AN APPLICATION OF THE PATH-SPACE TECHNIQUE TO THE THEORY OF TRIADS

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One of the most powerful tools in homotopy theory is the homotopy groups of a triad introduced by Blakers and Massey in [1]. Our aim here is to develop systematically the formal, elementary aspects of the theory of a generalized triad and the mapping track associated with it. This will be used in §5 to deduce a result (Theorem 5.5) which seems to be closely related to an exact sequence established by Brown [2].

There is an application of our theorem to the realization problem of Whitehead products. In this direction we obtain the following result: given $\theta \in H^{n'+1}(\pi, n ; \pi')$ and a pairing $W : \pi' \otimes \pi \to G$ such that the cup-product $\theta \cup \iota$ relative to W lies in the image of $\theta^* : H^{n+n'+1}(\pi', n'+1 ; G) \to H^{n+n'+1}(\pi, n ; G)$, there exists a space whose first invariant is θ and whose Whitehead product pairing is just W, where $\iota \in H^n(\pi, n ; \pi)$ is the basic class.

It will be assumed that all spaces and mappings occurring in this paper are taken from the category with base-points, and the notations introduced in [12] will be used without specific reference.

§1. The mapping track of a triad

In this paper we shall understand by a *triad* $(f : g)$ a pair of maps $A \xrightarrow{f} Y \xleftarrow{g} B$. For such a triad the following construction is basic:

$$E_{f,g} = \{(a, b, \beta) \in A \times B \times Y^I \mid f(a) = \beta(0),\ g(b) = \beta(1)\},$$
$$\mathrm{Ker}\ (f : g) = \{(a, b) \in A \times B \mid f(a) = g(b)\}.$$

These constructions give rise to the following diagrams:

(1.1)
$$\begin{tikzcd} & A \arrow[dr, "f"] & \\ \mathrm{Ker}\ (f : g) \arrow[ur, "\pi_1"] \arrow[dr, "\pi_2"'] & & Y \\ & B \arrow[ur, "g"'] & \end{tikzcd}$$

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(1.2)
$$\begin{tikzcd}
 & & \Omega A \arrow[dr, "\Omega f"] & & & A \arrow[dr, "f"] & \\
\cdots \arrow[r] & \Omega E_{f,g} \arrow[ur, "\Omega P_1"] \arrow[dr, "\Omega P_2"'] & & \Omega Y \arrow[r, "I"] & E_{f,g} \arrow[ur, "P_1"] \arrow[dr, "P_2"'] & & Y \\
 & & \Omega B \arrow[ur, "\Omega g"'] & & & B \arrow[ur, "g"'] &
\end{tikzcd}$$

where Ω is the loop functor; the maps are defined by setting $\pi_1(a, b) = a$, $\pi_2(a, b) = b$, $P_1(a, b, \beta) = a$, $P_2(a, b, \beta) = b$, $I(\beta) = (a_0, b_0, \beta)$, and Y^I denotes the space of paths $I = [0, 1] \to Y$ with CO-topology. We note that (1.1) is commutative and (1.2) is homotopy-commutative.

We shall call $E_{f,g}$ the *mapping track* of a triad $(f : g)$. In case f and g are inclusions this has been considered by Hu [5]. Various specializations of $(f : g)$ yield various spaces. For example, we have

$$\begin{aligned}
&EY = \{\beta \in Y^I \mid \beta(0) = y_0\}, && \text{for } y_0 \longrightarrow Y \xleftarrow{1} Y,\\
&E_f = \{(x, \beta) \in X \times EY \mid f(x) = \beta(1)\}, && \text{for } y_0 \longrightarrow Y \xleftarrow{f} X,\\
&Z_f = \{(x, \beta) \in X \times Y^I \mid f(x) = \beta(1)\}, && \text{for } Y \xrightarrow{1} Y \xleftarrow{f} X,\\
&E_f^- = \{(x, \beta) \in X \times Y^I \mid f(x) = \beta(0),\ \beta(1) = y_0\}, && \text{for } X \xrightarrow{f} Y \longleftarrow y_0,\\
&E^- Y = \{\beta \in Y^I \mid \beta(1) = y_0\}, && \text{for } Y \xrightarrow{1} Y \longleftarrow y_0,
\end{aligned}$$

We have furthermore that Ker $(f : g) = A \cap B$ for inclusions $A \xrightarrow{f} Y$, $B \xrightarrow{g} Y$ and, when g is a fibering, Ker $(f : g)$ is the fibering *induced* by f from g.

PROPOSITION 1.3. *$(a, b, \beta) \to (b, a, \beta^{-1})$ yields a homeomorphism $E_{f,g} \to E_{g,f}$.*

THEOREM 1.4. *If g is a fibering then $E_{f,g}$ is homotopically equivalent to the induced fibre space* Ker $(f : g)$.

Proof. Let $\Lambda : Z_g \to B^I (\lambda : Z_g \to B)$ be, respectively, a (path) lifting function for g (see [12], p. 113). Define Φ : Ker $(f : g) \to E_{f,g}$ and $\Psi : E_{f,g} \to$ Ker $(f : g)$ as follows:

$$\Phi(a, b) = (a, b, e_y),\ y = f(a) = g(b), \tag{1.5}$$

$$\Psi(a, b, \beta) = (a, \lambda(b, \beta)), \tag{1.6}$$

where e_y is the constant path at y. Since there exists a homotopy between $1_B : B \to B$ and the map $b \to \lambda(b, e_y)$ which moves points along fibres, it follows that $\Psi\Phi \simeq 1$. $\Phi\Psi \simeq 1$ is shown by considering a homotopy given by

$$(a,\ b,\ \beta)\to(a,\ \Lambda(b,\ \beta)(t),\ \beta_{0,t}),\qquad 0\leqq t\leqq 1.$$

THEOREM 1.7. *The sequence*

$$\begin{tikzcd}[column sep=small]
 & \pi(V,\ \Omega A) \arrow[dr, "(\Omega f)_*"] & & & \pi(V,\ A) \arrow[dr, "f_*"] & \\
{} \arrow[ur] \arrow[dr] & & \pi(V,\ \Omega Y) \arrow[r, "I_*"] & \pi(V,\ E_{f,g}) \arrow[ur, "P_{1*}"] \arrow[dr, "P_{2*}"'] & & \pi(V,\ Y) \\
 & \pi(V,\ \Omega B) \arrow[ur, "(\Omega g)_*"'] & & & \pi(V,\ B) \arrow[ur, "g_*"'] &
\end{tikzcd}$$

is exact for any space V in the following sense (cf. Olum [13]):

(i) *$a\in\pi(V,\ A)$ and $b\in\pi(V,\ B)$ have the same image in $\pi(V,\ Y)$ if, and only if, there exists $c\in\pi(V,\ E_{f,g})$ such that $a=P_{1*}(c)$ and $b=P_{2*}(c)$;*

(ii) $\operatorname{Ker} P_{1*}\cap\operatorname{Ker} P_{2*}=\operatorname{Im} I_*$;

(iii) *$d_1,\ d_2\in\pi(V,\ \Omega Y)$ satisfy $I_*(d_1)=I_*(d_2)$ if, and only if, there exist $a\in\pi(V,\ \Omega A)$ and $b\in\pi(V,\ \Omega B)$ such that $(\Omega f)_*(a)\cdot d_2=d_1\cdot(\Omega g)_*(b)$, where the dots denote the group operation in $\pi(V,\ \Omega Y)$ determined by the loop-multiplication.*

Proof. (i) Let $h_1,\ h_2$ represent $a,\ b$ respectively. If $f\circ h_1\simeq g\circ h_2$ we can find a homotopy H_t, $0\leqq t\leqq 1$, such that $H_0=f\circ h_1$, $H=g\circ h_2$: then it suffices to define a representative $k\colon\ V\to E_{f,g}$ for c as follows:

$$k(v)=(h_1(v),\ h_2(v),\ \beta(v)),\qquad v\in V,$$

where $\beta(v)$ is the path in Y given by $\beta(v)(t)=H_t(v)$, $0\leqq t\leqq 1$.

