

A REMARK ON DECOMPOSITIONS OF THE PERMUTATION REPRESENTATION OF A PERMUTATION GROUP

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TO RICHARD BRAUER on the occasion of his 60th birthday

Let \mathfrak{G} be a permutation group on n -letters $\underline{1}, \underline{2}, \dots, \underline{n}$. Let \mathfrak{G}_1 be the subgroup of \mathfrak{G} fixing suitable one letter, say $\underline{1}$. For any element G of \mathfrak{G} , a non-singular matrix $G^* = (g_{ij})$ of degree n is defined by the equation

$$(1) \quad \begin{pmatrix} \underline{1}^G \\ \cdot \\ \cdot \\ \cdot \\ \underline{n}^G \end{pmatrix} = G^* \begin{pmatrix} \underline{1} \\ \cdot \\ \cdot \\ \cdot \\ \underline{n} \end{pmatrix}.$$

Since g'_{ij} s are 0 or 1, we may assume that G^* is a matrix whose coefficients are in an arbitrary unitary ring K . Then if for any element G of \mathfrak{G} we take the mapping $G \rightarrow G^*$, this mapping will be a representation P_K of \mathfrak{G} by the non-singular $n \times n$ matrices over K . By the formula (1) the representation P_K is also the representation of \mathfrak{G} induced by the identity representation of \mathfrak{G}_1 over K . We call P_K the permutation representation of \mathfrak{G} over K . If K is a field of characteristic 0 (more generally, if the characteristic of K does not divide the order of \mathfrak{G}), then it is well known that \mathfrak{G} is a doubly transitive group when and only when P_K is directly decomposed into two irreducible constituents (see [2]). Now in the present note we consider decompositions of the permutation representation P_K of \mathfrak{G} over an arbitrary unitary ring K , instead of such a field of characteristic 0.

THEOREM 1. *Assume that \mathfrak{G} is a doubly transitive group.*

i) If n is an invertible element of K (e.g. K is a field whose characteristic does not divide n), then P_K is directly decomposed into two indecomposable

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constituents and one of them is the identity representation.

ii) If n is not an inversible element of K (e.g. K is rational integer ring or a field whose characteristic divides n), then P_K is a indecomposable representation.

Proof. Let M_K be the representation module of \mathfrak{G} corresponding to P_K . Then we may suppose that M_K has a basis $\underline{1}, \underline{2}, \dots, \underline{n}$ over K and, for any element G of \mathfrak{G} , G^* operates on this basis such that $\underline{i} \rightarrow \underline{i}^G$. If M_K is directly decomposed into a certain number of \mathfrak{G} -submodules of M_K , say M_1, \dots, M_r , then we have, for $u \in M$ uniquely, $u = \sum_{i=1}^r u_i$, $u_i \in M_i$ and the mappings $\delta_i: u \rightarrow u_i$, $i = 1, \dots, r$, are idempotent \mathfrak{G} -endomorphisms of M_K such that

$$(2) \quad \delta_i \delta_j = 0 \quad \text{for } i \neq j \text{ and } \sum_{i=1}^r \delta_i = \text{identity.}$$

Conversely, if there exist idempotent \mathfrak{G} -endomorphisms $\delta_1, \dots, \delta_r$ of M_K satisfying the relations (2), then it is easy to see that M_K is directly decomposed into r \mathfrak{G} -submodules of M_K . Therefore, in order to determine the direct decomposition of M_K , we need only to look for idempotent \mathfrak{G} -endomorphisms of M_K satisfying the relations (2). Let δ be a \mathfrak{G} -endomorphism of M_K and put $\underline{i}^\delta = \sum_{j=1}^n \lambda_{ij} \underline{j}$. Since $\underline{i}^{\delta G} = \underline{i}^{G\delta}$ for any element G of \mathfrak{G} , we have $\sum_{j=1}^n \lambda_{ij} \underline{j}^G = \sum \lambda_{iGj} \underline{j}$, hence $\lambda_{ij} = \lambda_{iG_j G}$ for any element G of \mathfrak{G} and for any integers $1 \leq i, j \leq n$. Since \mathfrak{G} is doubly transitive it is easy to see that $\lambda_{11} = \dots = \lambda_{nn} (= \lambda)$ and $\lambda_{ij} = \lambda_{k1} (= \mu)$ if $i \neq j$ and $k \neq 1$. Hence we have

$$\begin{pmatrix} 1^\delta \\ \cdot \\ \cdot \\ \cdot \\ n^\delta \end{pmatrix} = \Delta(\delta) \begin{pmatrix} 1 \\ \cdot \\ \cdot \\ \cdot \\ n \end{pmatrix} \quad \text{where } \Delta(\delta) = \begin{pmatrix} \lambda & \mu & \cdot & \cdot & \mu \\ \mu & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \mu & \cdot & \cdot & \cdot & \lambda \end{pmatrix}$$

