A PROOF OF BRAUER'S THEOREM ON GENERALIZED DECOMPOSITION NUMBERS

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To Professor RICHARD BRAUER on the occasion of his 60th birthday

In [3] R. Brauer gave a proof of his theorem on generalized decomposition numbers which was first announced in [1], and a simplification of it has been made by K. Iizuka [5]. In this note we shall show that the theorem may be proved from another point of view by using some results obtained by J. A. Green in [4].

After stating some results by Green and Osima in the first and second sections we first prove a theorem on characters (Theorem 1) and by using the theorem we prove Brauer's theorem in the fourth section.

1. The algebra $Z(\mathfrak{G} : \mathfrak{H})$

Let $\mathfrak{G}$ be a finite group. We consider the group ring $\Gamma(\mathfrak{G})$ of $\mathfrak{G}$ over the ring $\mathfrak{o}$ of $\mathfrak{p}$-adic integers, where $\mathfrak{p}$ is a prime ideal divisor of a fixed prime p in some algebraic number field.

If G is any element of $\mathfrak{G}$, γ any element of $\Gamma(\mathfrak{G})$, write $\gamma^G = G^{-1}\gamma G$. Then for a subgroup $\mathfrak{H}$ of $\mathfrak{G}$ the set

$$Z(\mathfrak{G} : \mathfrak{H}) = \{\gamma \in \Gamma(\mathfrak{G}) : \gamma^H = \gamma \text{ for all } H \in \mathfrak{H}\}$$

is a subalgebra of $\Gamma(\mathfrak{G})$. Let $\mathfrak{L}_1, \mathfrak{L}_2, \ldots, \mathfrak{L}_s$ be the classes of $\mathfrak{H}$-conjugate elements in $\mathfrak{G}$, where two elements X and Y of $\mathfrak{G}$ are called $\mathfrak{H}$-conjugate if there exists an element H in $\mathfrak{H}$ such that $Y = X^H$. If $L_1, L_2, \ldots, L_s$ denote the sums of the elements in $\mathfrak{L}_1, \mathfrak{L}_2, \ldots, \mathfrak{L}_s$ respectively, these sums form an $\mathfrak{o}$-basis of $Z(\mathfrak{G} : \mathfrak{H})$.

For a fixed $\mathfrak{H}$-conjugacy class $\mathfrak{L}_\alpha$, a Sylow p-subgroup of the normalizer $\mathfrak{N}_{\mathfrak{H}}(U_\alpha)$ of some element $U_\alpha \in \mathfrak{L}_\alpha$ in $\mathfrak{H}$ is called the p-defect group of $\mathfrak{L}_\alpha$, and is

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denoted by $\mathfrak{P}_\alpha$. It is determined up to $\mathfrak{H}$-conjugacy.

Let $\mathfrak{P}$ be a p-subgroup of $\mathfrak{H}$ and $I(\mathfrak{P})$ the set of those $\alpha \in \{1, 2, \ldots, s\}$ such that $\mathfrak{P}_\alpha \underset{\mathfrak{H}}{\leq} \mathfrak{P}$, i.e. $\mathfrak{P}_\alpha \leq H^{-1}\mathfrak{P}H$ for some $H \in \mathfrak{H}$. The set of all $z \in Z(\mathfrak{G} : \mathfrak{H})$ of the form

$$z \equiv \sum_{\alpha \in I(\mathfrak{P})} a_\alpha L_\alpha \bmod \mathfrak{p}Z(\mathfrak{G} : \mathfrak{H}) \qquad (a_\alpha \in \mathfrak{o})$$

is denoted by $Z_{\mathfrak{P}}(\mathfrak{G} : \mathfrak{H})$.