(ii) Let $k\colon\ V\to E_{f,g}$ be expressed as $k(v)=(h_1(v),\ h_2(v),\ \gamma(v))$, $v\in V$, and let $h_1\simeq 0$, $h_2\simeq 0$. We denote by $\alpha(v)$ and $\beta(v)$ the elements of EA and EB determined by the contractions of $h_1(v)$ and $h_2(v)$. Then it is easy to see that $k(v)\simeq I\{f\alpha(v)\cdot\gamma(v)\cdot g\beta(v)^{-1}\}$.

(iii) Let $\bar d_1,\ \bar d_2\colon\ V\to\Omega Y$ represent $d_1,\ d_2$ respectively, and let $H_t:\ V\to E_{f,g}$ be such that $H_0=I\bar d_1$ and $H_1=I\bar d_2$. Then we have only to take for $a.\ b$ the elements represented by P_1H_t and P_2H_t, $0\leqq t\leqq 1$.

THEOREM 1.8. *If g is a fibering, then* (1.1) *induces an exact diagram:*

$$\begin{tikzcd}[column sep=small]
 & & & \pi(V,\ A) \arrow[dr, "f_*"] & \\
{} \arrow[r] & \pi(V,\ \Omega Y) \arrow[r, "J_*"] & \pi(V,\ \operatorname{Ker}(f:\ g)) \arrow[ur, "\pi_{1*}"] \arrow[dr, "\pi_{2*}"'] & & \pi(V,\ Y), \\
 & & & \pi(V,\ B) \arrow[ur, "g_*"'] &
\end{tikzcd}$$

where $J=\Psi\circ I$, Ψ: $E_{f,g}\rightarrow \mathrm{Ker}\ (f:g)$ *being an equivalence in the proof of Theorem* 1.4.

Proof. This follows from Theorems 1.4 and 1.7, since $P_1=\pi_1\Psi$ and $P_2\simeq\pi_2\Psi$.

PROPOSITION 1.9. P_1: $E_{f,g}\rightarrow A$, P_2: $E_{f,g}\rightarrow B$ *and* $P_1\times P_2$: $E_{f,g}\rightarrow A\times B$ *are fiberings with fibres* E_g, $E_{\bar f}$ *and* ΩY *respectively.*

Proof. A path lifting function Λ for P_1 is defined by setting

$$\Lambda(a,\ b,\ \beta,\ \alpha)(s)=(\alpha(s),\ b,\ \beta_s),$$

for $0\leqq s\leqq 1$, $\alpha\in A^I$, $\alpha(1)=a$, in which β_s is a path in Y given by

$$\beta_s(t)=\begin{cases} f\alpha(2t+s), & 0\leqq t\leqq \dfrac{1-s}{2}, \\ \beta\left(\dfrac{2t+s-1}{1+s}\right), & \dfrac{1-s}{2}\leqq t\leqq 1. \end{cases}$$

Similarly for P_2 and $P_1\times P_2$.

§ 2. Transformation between triads

Let the following diagram be given:

(2.1)
$$\begin{array}{ccccc} A & \xrightarrow{f} & Y & \xleftarrow{g} & B \\ {\scriptstyle\psi_1}\downarrow & & {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\psi_2} \\ A' & \xrightarrow[f']{} & Y' & \xleftarrow[g']{} & B' \end{array}$$

If (2.1) is homotopy-commutative, then we say that (2.1) is a *transformation* from a triad $(f:g)$ to a triad $(f':g')$. We call it a *map* if it is strictly commutative.

Let now G_t, H_t, $0\leqq t\leqq 1$, be fixed homotopies such that $G_0=f'\psi_1$, $G_1=\varphi f$, $H_0=g'\psi_2$, $H_1=\varphi g$. We define $\chi=E(\psi_1,\ \varphi,\ \psi_2\ ;\ G,\ H)$: $E_{f,g}\rightarrow E_{f',g'}$ by setting

(2.2)
$$\chi(a,\ b,\ \beta)=(\psi_1(a),\ \psi_2(b),\ \beta')$$

where β' is the path in Y' given by

$$\beta'(s)=\begin{cases} G_{3s}(a), & 0\leqq s\leqq \dfrac{1}{3}, \\ \varphi\beta(3s-1), & \dfrac{1}{3}\leqq s\leqq \dfrac{2}{3}. \\ H_{3-3s}(b), & \dfrac{2}{3}\leqq s\leqq 1. \end{cases}$$

For a map (2.1) we shall set $\beta' = \varphi\beta$ in (2.2), and denote simply by $E(\psi_1, \varphi, \psi_2)$.

Further let

$$\begin{array}{ccccc} A' & \xrightarrow{f'} & Y' & \xleftarrow{g'} & B' \\ \psi_1' \downarrow & & \varphi' \downarrow & & \downarrow \psi_2' \\ A'' & \xrightarrow[f'']{} & Y'' & \xleftarrow[g'']{} & B'' \end{array}$$

be another transformation with homotopies G_t', H_t' such that $G_0' = f''\psi_1'$, $G_1' = \varphi' f'$, $H_0' = g''\psi_2'$, $H_1' = \varphi' g'$. Consider the homotopies $(G' \circ G)$, $(H' \circ H)$ which are given by

$$(G' \circ G)_t(a) = \begin{cases} G_{2t}'\psi_1(a), & 0 \leqq t \leqq \frac{1}{2}, \\ \varphi' G_{2t-1}(a), & \frac{1}{2} \leqq t \leqq 1, \end{cases}$$

$$(H' \circ H)_t(b) = \begin{cases} H_{2t}'\psi_2(b), & 0 \leqq t \leqq \frac{1}{2}, \\ \varphi' H_{2t-1}(b), & \frac{1}{2} \leqq t \leqq 1, \end{cases}$$

for $a \in A$, $b \in B$. Then it is immediate to verify

PROPOSITION 2.3. *$E(\psi_1'\psi_2, \varphi'\varphi, \psi_2'\psi_2;\ G' \circ G, H' \circ H)$ is homotopic to $E(\psi_1', \varphi', \psi_2';\ G', H') \circ E(\psi_1, \varphi, \psi_2;\ G, H)$.*

PROPOSITION 2.4. *Let (2.1) be given and let $\varphi \simeq \bar\varphi$, $\psi_1 \simeq \bar\psi_1$, $\psi_2 \simeq \bar\psi_2$. Then there exist homotopies $\bar G : f'\bar\psi_1 \simeq \bar\varphi f$ and $\bar H : g'\bar\psi_2 \simeq \bar\varphi g$ such that $E(\bar\psi_1, \bar\varphi, \bar\psi_2;\ \bar G, \bar H) \simeq E(\psi_1, \varphi, \psi_2;\ G, H)$.*

Proof. Let $\varphi^\tau : \varphi \simeq \bar\varphi$, $\psi_1^\tau : \psi_1 \simeq \bar\psi_1$, $\psi_2^\tau : \psi_2 \simeq \bar\psi_2$. Define $G_t^\tau : A \to Y'$ by

$$G_t^\tau = \begin{cases} f' \circ \psi_1^{\tau - 3t}, & 0 \leqq t \leqq \frac{\tau}{3}, \\ G_{(3t-\tau)(3-2\tau)^{-1}}, & \frac{\tau}{3} \leqq t \leqq 1 - \frac{\tau}{3}, \\ \varphi^{3t+\tau-3} \circ f, & 1 - \frac{\tau}{3} \leqq t \leqq 1, \end{cases}$$

and define H_t similarly. Then $E(\psi_1^\tau, \varphi^\tau, \psi_2^\tau;\ G_t^\tau, H_t^\tau)$ gives the desired homotopy.

