Omega \cap B_r(x_{r,n})} \mu_n \geq 1 - r \tag{A.1}$$

if n is sufficiently large.

(2.1) is equivalent to the inequality

$$\frac{\rho - 16\pi + (\varepsilon/2)}{32\pi - \varepsilon} \int_\Omega |\nabla u|^2 + J_\rho(u) \geq -\rho \log c - \rho c. \tag{A.2}$$

Since we assumed that $\rho < 16\pi$, we are able to take a sufficiently small ε such that $\rho - 16\pi + (\varepsilon/2) < 0$. Then (A.2) with this ε does not hold for u_n with sufficiently large n because $J_\rho(u_n) \longrightarrow -\infty$. Accordingly, for every $\delta_0 > 0$, every two subsets S_1 and S_2 of $\overline{\Omega}$ satisfying $\mathrm{dist}(S_1, S_2) \geq \delta_0 > 0$, and every $\gamma_0 \in (0, 1/2)$, we obtain

$$\min\left(\frac{\int_{S_1} e^{u_n}}{\int_\Omega e^{u_n}}, \frac{\int_{S_2} e^{u_n}}{\int_\Omega e^{u_n}} \right) = \min\left(\int_{S_1} \mu_n, \int_{S_2} \mu_n \right) < \gamma_0 \tag{A.3}$$

if n is sufficiently large.

Let $Q_n(r)$ be the concentration function of μ_n, that is,

$$Q_n(r) = \sup_{x \in \Omega} \int_{\Omega \cap B_r(x)} \mu_n.$$

For every $r > 0$, take $x_{r,n} \in \overline{\Omega}$ such that $\int_{\Omega \cap B_{r/2}(x_{r,n})} \mu_n = Q_n(r/2)$. Applying (A.3) for $\delta_0 = r/2$, $S_1 = \Omega \cap B_{r/2}(x_{r,n})$, and $S_2 = \Omega \backslash B_r(x_{r,n})$, we obtain that, for every $\gamma_0 \in (0, 1/2)$,

$$\min\left(\int_{S_1} \mu_n, \int_{S_2} \mu_n \right) = \min\left(Q_n\left(\frac{r}{2}\right), 1 - \int_{\Omega \cap B_r(x_{r,n})} \mu_n \right) < \gamma_0 \tag{A.4}$$

if n is sufficiently large.

Since $\int_\Omega \mu_n \equiv 1$, it is easy to see that there exists a constant C independent of n such that

$$Q_n(r) \geq Cr^2 \quad \text{for every} \quad 0 < r \leq \mathrm{diam}(\Omega).$$

Taking sufficiently small γ_0 such that $Q_n(r/2) \geq \gamma_0 > 0$, we obtain (A.1) from (A.4). □

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