Lemma 1 (Osima [6], Green [4], Lemma 3.2 c). *If $\mathfrak{P}$ is a p-subgroup of $\mathfrak{H}$, then $Z_{\mathfrak{P}}(\mathfrak{G} : \mathfrak{H})$ is an ideal of $Z(\mathfrak{G} : \mathfrak{H})$.*

2. Characters

If a right $\Gamma(\mathfrak{G})$-module M is free and finitely generated over $\mathfrak{o}$ and unitary, i.e. $m1 = m$ for all $m \in M$, we call M a *representation module of $\mathfrak{G}$ over $\mathfrak{o}$ or $\mathfrak{G}$-representation module* for short. A $\mathfrak{G}$-representation module M has an $\mathfrak{o}$-basis, and hence a matrix representation is associated with M. The character of the matrix representation associated with M is denoted by χ_M.

A $\mathfrak{G}$-representation module M is said to be *$\mathfrak{H}$-projective* if M is a direct summand of the induced module $N \otimes_{\Gamma(\mathfrak{H})} \Gamma(\mathfrak{G})$ of some $\mathfrak{H}$-representation module N.

Lemma 2 (Green [4], Lemma 4.1 a). *Let $\mathfrak{H}$ be a subgroup of $\mathfrak{G}$, $\mathfrak{P}$ a p-subgroup of $\mathfrak{H}$ and let M be a $\mathfrak{G}$-representation module. If e is an idempotent in $Z_{\mathfrak{P}}(\mathfrak{G} : \mathfrak{H})$, then $\mathfrak{H}$-representation module Me is $\mathfrak{P}$-projective.*

If for an element X of $\mathfrak{G}$ $X = PV = VP$, where P has order a power of p and V has order prime to p, P and V are called *p-factor* and *p-regular factor* of X, respectively. The following is one of the main theorems by Green in [4].

Lemma 3 (Green [4], Theorem 3). *Let $\mathfrak{P}$ be a p-subgroup of $\mathfrak{G}$ and M a $\mathfrak{G}$-representation module. If M is $\mathfrak{P}$-projective and the p-factor of an element X does not lie in any conjugate of $\mathfrak{P}$, then*

$$\chi_M(X) = 0.$$

3. Brauer homomorphisms

Let $\mathfrak{P}$ be a given p-subgroup of $\mathfrak{G}$ and let $\mathfrak{H}$ be a subgroup such that $\mathfrak{P}\mathfrak{C}(\mathfrak{P}) \leq \mathfrak{H} \leq \mathfrak{N}(\mathfrak{P})$, where $\mathfrak{C}(\mathfrak{P})$ and $\mathfrak{N}(\mathfrak{P})$ are the centralizer and normalizer of $\mathfrak{P}$, respectively. For a $\mathfrak{G}$-conjugacy class $\mathfrak{K}_\alpha$, let $\mathfrak{K}'_\alpha=\mathfrak{K}_\alpha\cap\mathfrak{C}(\mathfrak{P})$ and $\mathfrak{K}''_\alpha=\mathfrak{K}_\alpha-\mathfrak{K}'_\alpha$. Denote by K'_α, K''_α the sums of the elements in $\mathfrak{K}'_\alpha$, $\mathfrak{K}''_\alpha$, respectively. Then $\mathfrak{K}'_\alpha$ and $\mathfrak{K}''_\alpha$ are collections of $\mathfrak{H}$-conjugacy classes, and hence K'_α and K''_α are in $Z(\mathfrak{G}:\mathfrak{H})$. Each $\mathfrak{H}$-conjugacy class in $\mathfrak{K}''_\alpha$ has the defect group $\mathfrak{Q}$ such that $\mathfrak{P}\not\leqq\mathfrak{Q}$.

Let $Z(\mathfrak{G})$ be the center of $\Gamma(\mathfrak{G})$ and $Z^*(\mathfrak{G})$ the residue algebra $Z(\mathfrak{G})/\mathfrak{p}Z(\mathfrak{G})$. Then Brauer [2] has shown that the linear mapping $s^*: Z^*(\mathfrak{G})\to Z^*(\mathfrak{H})$ which is defined by $s^*(K_\alpha)=K'_\alpha$ is an algebra homomorphism. We shall call this the *Brauer homomorphism.*