PROPOSITION 2.5. *$E(1_A, 1_Y, 1_B;\ G, H)$ is a homotopy equivalence.*

Proof. Let G^-, H^- be defined by $G_t^- = G_{1-t}$, $H_t^- = H_{1-t}$, $0 \leqq t \leqq 1$. By

Proposition 2.3 we have $E(1_A, 1_Y, 1_B; G^-, H^-) \circ E(1_A, 1_Y, 1_B; G, H) \simeq E(1_A, 1_Y, 1_B; G^- \circ G, H^- \circ H)$. If G_t^τ, H_t^τ, $0 \leqq \tau \leqq 1$, are defined by

$$G_t^\tau = \begin{cases} G_{1-2\tau t}, & 0 \leqq t \leqq \frac{1}{2}, \\ G_{1-2\tau(1-t)}, & \frac{1}{2} \leqq t \leqq 1, \end{cases}$$

and similarly for H_t, then we have

$$E(1, 1, 1; G^- \circ G, H^- \circ H) \simeq E(1, 1, 1; f, g)$$

by the homotopy $E(1, 1, 1; G_t^\tau, H_t^\tau)$. Since $E(1, 1, 1; f, g)$ is homotopic to the identity map of $E_{f,g}$, it follows that $E(1, 1, 1; G^-, H^-)$ is a left homotopy inverse of $E(1, 1, 1; G, H)$. We see similarly that $E(1, 1, 1; G^-, H^-)$ is a right homotopy inverse, and this completes the proof.

As an immediate consequence of the above three propositions we have

Theorem 2.6. *Let a transformation* (2.1) *be given, and suppose that vertical maps are homotopy equivalences. Then* $E(\psi_1, \varphi, \psi_2; G, H)$ *is also an equivalence, that is,* $E_{f,g}$ *is an invariant object under homotopy equivalences.*

Now we see that a transformation (2.1) gives rise to a new map:

$$(2.7)\qquad \begin{array}{ccccc} E_{\psi_1} & \xrightarrow{\chi_1} & E_\varphi & \xleftarrow{\chi_2} & E_{\psi_2} \\ {\scriptstyle P\psi_1}\downarrow & & \downarrow{\scriptstyle P\varphi} & & \downarrow{\scriptstyle P\psi_2} \\ A & \xrightarrow[f]{} & Y & \xleftarrow[g]{} & B \end{array}$$

where $\chi_1 = E(f', f; G)$ and $\chi_2 = E(g', g; H)$. From (2.1) and (2.7) we obtain a sequence

$$(2.8)\qquad E_{\chi_1, \chi_2} \xrightarrow{E(P\psi_1,\ P\phi,\ P\psi_2)} E_{f,g} \xrightarrow{E(\psi_1,\ \varphi,\ \psi_2;\ G,\ H)} E_{f',g'}$$

We shall prove

Proposition 2.9. (2.8) *induces, for any space* V, *an exact sequence*

$$\pi(V, E_{\chi_1, \chi_2}) \longrightarrow \pi(V, E_{f,g}) \xrightarrow{\chi_*} \pi(V, E_{f',g'}).$$

Proof. First we shows that $\chi \circ E(P\psi_1, P\varphi, P\psi_2)$ is nullhomotopic. Take any point of E_{χ_1, χ_2}, that is $(a, \alpha', b, \beta', \gamma, \tilde{\gamma}) = x \in A \times EA' \times B \times EB' \times Y^I \times (EY')^I$ such that

$\psi_1(a) = \alpha'(1)$, $\psi_2(b) = \beta'(1)$, $\varphi\gamma(s) = \tilde{\gamma}(1, s)$, $0 \leqq s \leqq 1$,
$f(a) = \gamma(0)$, $g(b) = \gamma(1)$, $\tilde{\gamma}(0, t) = y_0'$,

$$\tilde{\gamma}(s, 0) = \begin{cases} f'\alpha'(2s), & 0 \leqq s \leqq \frac{1}{2}, \\ G_{2s-1}(a), & \frac{1}{2} \leqq s \leqq 1, \end{cases}$$

$$\tilde{\gamma}(s, 1) = \begin{cases} g'\beta'(2s), & 0 \leqq s \leqq \frac{1}{2}, \\ H_{2s-1}(b), & \frac{1}{2} \leqq s \leqq 1. \end{cases}$$

Therefore

$$\chi \circ E(P\psi_1, P\varphi, P\psi_2)(a, \alpha', b, \beta', \gamma, \tilde{\gamma}) = (\alpha'(1), \beta'(1), \delta),$$

where δ is the path in Y' given by

$$\delta(t) = \begin{cases} G_{3t}(a), & 0 \leqq t \leqq \frac{1}{3}, \\ \varphi\gamma(3t-1), & \frac{1}{3} \leqq t \leqq \frac{2}{3}, \\ H_{3-3t}(b), & \frac{2}{3} \leqq t \leqq 1. \end{cases}$$

Let $\rho : I \times I \to I \times I$ be a homeomorphism such that $\rho(0 \times I) = 0 \times I$, $\rho(I \times i) = \left[0, \frac{1}{2}\right] \times i$, $i = 0$ or 1, $\rho\left(1 \times \left[0, \frac{1}{3}\right]\right) = \left[\frac{1}{2}, 1\right] \times 0$, $\rho\left(1 \times \left[\frac{2}{3}, 1\right]\right) = \left[\frac{1}{2}, 1\right] \times 1$, $\rho\left(1 \times \left[\frac{1}{3}, \frac{2}{3}\right]\right) = 1 \times I$ and ρ is linear on the indicated segments. Then it is clear that $(x, \tau) \to (\alpha'(\tau), \beta'(\tau), \tilde{\gamma}\rho | \tau \times I)$ is a homotopy deforming $x \to (\alpha'(1), \beta'(1), \delta)$ into the constant map.

Conversely, let $k : V \to E_{f,g}$ be expressed by

$$k(v) = (a(v), b(v), \gamma(v)), \qquad v \in V$$

and let $(A_t(v), B_t(v), C_t(v)) : 0 \simeq \chi \circ k(v)$. We denote by $\alpha'(v)$ and $\beta'(v)$ the paths determined by $A_t(v)$, $B_t(v)$ respectively, and we define $\tilde{\gamma}(v) : I \times I \to Y'$ by $\tilde{\gamma}(v)(t, s) = C_{t'}(v)(s')$, where $(t', s') = \rho^{-1}(t, s)$. It is obvious that $h : V \to E_{\chi_1, \chi_2}$, given by

$$h(v) = (a(v), \alpha'(v), b(v), \beta'(v), \gamma(v), \tilde{\gamma}(v)),$$

satisfies $k = E(P\psi_1, P\varphi, P\psi_2) \circ h$. Thus the proof is complete.

Applying the above proposition and Theorem 2.6, and noting that $E_{\Omega f, \Omega g}$ is homeomorphic to $\Omega E_{f,g}$, we reach the final result.

THEOREM 2.10. *Every transformation* (2.1) *induces, for any space V, an exact sequence*

$$\cdots \xrightarrow{(\Omega\chi)_*} \pi(V, \Omega E_{f',g'}) \longrightarrow \pi(V, E_{\chi_1,\chi_2}) \longrightarrow \pi(V, E_{f,g}) \xrightarrow{\chi_*} \pi(V, E_{f',g'}).$$

COROLLARY 2.11. *Let $A \xrightarrow{f} Y \xleftarrow{g} B$ be a triad and suppose there exists a map $h : A \to B$ such that $g \circ h = f$. Then the sequence*

$$\cdots \to \pi(V, \Omega E_g) \to \pi(V, E_h) \to \pi(V, E_f) \to \pi(V, E_g)$$

is exact.

Proof. Apply Theorem 2.10 to the map

$$\begin{array}{ccccc} y_0 & \longrightarrow & Y & \xleftarrow{f} & A \\ \downarrow & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle h} \\ y_0 & \longrightarrow & Y & \xleftarrow{g} & B \end{array}$$

Finally, we prove

LEMMA 2.12. *For an arbitrary triad $A \xrightarrow{f} Y \xleftarrow{g} B$, there exists a homotopically equivalent triad $A \xrightarrow{j_1} M \xleftarrow{j_2} B$ such that j_1 and j_2 are both inclusions and cofiberings.*

Proof. It suffices to take for M the mapping cylinder of $f \vee g : A \vee B \to Y$, j_1 and j_2 being natural inclusions.