If δ is a idempotent \mathfrak{G} -endomorphism of M_K , i.e. $\delta^2 = \delta$, then

$$\Delta(\delta) = \Delta(\delta)^2 = \begin{pmatrix} \lambda^2 + (n-1)\mu^2 & \cdot & 2\lambda\mu + (n-2)\mu^2 \\ \cdot & \cdot & \cdot \\ 2\lambda\mu + (n-2)\mu^2 & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \lambda^2 + (n-1)\mu^2 & \cdot & \cdot \end{pmatrix},$$

therefore we have the equations $\lambda = \lambda^2 + (n-1)\mu^2$, $\mu = 2\lambda\mu + (n-2)\mu^2$. From these equations we see that $\lambda = \mu = 0$, $\lambda = \mu$ and $\lambda n = 1$, $\lambda = 1$ and $\mu = 0$, or $\lambda = \mu + 1$ and $n\lambda = n - 1$. Hence if n is not a inversible element of K then we

have no non trivial idempotent \mathbb{G} -endomorphisms of M_K and if n is a invertible element of K then there exist exactly two non trivial \mathbb{G} -endomorphisms δ_1, δ_2 of M_K where

$$\Delta(\delta_1) = \begin{pmatrix} \frac{1}{n} & & & \\ & \cdot & & \\ & & \cdot & \\ \frac{1}{n} & & & \frac{1}{n} \end{pmatrix} \text{ and } \Delta(\delta_2) = \begin{pmatrix} \frac{n-1}{n} & & & -\frac{1}{n} \\ & \cdot & & \\ & & \cdot & \\ -\frac{1}{n} & & & \frac{n-1}{n} \end{pmatrix}.$$

It is easy to see that $\delta_1 \delta_2 = \delta_2 \delta_1 = 0$, $\delta_1 + \delta_2 = \text{identity}$ and $M_K^\mathbb{G} = K \sum_{i=1}^n \underline{i}$. The proof is complete.

It seems to us of interest to determine irreducible constituents of M_K . When \mathbb{G} is the symmetric groups, H. K. Farahat determined irreducible constituents of M_K (see [1]). Using Farahat's method we can prove a following theorem.

THEOREM 2. *Let \mathbb{G} be a triply transitive group. If K is a field whose characteristic p divides n and does not divide the order of \mathbb{G}_1 then, for the \mathbb{G} -module M_K , we have a composition series $M_K \supset M_1 \supset M_2 \supset 0$ where $\dim_K M_2 = \dim_K M_K / M_1 = 1$.*

Proof. Put $M_1 = \sum_{i=2}^n K(i-1)$ and $M_2 = K \sum_{i=1}^n \underline{i}$. Then M_1, M_2 are \mathbb{G} -submodules of M_K and, by our assumption $p \mid n$, $M_1 \supset M_2$ and $\dim_K M_2 = \dim_K M_K / M_1 = 1$. Put $M_1^* = \sum_{i=2}^n K \underline{i}$. Then, since $\underline{1}^G = \underline{1}$ for any element G of \mathbb{G}_1 , we see that M_1^* is a \mathbb{G}_1 -module and there is a \mathbb{G}_1 -isomorphism θ of M_1 onto M_1^* , for which $\theta(i-1) = \underline{i}$. Furthermore, since $n \cdot 1 = 0$ in K , θ carries M_2 onto $M_2^* = K \sum_{i=2}^n \underline{i}$. It follows that θ induces a \mathbb{G}_1 -isomorphism of the factor module M_1/M_2 onto the factor module M_1^*/M_2^* . Since p does not divide the order of \mathbb{G}_1 and \mathbb{G}_1 is doubly transitive, M_1^*/M_2^* is an irreducible \mathbb{G}_1 -module. It follows that M_1/M_2 is an irreducible \mathbb{G}_1 -module. Hence M_1/M_2 is an irreducible \mathbb{G} -module. The proof is complete.

REFERENCES

- [1] H. K. Farahat, On the natural representation of the symmetric groups, Proc. Glasgow Math. Assoc. 5 (1962), 121-136.

- [2] G. Frobenius, Über die Charaktere der mehrfach transitiven Gruppe, Sitzungsber, Preuss. Akad. (1904), 558-571.

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