Let E be an idempotent in $Z(\mathfrak{G})$ and E^* the image of E under the natural mapping $Z(\mathfrak{G})\to Z^*(\mathfrak{G})$. As is well known, the idempotent $s^*(E^*)$ in $Z^*(\mathfrak{H})$ can be lifted to an idempotent e of $Z(\mathfrak{H})$, i.e. $e^*=s^*(E^*)$. Now, we consider the situation where $\mathfrak{P}$ is the cyclic subgroup generated by an element P of order a power of p and $\mathfrak{H}$ is the centralizer $\mathfrak{C}(\mathfrak{P})=\mathfrak{N}(P)$ of $\mathfrak{P}$. Then we have

THEOREM 1. *Let P be an element of order a power of p, E an idempotent of $Z(\mathfrak{G})$ and let e be the idempotent of $Z(\mathfrak{N}(P))$ such that $s^*(E^*)=e^*$, where $s^*: Z^*(\mathfrak{G})\to Z^*(\mathfrak{N}(P))$ is the Brauer homomorphism. If M is a $\mathfrak{G}$-representation module such that $ME=M$, then for any p-regular element V in $\mathfrak{N}(P)$, we have*

$$\chi_M(PV)=\chi_{Me}(PV).$$

Proof. If $E=\sum_\alpha b_\alpha K_\alpha$ then

$$e\equiv\sum_\alpha b_\alpha K'_\alpha \qquad \mathrm{mod}\ \mathfrak{p}Z(\mathfrak{G}:\mathfrak{N}(P)),$$

therefore

$$E-e\equiv\sum_\alpha b_\alpha K''_\alpha \qquad \mathrm{mod}\ \mathfrak{p}Z(\mathfrak{G}:\mathfrak{N}(P)).$$

Since each $\mathfrak{N}(P)$-conjugacy class in $\mathfrak{K}''_\alpha$ has the defect group $\mathfrak{Q}$ such that $P\notin\mathfrak{Q}$, $E-e$ lies in the ideal

$$\Lambda=\sum_{P\notin\mathfrak{Q}} Z_{\mathfrak{Q}}(\mathfrak{G}:\mathfrak{N}(P))$$

of $Z(\mathfrak{G}:\mathfrak{N}(P))$, where the sum is over all p-subgroups $\mathfrak{Q}$ of $\mathfrak{N}(P)$ which do not contain P. Let $f=E(E-e)$. Then $f\in\Lambda$, and $E e$ and f are mutually orthogonal idempotents such that $E=Ee+f$. Since E e and f commute with all elements of $\mathfrak{N}(P)$, MEe and Mf are $\mathfrak{N}(P)$-representation modules. By the assumption $ME = M$, therefore M is the direct sum of two $\mathfrak{N}(P)$-submodules $MEe = Me$ and Mf;

$$M = Me \oplus Mf.$$

Let $f = \sum f_i$, where $\{f_i\}$ is a set of mutually orthogonal primitive idempotents in $Z(\mathfrak{G} : \mathfrak{N}(P))$. Since $f_i = f_i f \in \Lambda$, by a theorem of Rosenberg (cf. Green [4], Lemma 3.3 a) there is a p-subgroup $\mathfrak{Q}_i$ of $\mathfrak{N}(P)$ such that $P \notin \mathfrak{Q}_i$ and $f_i \in Z_{\mathfrak{Q}_i}(\mathfrak{G} : \mathfrak{N}(P))$, and then Mf_i is $\mathfrak{Q}_i$-projective by Lemma 2. For any p-regular element V of $\mathfrak{N}(P)$, the p-factor of PV is P and P does not lie in any subgroup $\mathfrak{N}(P)$-conjugate to $\mathfrak{Q}_i$, therefore by Lemma 3 $\chi_{Mf_i}(PV) = 0$. Since

$$Mf = Mf_1 \oplus \cdots \oplus Mf_r,$$

$\chi_{Mf}(PV) = 0$, and hence $\chi_M(PV) = \chi_{Me}(PV)$.