§ 3. Some exact sequences

In this section we extend exact sequences established by Massey [9] and Hu [5]. Given a triad $A \xrightarrow{f} Y \xleftarrow{g} B$, let

$$T_{f,g} = \{(\alpha, \beta, \gamma) \in EA \times EB \times EEY \mid \gamma(s, 1) = f\alpha(s),\ \gamma(1, t) = g\beta(t)\},$$
$$S_{f,g} = \{(\alpha, \beta) \in EA \times EB \mid f\alpha(1) = g\beta(1)\}.$$

We observe that $S_{f,g}$ is just E_i for the natural inclusion $i = \pi_1 \times \pi_2 : \mathrm{Ker}\,(f : g) \to A \times B$. Corresponding to these constructions we consider the following maps:

p: $T_{f,g} \to S_{f,g}$ defined by $p(\alpha, \beta, \tilde{\gamma}) = (\alpha, \beta)$,
m: $S_{f,g} \to \Omega Y$ defined by $m(\alpha, \beta) = (f\alpha)\cdot(g\beta)^{-1}$,
n: $\Omega^2 Y \to T_{f,g}$ defined by $n(\tilde{\gamma}) = (e_{a_0}, e_{b_0}, \tilde{\gamma})$,
r_1: $S_{f,g} \to E_{\pi_1}$ defined by $r_1(\alpha, \beta) = (\alpha(1), \beta(1), \alpha)$,
r_2: $S_{f,g} \to E_{\pi_2}$ defined by $r_2(\alpha, \beta) = (\alpha(1), \beta(1), \beta)$.

These maps are obviously imbedded into the following sequences

$$\cdots \xrightarrow{\Omega m} \Omega^2 Y \xrightarrow{n} T_{f,g} \xrightarrow{p} S_{f,g} \xrightarrow{m} \Omega Y, \tag{3.1}$$

$$\cdots \xrightarrow{\Omega r_1} \Omega E_{\pi_1} \xrightarrow{q_2} \Omega B \xrightarrow{u_2} S_{f,g} \xrightarrow{r_1} E_{\pi_1}, \tag{3.2}$$

$$\cdots \xrightarrow{\Omega r_2} \Omega E_{\pi_2} \xrightarrow{q_1} \Omega A \xrightarrow{u_1} S_{f,g} \xrightarrow{r_2} E_{\pi_2},$$

in which $u_2 : \Omega B \to S_{f,g}$ and $q_2 : \Omega E_{\pi_1} \to \Omega B$ are defined by

$u_2(\beta) = (e_{a_0}, \beta)$,
$q_2(\alpha, \beta, \tilde{\alpha}) = \beta^{-1}$ for $\alpha \in \Omega A$, $\beta \in \Omega B$, $\tilde{\alpha} \in \Omega EA$
such that $f\alpha = g\beta$, $\tilde{\alpha}(1, t) = \alpha(t)$,

and u_1 and q_1 are similarly defined.

It is easily seen that E_m is homeomorphic to $T_{f,g}$. Thus we have

PROPOSITION 3.3. *The sequence* (3.1) *induces an exact sequence*

$$\cdots \xrightarrow{n_*} \pi(V, T_{f,g}) \xrightarrow{p_*} \pi(V, S_{f,g}) \xrightarrow{m_*} \pi(V, \Omega Y).$$

Now we consider $l_1 : \Omega B \to E_{r_1}$ and $l_2 : \Omega A \to E_{r_2}$ defined by

$$l_1(\beta) = (e_{a_0}, \beta;\ e_{a_0}, e_{b_0}, \tilde{e}),\ l_2(\alpha) = (\alpha, e_{b_0};\ e_{a_0}, e_{b_0}, \tilde{e})$$

where $\tilde{e} : I \times I \to A$ (or B) is the constant map. Then we prove

LEMMA 3.4. *l_1 and l_2 are homotopy equivalencs.*

Proof. Every point of E_{r_1} is of the form $(\alpha, \beta;\ \alpha', \beta', \tilde{\gamma}) \in EA \times EB \times EA \times EB \times EEA$, where $f\alpha(1) = g\beta(1)$, $\alpha'(1) = \alpha(1)$, $\beta'(1) = \beta(1)$, $\tilde{\gamma}(s, 1) = \alpha(s)$, $\tilde{\gamma}(1, t) = \alpha'(t)$. We define $h_1 : E_{r_1} \to \Omega B$ by

$$h_1(\alpha;\ \beta;\ \alpha', \beta', \tilde{\gamma}) = \beta \cdot (\beta')^{-1}.$$

Clearly $h_1 \circ l_1 \simeq 1$. $l_1 \circ h_1$ is also deformed into the identity map by the following homotopy:

$$(\alpha, \beta;\ \alpha', \beta', \tilde{\gamma}) \to (\alpha_\tau, \beta_\tau;\ \alpha'_{0,\tau}, \beta'_{0,\tau}, \tilde{\gamma}_\tau), \qquad 0 \leqq \tau \leqq 1,$$

where

$$\alpha_\tau(s) = \tilde{\gamma}(s, \tau), \ \tilde{\gamma}_\tau(t, s) = \tilde{\gamma}(t, \tau s),$$

$$\beta_\tau(s) = \begin{cases} \beta\left(\frac{2s}{1+\tau}\right), & 0 \leqq s \leqq \frac{1+\tau}{2}, \\ \beta'(\tau + 2 - 2s), & \frac{1+\tau}{2} \leqq s \leqq 1. \end{cases}$$

By Lemma 3.4 we have

PROPOSITION 3.5. (3.2) *induce exact sequences*

$$\pi(V, \Omega E_{\pi_1}) \xrightarrow{q_{2*}} \pi(V, \Omega B) \xrightarrow{u_{2*}} \pi(V, S_{f,g}) \xrightarrow{r_{1*}} \pi(V, E_{\pi_1})$$

and

$$\pi(V, \Omega E_{\pi_2}) \xrightarrow{q_{1*}} \pi(V, \Omega A) \xrightarrow{u_{1*}} \pi(V, S_{f,g}) \xrightarrow{r_{2*}} \pi(V, E_{\pi_2})$$

The above Propositions 3.3 and 3.5 may be regarded as an extension of the exact sequences established by Massey [9].

We now observe that (1.1) yields maps $\chi_1 = E(f, \pi_2) : E_{\pi_1} \to E_g$ and $\chi_2 = E(g, \pi_1) : E_{\pi_2} \to E_f$, and that E_{χ_1} and E_{χ_2} can be identified with $T_{f,g}$. Thus we conclude

PROPOSITION 3.6. *The sequences*

$$\to \pi(V, \Omega E_g) \to \pi(V, T_{f,g}) \to \pi(V, E_{\pi_1}) \xrightarrow{\chi_{1*}} \pi(V, E_g)$$

and

$$\to \pi(V, \Omega E_f) \to \pi(V, T_{f,g}) \to \pi(V, E_{\pi_2}) \xrightarrow{\chi_{2*}} \pi(V, E_f)$$

are exact.

This is a generalization of exact sequences of a usual triad [1].

Finally we prove

PROPOSION 3.7. *Let* $A \xrightarrow{f} Y \xleftarrow{g} B$ *be a triad in which* g *is a fibering. Then*

(i) $\chi_1 : E_{\pi_1} \to E_g$ *is a homotopy equivalence:*

(ii) $T_{f,g}$ *is contractible;*

(iii) $\chi_2 : E_{\pi_2} \to E_f$ *has a right inverse.*

Proof. χ_1 is given by $\chi_1(a, b, \alpha) = (b, f\alpha)$ for $(a, b, \alpha) \in A \times B \times EA$ with $f(a) = g(b)$, $\alpha(1) = a$. Let $\lambda : Z_g \to B$ denote a lifting function for g. We define

$\Gamma_1 : E_g \to E_{\pi_1}$ by setting $\Gamma_1(b, \gamma) = (a_0, \lambda(b, \gamma), e_{a_0})$ for $\gamma \in EY$, $g(b) = \gamma(1)$. It follows at once that Γ_1 is a homotopy inverse of χ_1, which prove (i). (ii) is an immediate consequence of (i) and Proposition 3.6. To prove (iii), consider $\Gamma_2 : E_f \to E_{\pi_2}$ which is defined by $\Gamma_2(a, \gamma) = (a, \lambda(b_0, \gamma^{-1}), \Lambda(b_0, \gamma^{-1})^{-1})$, where $\Lambda : Z_g \to B^I$ is the path lifting function with which λ is associated. Clearly $\chi_2 \circ \Gamma_2 = 1$, as we wish to prove.

§ 4. Cotriad

In order to dualize the preceding results, we shall call $A \xleftarrow{f} X \xrightarrow{g} B$ a *cotriad* and denote by $\langle f : g \rangle$. Then the argument is quite automatic, but briefly indicated.

With a given cotriad $A \xleftarrow{f} X \xrightarrow{g} B$, we associate the following spaces:

$C_{f,g}$ = the space obtained from $A \cup X \times I \cup B$ by the identifications

$$(x, 0) = f(x),\ (x, 1) = g(x),\ (x_0, s) = (x_0, t),\ x \in X,\ s, t \in I,$$

Coker $\langle f : g \rangle$ = the space obtained from $A \cup B$ by the identifications

$$f(x) = g(x),\ x \in X.$$

In case f and g are inclusions Coker $\langle f : g \rangle$ is the union of A and B, and in case g is a cofibering it is the cofiber space induced by f. Further, $C_{f,g}$, which may be called the mapping cylinder of a co-triad $\langle f : g \rangle$, has already appeared in the book of Eilenberg-Steenrod [4], p. 51, G, 4 for inclusions f and g.

We have now the (homotopy-) commutative diagrams

$$\text{(4.1)}\qquad \begin{tikzcd} & A \arrow[dr, "i_1"] & \\ X \arrow[ur, "f"] \arrow[dr, "g"'] & & \mathrm{Coker}\langle f : g\rangle \\ & B \arrow[ur, "i_2"'] & \end{tikzcd}$$

$$\text{(4.2)}\qquad \begin{tikzcd} & A \arrow[dr, "I_1"] & & & SA \arrow[dr, "SI_1"] & & \\ X \arrow[ur, "f"] \arrow[dr, "g"'] & & C_{f,g} \arrow[r, "Q"] & SX \arrow[ur, "Sf"] \arrow[dr, "Sg"'] & & SC_{f,g} \arrow[r] & {} \\ & B \arrow[ur, "I_2"'] & & & SB \arrow[ur, "SI_2"'] & & \end{tikzcd}$$

where i_1, i_2, I_1 and I_2 are appropriate injections, and Q is the map which pinches $A \cup B$ to a point.

(4.3) (4.2) induces, for any space V, an exact diagram:

$$\begin{tikzcd}
 & \pi(A, V) \arrow[dl, "f^*"'] & & \\
\pi(X, V) & & \pi(C_{f,g}, V) \arrow[ul, "I_1^*"'] \arrow[dl, "I_2^*"] & \pi(SX, V) \arrow[l, "Q^*"'] \\
 & \pi(B, V) \arrow[ul, "g^*"] & &
\end{tikzcd}$$

Let us now suppose that g is a cofibering with an extension function λ': $B \to M_g$. Let Φ' : $C_{f,g} \to \mathrm{Coker}\ \langle f : g \rangle$ and Ψ' : $\mathrm{Coker}\ \langle f : g \rangle \to C_{f,g}$ be the maps defined by

$$\Phi'(a) = a,\ \Phi'(b) = b,\ \Phi'(x, s) = f(x) = g(x), \quad \text{for } a \in A,\ b \in B,\ x \in X,\ 0 \leqq s \leqq 1,$$
$$\Psi'(a) = a,\ \Psi'(b) = \bar{\lambda}'(b) \quad \text{for } a \in A,\ b \in B,$$

where $\bar{\lambda}'$ denotes the composition $B \xrightarrow{\lambda'} M_g \dashrightarrow C_{f,g}$.

(4.4) The above Φ' and Ψ' are mutually inverse homotopy equivalences.

(4.5) The following diagram is exact:

$$\begin{tikzcd}
 & \pi(A, V) \arrow[dl, "f^*"'] & & \\
\pi(X, V) & & \pi(\mathrm{Coker}\ \langle f : g \rangle, V) \arrow[ul, "i_1^*"'] \arrow[dl, "i_2^*"] & \pi(SX, V) \arrow[l, "(Q \circ \Psi')^*"'] \\
 & \pi(B, V) \arrow[ul, "g^*"] & &
\end{tikzcd}$$

We note that this may be considered as a generalization of the Mayer-Vietoris cohomology sequence of a proper triad [4], p. 43.

Let

$$\begin{tikzcd}
A \arrow[d, "\psi_1"'] & X \arrow[l, "f"'] \arrow[r, "g"] \arrow[d, "\varphi"] & B \arrow[d, "\psi_2"] \\
A' & X' \arrow[l, "f'"'] \arrow[r, "g'"] & B'
\end{tikzcd}$$

be homotopy-commutative, and let G_t : $f'\varphi \simeq \psi_1 f$ and H_t : $g'\varphi \simeq \psi_2 g$ be homotopies. We define $\chi' = C(\psi_1, \varphi, \psi_2 : G, H)$: $C_{f,g} \to C_{f',g'}$ by

$$\chi'(a)=\psi_1(a),\ \chi'(b)=\psi_2(b), \qquad a\in A,\ b\in B,$$

$$\chi'(x, s)=\begin{cases} G_{1-3s}(x), & 0\leqq 3s\leqq 1,\\ (\varphi(x),\ 3s-1), & 1\leqq 3s\leqq 2,\\ H_{3s-2}(x), & 2\leqq 3s\leqq 3,\end{cases}$$

(4.6) If ψ_1, φ and ψ_2 are homotopy equivalences, then so is χ'.

Let $A\xleftarrow{f}X\xrightarrow{g}B$ be a cotriad, and let us consider

$T'_{f,g}$ = the space obtained from $CA\cup CCX\cup CB$ by the identifications:

$$(x,\ s,\ 1)=(f(x),\ s),\ (x,\ 1,\ t)=(g(x),\ t),\ x\in X,\ 0\leqq s,\ t\leqq 1,$$

$S'_{f,g}$ = the space obtained from $CA\cup CB$ by the identifications:

$$(f(x),\ 1)=(g(x),\ 1),\ x\in X.$$

Then the following sequences are obviously defined:

(4.7) $\quad SX\xrightarrow{m'}S'_{f,g}\xrightarrow{p'}T'_{f,g}\xrightarrow{n'}S^2X\longrightarrow\cdots$

(4.8) $\quad C_{i_1}\longrightarrow S'_{f,g}\longrightarrow SB\longrightarrow SC_{i_1}\longrightarrow\cdots$ and $C_{i_2}\longrightarrow S'_{f,g}\longrightarrow SA\longrightarrow SC_{i_2}\longrightarrow\cdots$

(4.9) $\quad C_f\longrightarrow C_{i_2}\longrightarrow T'_{f,g}\longrightarrow SC_f\longrightarrow\cdots$ and $C_g\longrightarrow C_{i_1}\longrightarrow T'_{f,g}\longrightarrow SC_g\longrightarrow\cdots$

It is easy to verify

(4.10) The above sequences (4.7)-(4.9) induces exact sequences.

(4.11) Let $A\xleftarrow{f}X\xrightarrow{g}B$ be a cotriad in which g is a cofibering. Then

(i) $C(f,\ i_2)\ :\ C_g\longrightarrow C_{i_1}$ is a homotopy equivalence;

(ii) $T'_{f,g}$ is contractible;

(iii) $C(g,\ i_1)\ :\ C_f\longrightarrow C_{i_2}$ has a left inverse.