4. Proof of Brauer's theorem

Let $\{\chi_i\}$ be the set of absolutely irreducible ordinary characters of $\mathfrak{G}$, P an element of order a power of p and let $\{\tilde{\varphi}_j\}$ be the set of absolutely irreducible ordinary characters of $\mathfrak{N}(P)$. Let

$$\chi_i | \mathfrak{N}(P) = \sum_j r_{ij} \tilde{\chi}_j \tag{1}$$

be the decomposition of the restriction of χ_i to $\mathfrak{N}(P)$, and let

$$\tilde{\chi}_j = \sum_\mu \tilde{d}_{j\mu} \tilde{\varphi}_\mu \tag{2}$$

be the p-modular decomposition of $\tilde{\chi}_j$, where the $\tilde{\varphi}_\mu$ are the irreducible p-modular characters of $\mathfrak{N}(P)$ and the $\tilde{d}_{j\mu}$ are the decomposition numbers of $\mathfrak{N}(P)$. Since P is in the center of $\mathfrak{N}(P)$

$$\tilde{\chi}_j(PV) = \varepsilon_j \tilde{\chi}_j(V) = \sum_\mu \varepsilon_j \tilde{d}_{j\mu} \tilde{\varphi}_\mu(V) \tag{3}$$

for any p-regular element V in $\mathfrak{N}(P)$, where $\varepsilon_j = \dfrac{\tilde{\chi}_j(P)}{\tilde{\chi}_j(1)}$. From (1), (2) and (3)

$$\chi_i(PV) = \sum_\mu d^P_{i\mu} \tilde{\varphi}_\mu(V)$$

for any p-regular element V of $\mathfrak{N}(P)$, where $d^P_{i\mu} = \sum_j r_{ij} \varepsilon_j \tilde{d}_{j\mu}$. The $d^P_{i\mu}$ are called the *generalized decomposition numbers of* $\mathfrak{G}$.

Now suppose that $\mathfrak{o}$ contains a primitive g-th root of unity, where g is the order of $\mathfrak{G}$. Let E be a primitive idempotent of $Z(\mathfrak{G})$. Any χ_i is the character of some representation module M_i of $\mathfrak{G}$ over $\mathfrak{o}$. If $M_iE = M_i$ then we say that χ_i belongs to the p-block B associated with E.

Let e be the idempotent in $Z(\mathfrak{N}(P))$ such that $e^* = s^*(E^*)$, where $s^* : Z^*(\mathfrak{G}) \to Z^*(\mathfrak{N}(P))$ is the Brauer homomorphism. If $\tilde{B}$ is the set of $\tilde{\chi}_j$ such that the associated representation medule $\tilde{M}_j$ of $\mathfrak{N}(P)$ over $\mathfrak{o}$ satisfies $\tilde{M}_je = \tilde{M}_j$, then $\tilde{B}$ is a collection of p-blocks of $\mathfrak{N}(P)$. We shall also denote by $\tilde{B}$ the set of p-modular characters $\tilde{\varphi}_\mu$ of $\mathfrak{N}(P)$ such that $\tilde{d}_{j\mu} \neq 0$ for some $\tilde{\chi}_j \in \tilde{B}$. Then the Brauer's theorem reads as follows:

THEOREM 2. *If χ_i belongs to a p-block B of $\mathfrak{G}$, then the generalized decomposition numbers $d_{i\mu}^P$ can be different from zero only for $\tilde{\varphi}_\mu$ which belongs to $\tilde{B}$.*

Proof. Let V be any p-regular element of $\mathfrak{N}(P)$. Let

$$\chi_i = \sum_j{}' r_{ij}\tilde{\chi}_j + \sum_k{}'' r_{ik}\tilde{\chi}_k,$$

where the sum $\sum'$ is over all $\tilde{\chi}_j$ in $\tilde{B}$ and the sum $\sum''$ is over all other $\tilde{\chi}_k$. Then from Theorem 1 we have

$$\begin{aligned} \chi_i(PV) &= \sum_j{}' r_{ij}\tilde{\chi}_j(PV) \\ &= \sum_\mu{}' d_{i\mu}^P \tilde{\varphi}_\mu(V), \end{aligned}$$

where μ ranges over the suffices such that $\tilde{\varphi}_\mu \in \tilde{B}$. Since the $\tilde{\varphi}_\mu$ are linearly independent, we have the therem.

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