This proposition shows that $\pi(T'_{f,g},\ K(\pi,\ n))$ is an analogue of cohomology groups of a triad (cf. [4], p. 204, Theorem 11.3)

§ 5. Cohomology of induced fibrations

Let $A\xrightarrow{f}Y\xleftarrow{g}B$ be a triad in which all spaces are assumed to be path-connected. We now define

$$\mu,\ \Pi_1\ :\ (E_f^-\times E_g,\ \Omega Y\times E_g)\rightarrow(E_{f,g},\ E_g)$$

by setting

$$\mu(a,\ \gamma,\ b,\ \delta)=(a,\ b,\ \gamma\cdot\delta),$$

$$\Pi_1(a,\ \gamma,\ b,\ \delta)=(a,\ b_0,\ \gamma)$$

for $(a, \gamma, b, \delta) \in A \times E^{-}Y \times B \times EY$ with $f(a) = \gamma(0)$, $g(b) = \delta(1)$.

THEOREM 5.1. $(\mu^* - \Pi_1^*) \circ P_1^* : H^q(A, a_0) \to H^q(E_f^- \times E_g, \Omega Y \times E_g)$ *is trivial for all* $q \geqq 0$.

Proof. This is clear, since we have $P_1 \circ \mu(a, \gamma, b, \delta) = a = P_1 \circ \Pi_1(a, \gamma, b, \delta)$.

The goal of this section is to prove

THEOREM 5.2. *Let A be a r-connected space $(r \geqq 2)$ with non-degenerate base-point a_0 and let Y be a t-connected space $(t \geqq 2)$ with non-degenerate base-point y_0. Suppose further that E_g is s-connected, $s \geqq 1$. Then the sequence*

$$H^q(A, a_0) \xrightarrow{P_1^*} H^q(E_{f,g}, E_g) \xrightarrow{\mu^* - \Pi_1^*} H^q(E_f^- \times E_g, \Omega Y \times E_g)$$

is exact for $q \leqq r + s + t + 2$.

Proof. Given a transformation (2.1), we have $\mu \circ (\chi_1 \times \chi_2) \simeq \chi \circ \mu$, $\Pi_1 \circ (\chi_1 \times \chi_2) = \chi \circ \Pi_1$ and $\psi_1 \circ P_1 = P_1 \circ \chi$, where $\chi = E(\psi_1, \varphi, \psi_2;\ G, H)$, $\chi_1 = E(\psi_1, \varphi, 0;\ G, 0)$, $\chi_2 = E(0, \varphi, \psi_2;\ 0, H)$. Therefore we can assume, by Lemma 2.12, that f and g are inclusions.

Let now $(Y;\ A, B)$ be a usual triad with base-point y_0. For subspaces K and L of Y, let $E_{K,L}$ denote the space of paths γ in Y such that $\gamma(0) \in K$ and $\gamma(1) \in L$. We shall write μ for multiplication of paths in Y, Π_1 for the projection on the first factor and P_1, P_2 for the maps taking, respectively, the initial and final point of paths. Let $W = \left\{\gamma \in E_{A,Y} \middle| \gamma\left(\frac{1}{2}\right) = y_0\right\}$. We need the following two lemmas:

LEMMA 5.3. (*a*) *There exists a neighborhood V_1 of $E_{y_0,B}$ in $E_{A,B}$ such that $E_{y_0,B}$ is a strong deformation retract of V_1.*

(*b*) *There exists a neighborhood V_2 of ΩY in E_{A,y_0} such that ΩY is a strong deformation retract of V_2.*

(*c*) *There exists a neighborhood V_3 of W in $E_{A,Y}$ such that $(W, W \cap (E_{A,B} \cup EY))$ is a strong deformation retract of $(V_3, V_3 \cap (E_{A,B} \cup EY))$.*

LEMMA 5.4. $(E_{A,Y}, E_{A,B} \cup EY \cup W)$ *is* $(r+s+t+3)$*-connected.*

The first lemma is easily checked in a manner similar to those in [17] (cf. [15]), and the proof of the second will be postponed later.

Consider now the following commutative diagram

$$
\begin{tikzcd}
H^q(A, a_0) \arrow[r, "P_1^*"] \arrow[d, "P_1^*"', "\wr"] & H^q(E_{A,B}, E_{y_0,B}) \arrow[r, "\mu^*-\Pi_1^*"] & H^q(E_{A,y_0} \times E_{y_0,B}, \Omega Y \times E_{y_0,B}) \\
H^q(E_{A,Y}, EY) \arrow[r, "i^*"] & H^q(E_{A,B} \cup EY, EY) \arrow[r, "\mu^*-\Pi_1^*"] \arrow[u, "k_1^*", "\wr"'] \arrow[d, "\delta"'] & H^q(E_{A,y_0} \times E_{y_0,B} \cup \Omega Y \times EY, \Omega Y \times EY) \arrow[u, "k_2^*", "\wr"'] \arrow[d, "\delta"'] \\
 & H^{q+1}(E_{A,Y}, E_{A,B} \cup EY) \arrow[r, "\mu^*"] \arrow[d, equal] & H^{q+1}(E_{A,y_0} \times EY, E_{A,y_0} \times E_{y_0,B} \cup \Omega Y \times EY) \\
H^{q+1}(E_{A,Y}; E_{A,B} \cup EY, W) \arrow[r] & H^{q+1}(E_{A,Y}, E_{A,B} \cup EY) \arrow[r, "j^*"] & H^{q+1}(W, W \cap (E_{A,B} \cup EY)), \arrow[u, "\mu^*", "\wr"']
\end{tikzcd}
$$

in which δ are coboundary homomorphisms, i, j, k_1, k_2 are appropriate inclusions and μ at the right lower corner is a homeomorphism. Since the vertical P_1 is a homotopy equivalence, P_1^* is an isomorphism onto. By Lemma 5.3 and Theorem 11.3 of Eilenberg-Steenrod [4], k_1^* and k_2^* are exision isomorphisms, and moreover we see from Lemma 5.4 that

$$H^{q+1}(E_{A,Y};\ E_{A,B} \cup EY,\ W) \approx H^{q+1}(E_{A,Y},\ E_{A,B} \cup EY \cup W) = 0$$

for $q \leqq r+s+t+2$.

We next remark that the bottom line is a triadic cohomology sequence of a triad $(E_{A,Y};\ E_{A,B} \cup EY,\ W)$ and hence exact. Take any element $x \in H^q(E_{A,B}, E_{y_0,B})$ such that $(\mu^* - \Pi_1^*)(x) = 0$ for $q \leqq r+s+t+2$; then $j^*\delta k_1^{*-1}(x) = 0$. Since j^* is a momomorphism, there exists a $y \in H^q(A, a_0)$ such that $k_1^{*-1}(x) = i^*P_1^*(y)$, noting that Ker $\delta = \mathrm{Im}\ i^*$. Thus $x = P_1^*(y)$ which completes the proof of Theorem 5.2.

Suppose now that there is given a triad $A \xrightarrow{f} Y \xleftarrow{g} B$ such that g is a fibering with fibre F. We denote by $\lambda : Z_g \to B$ a lifting function for g (see [12], p. 113). We define

$$\tilde{\mu},\ \tilde{\Pi}_1 : (E_f^- \times F,\ \Omega Y \times F) \to (\mathrm{Ker}(f : g),\ F)$$

by $\tilde{\mu}(a, \gamma, b) = (a, \lambda(b, \gamma))$, $\tilde{\Pi}_1(a, \gamma, b) = (a, \lambda(b_0, \gamma))$ for $a \in A$, $\gamma \in E^-Y$, $b \in B$ with $f(a) = \gamma(0)$, $g(b) = y_0$. In view of Theorem 1.4 we see that these maps correspond to μ, Π_1 in Theorem 5.2. Thus we conclude

THEOREM 5.5. *Let $(f : g)$ be as above. Suppose further that A, F, Y are respectively r-, s-, t-connected, $r \geqq 2$, $s \geqq 1$, $t \geqq 2$, and that A and Y have non-degenerate base-points. Then the sequence*

$$H^q(A,\ a_0)\xrightarrow{\pi_1^*}H^q(\mathrm{Ker}(f\ :\ g),\ F)\xrightarrow{\tilde{\mu}^*-\tilde{\Pi}_1^*}H^q(E_f^-\times F,\ \Omega Y\times F)$$

is exact for $q\leqq r+s+t+2$.

In case $s=t-1$, it seems likely that the above theorem gives a geometric version to a part of an exact sequence obtained by E. H. Brown ([2], p. 240).

Finally, we shall give a proof of Lemma 5.4.

Proof of Lemma 5.4. Since $P_1\times P_2$ are both fibre maps in the diagram

$$\begin{tikzcd}[column sep=small]
\cdots \arrow[r] & \pi_i(W,\ W\cap(E_{A,B}\cup EY)) \arrow[rr, "j^*"] \arrow[dr, "(P_1\times P_2)_*"'] & & \pi_i(E_{A,Y},\ E_{A,B}\cup EY) \arrow[r] \arrow[dl, "(P_1\times P_2)_*"] & \cdots \\
& & \pi_i(A\times Y,\ A\times B\cup y_0\times Y) & &
\end{tikzcd}$$

exactness of the horizontal line implies $\pi_i(E_{A,Y};\ E_{A,B}\cup EY,\ W)=0$ for $i\geqq 2$. Hence, by considering the homotopy sequence of a tetrad $(E_{A,Y};\ E_{A,B}\cup EY\cup W,\ E_{A,B}\cup EY,\ W)$, we have

$$\begin{aligned}\pi_{i+1}(E_{A,Y},\ E_{A,B}\cup EY\cup W)&\approx\pi_{i+1}(E_{A,Y};\ E_{A,B}\cup EY\cup W,\ E_{A,B}\cup EY,\ W)\\&\approx\pi_i(E_{A,B}\cup EY\cup W;\ E_{A,B}\cup EY,\ W)\end{aligned}$$

for $i\geqq 2$. But it follows from the Künneth theorem that

$$\pi_i(W,\ W\cap(E_{A,B}\cup EY))\approx\pi_i(A\times Y,\ A\times B\cup y_0\times Y)=0$$

for $i\leqq r+s+2$, since $(A,\ y_0)$ is r-connected and $E_{y_0,B}=E_g$ is s-connected. On the other hand $\pi_i(E_{A,B}\cup EY,\ W\cap(E_{A,B}\cup EY))\approx\pi_i(E_{A,Y},\ W)$ and, moreover, we see that $\gamma\to\gamma\cdot e_y$, $y=\gamma(1)$, yields a homotopy equivalence $((E_{A,Y},\ E_{A,y_0})\to(E_{A,Y},\ W)$. Therefore it follows from $(P_2)_*:\pi_i(E_{A,Y},\ E_{A,y_0})\approx\pi_i(Y,\ y_0)$ that $(E_{A,B}\cup EY,\ W\cap(E_{A,B}\cup EY))$ is t-connected. Applying the Blakers-Massey theorem [1], we have that $(E_{A,B}\cup EY\cup W;\ E_{A,B}\cup EY,\ W)$ is $(r+s+t+2)$-connected and hence

$$\pi_i(E_{A,Y},\ E_{A,B}\cup EY\cup W)=0\qquad\text{for } 3\leqq i\leqq r+s+t+3. \tag{5.6}$$

Consider now the exact sequence

$$\begin{aligned}\pi_2(E_{A,Y})\to\pi_2(E_{A,Y},\ E_{A,B}\cup EY\cup W)\to\pi_1(E_{A,B}\cup EY\cup W)\to\pi_1(E_{A,Y})\\\to\pi_1(E_{A,Y},\ E_{A,B}\cup EY\cup W)\to 0,\end{aligned}$$

where $\pi_i(E_{A,Y})\approx\pi_i(E_{A,Y},\ EY)\approx\pi_i(A)=0$ for $i\leqq 2$. Upon noticing that $\pi_1(W)\approx\pi_1(E_{A,y_0})\approx\pi_2(Y,\ A)=0$ and $\pi_1(E_{A,B})\approx\pi_1(E_{A,B},\ E_{y_0,B})\approx\pi_1(A)=0$, it follows from van Kampen's theorem [13] that $\pi_1(E_{A,B} \cup EY \cup W) = 0$. Hence $\pi_i(E_{A,Y}, E_{A,B} \cup EY \cup W) = 0$ for $i = 1, 2$. Combining this with (5.6), we obtain the desired conclusion.

§ 6. Realizability of Whitehead products

In this section we shall state a result which is dual to a theorem of I. M. James [8] as an application of Theorem 5.2. See also [6, 7, 11, 18].

Let $f : X \to Y$ be a map in which Y possesses a non-degenerate base-point. $\hat{\Psi} : Z_f \to X$ denote the homotopy equivalence given by $\hat{\Psi}(x, \beta) = x$, $x \in X$, $\beta \in Y^I$, $f(x) = \beta(1)$. We set $Pf = \Psi | E_f$. Let $l : (E^- Y, \Omega Y) \to (Z_f, E_f)$ be the inclusion, $l(\beta) = (x_0, \beta)$, and let $P : (Z_f, E_f) \to (X, y_0)$ denote the map defined by $P(x, \beta) = \beta(0)$

There is defined the path-multiplication $\mu : E^- Y \times EY \to Y^I$ in an obvious manner. This induces maps $E^- Y \times E_f \to Z_f$ and $\Omega Y \times E_f \to E_f$ which are denoted by the same letter μ. In what follows, we use π_1, π_2 to denote projections on the first and the second factors respectively.

We have then the following commutative diagram

$$\begin{tikzcd}
 & & & H^{q-1}(E_f) & \\
 & & H^q(E^- Y \times E_f, \Omega Y \times E_f) & H^{q-1}(\Omega Y \times E_f) \arrow[l, "\delta"'] \arrow[u, "i_2^*"] & H^{q-1}(E^- Y \times E_f) \arrow[l, "i_1^*"'] \arrow[ul, "i_3^*", "\approx"'] \\
H^q(E_f) & H^q(Z_f) \arrow[l, "i^*"'] & H^q(Z_f, E_f) \arrow[l, "j^*"'] \arrow[u, "\bar{\nu}"] & H^{q-1}(E_f) \arrow[l, "\delta"'] \arrow[u, "\nu"] & \\
 & H^q(X) \arrow[ul, "(Pf)^*"] \arrow[u, "\hat{\Psi}^*", "\wr"'] & H^q(Y) \arrow[l, "f^*"'] \arrow[u, "P^*"] & &
\end{tikzcd}$$

where $\bar{\nu} = \mu^* - \pi_1^* l^*$, $\nu = \mu^* - \pi_2^* - \pi_1^*(If)^*$, δ are coboundary homomorphisms, and i, j, i_1, i_2, i_3 are appropriate injections.

THEOREM 6.1. *(a)* $\bar{\nu} \circ P^* = 0$.

(b) If Y and E_f are respectively r-and s-connected, $r \geqq 2$, $s \geqq 1$, then P^*: $H^q(Y) \to H^q(Z_f, E_f)$ is a monomorphism for $q \leqq r+s+2$ and the sequence

$$H^q(Y) \xrightarrow{P^*} H^q(Z_f, E_f) \xrightarrow{\bar{\nu}} H^q(E^- Y \times E_f, \Omega Y \times E_f)$$

is exact for $q \leqq 2r+s+2$.

(*c*) δ *is monomorphic on* Ker i_2^* *and* Im $\nu \subset$ Ker i_2^*.

The first half of (*b*) follows from Serre's theorem since P is a fibre map with fibre E_f. (*a*) and (*b*) are obtained by applying Theorems 5.1 and 5.2 to a triad $Y \xrightarrow{1} Y \xleftarrow{f} X$. (*c*) is an immediate consequence of the fact that i_3^* is an isomorphism.

In the sequel assume that all spaces considered have the same homotopy type of a CW-complex. To simplify the notation we do not distinguish between a map and the homotopy class or the cohomology class it represents.

Now we shall take $\theta : K(\pi, n) \to K(\pi', n'+1)$ instead of $f : X \to Y$ in the foregoing consideration, where $2 \leqq n < n'$, and consider $\phi : E_\theta \to K(G, n+n')$. Let W denote the Whitehead product pairing $\pi' \otimes \pi \to G$ in E_ϕ. We call E_ϕ a space *of type* (W, θ). Let $\iota \in H^n(\pi, n; \pi)$, $\iota' \in H^{n'+1}(\pi', n'+1; \pi')$ be basic classes respectively. In these situations it is proven that

LEMMA 6.2. (Meyer [10] and Peterson-Stein [14]) $\nu(\phi) = \pi_1^*({}^1\iota') \cup \pi_2^*(P\theta)^*(\iota)$, *where* ${}^1\iota'$ *denotes the suspension of* ι' *and the cup-product is with respect to* W.

The proof of our result stated in the introduction is based on the following theorem.

THEOREM 6.3. $\delta(\phi) = P^*(\iota') \cup \hat{\Psi}^*(\iota) + P^*(\rho)$ *for unique* $\rho \in H^{n+n'+1}(\pi', n'+1; G)$, *where the cup-product is relative to* W.

Proof. For convenience we consider the projection $p_2 : E^- Y \times E_\theta \to E_\theta$ and the injection $k : \Omega Y \times E_\theta \to E^- Y \times E_\theta$, and let $p : (E^- Y, \Omega Y) \to (Y, y_0)$ be the fibre map given by $p(\beta) = \beta(0)$. $l_0 : E^- Y \to Z_\theta$ denotes the map determined by l. Since $l_0^* \hat{\Psi}^*(\iota) = 0$, we have $\pi_1^* l^* [P^*(\iota') \cup \hat{\Psi}^*(\iota)] = 0$. Further,

$$
\begin{aligned}
\bar{\nu}\delta(\phi) &= \delta\nu(\phi) = \delta[\pi_1^*({}^1\iota') \cup \pi_2^*(P\theta)^*(\iota)] && \text{by Lemma 6.2,} \\
&= \delta[\pi_1^*({}^1\iota') \cup k^* p_2^*(P\theta)^*(\iota)] \\
&= \delta\pi_1^*({}^1\iota') \cup p_2^*(P\theta)^*(\iota) && \text{by [16], (3.2),} \\
&= \pi_1^*\delta({}^1\iota') \cup \pi_2^*\hat{\Psi}^*(\iota), && \text{since } i \circ p_2 = \pi_2, \\
&= \pi_1^* p^*(\iota') \cup \pi_2^*\hat{\Psi}^*(\iota) \\
&= \pi_1^* l^* P^*(\iota') \cup \pi_2^*\hat{\Psi}^*(\iota) \\
&= \mu^* P^*(\iota') \cup \mu^*\hat{\Psi}^*(\iota) && \text{by Theorem 6.1 } (a), \\
&= \mu^*[P^*(\iota') \cup \hat{\Psi}^*(\iota)].
\end{aligned}
$$

This calculation leads to $\bar{\nu}[\delta(\phi) - P^*(\iota') \cup \hat{\Psi}^*(\iota)] = 0$. Hence, by Theorem 6.1 (b), we see that there exists a unique $\rho \in H^{n+n'+1}(\pi', n'+1; G)$ with the desired property.

THEOREM 6.4. *Let $\theta : K(\pi, n) \to K(\pi', n'+1)$, where $2 \leqq n < n'$, and let $\tilde{W}$: $\pi' \otimes \pi \to G$ be a given homomorphism. Let $\theta \cup \iota$ denote the cup-product of θ and the basic class of $K(\pi, n)$ relative to $\tilde{W}$. Then there exists a space of type $(\tilde{W}, \theta)$ if, and only if, $\theta \cup \iota$ is contained in the image of the homomorphism*

$$\theta^* : H^{n+n'+1}(\pi', n'+1; G) \to H^{n+n'+1}(\pi, n; G).$$

Proof. Applying $(\hat{\Psi}^*)^{-1} j^*$ to the formula in Theorem 6.3, we obtain $0 = \theta^*(\iota') \cup \iota + \theta^*(\rho)$, i.e., $\theta \cup \iota = -\theta^*(\rho)$, which proves the "only if" part. Conversely, suppose there exists $\rho \in H^{n+n'+1}(\pi', n'+1; G)$ such that $-\theta^*(\rho) = \theta \cup \iota$ rel $\tilde{W}$. Here "rel $\tilde{W}$" indicates that the cup-product is to be taken relative to $\tilde{W}$. This shows that $P^*(\iota') \cup \hat{\Psi}^*(\iota)$ rel $\tilde{W} + P^*(\rho)$ lies in the kernel of j^*, so that, by exactness of the cohomology sequence of the pair (Z_θ, E_θ), there is $\phi \in H^{n+n'}(E_\theta)$ such that $\delta(\phi) = P^*(\iota') \cup \hat{\Psi}^*(\iota)$ rel $\tilde{W} + P^*(\rho)$. We shall show that the space E_ϕ is of type $(\tilde{W}, \theta)$. Let W denote the Whitehead product pairng in E_ϕ. Now

$$\begin{aligned} &\delta[\pi_1^*({}^1\iota') \cup \pi_2^*(P\theta)^*(\iota) \text{ rel } \tilde{W}] \\ &\quad = \bar{\nu}[P^*(\iota') \cup \hat{\Psi}^*(\iota) \text{ rel } \tilde{W}] && \text{from the proof of Th. 6.3,} \\ &\quad = \bar{\nu}\delta(\phi) = \delta\nu(\phi) && \text{by Theorem 6.1, (a),} \\ &\quad = \delta[\pi_1^*({}^1\iota') \cup \pi_2^*(P\theta)^*(\iota) \text{ rel } W] && \text{by Lemma 6.2.} \end{aligned}$$

Therefore, Theorem 6.1, (c), implies $\pi_1^*({}^1\iota') \cup \pi_2^*(P\theta)^*(\iota)$ rel $\tilde{W} = \pi_1^*({}^1\iota') \cup \pi_2^*(P\theta)^*(\iota)$ rel W. This means that $\tilde{W} = W$.

COROLLARY 6.5. *There always exists a space of type* $(W, 0)$.

COROLLARY 6.6. *Under the same notation as in Theorem* 6.3, *we have* $(I\theta)^*(\phi) = {}^1\rho$.

This is deduced by applying $\delta^{-1} l^*$ to the formula in Theorem 6.3, where $\delta : H^{n+n'}(\pi', n'; G) \approx H^{n+n'+1}(E^- K(\pi', n'+1), \Omega K(\pi', n'+1); G)$.

COROLLARY 6.7. *Let $\theta : K(\pi, n) \to K(\pi', 2n)$, $n \geqq 2$, and let W_1, W_2 be, respectively, given pairings $\pi \otimes \pi \to \pi'$, $\pi' \otimes \pi \to G$. Then there exists a space whose first invariant is θ and whose Whitehead product pairings are just W_1 and W_2, if, and only if,* $(\mu^* - \pi_1^* - \pi_2^*)(\theta) = \pi_1^*(\iota) \cup \pi_2^*(\iota)$ rel W_1 *and* $\theta \cup \iota$

rel $W_2 \in \theta^* H^{3n}(\pi', 2n; G)$, *where* $\mu: K(\pi, n) \times K(\pi, n) \to K(\pi, n)$ *is the H-structure map.*

This follows from a result proved by Copeland [3].

COROLLARY 6.8. *If* cat $K(\pi, n) \leqq 2$, *then there always exists a space of type* (W, θ).

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