BLOCKS AND NORMAL SUBGROUPS OF FINITE GROUPS

W. F. REYNOLDS

To RICHARD BRAUER on his Sixtieth Birthday

Introduction

Let H be a normal subgroup of a finite group G, and let ζ be an (absolutely) irreducible character of H. In [7], Clifford studied the irreducible characters χ of G whose restrictions to H contain ζ as a constituent. First he reduced this question to the same question in the so-called inertial subgroup S of ζ in G, and secondly he described the situation in S in terms of certain projective characters of S/H. In section 8 of [10], Mackey generalized these results to the situation where all the characters concerned are projective.

Fong proved analogues of both of Clifford's results, in which the characters of G are taken not individually, but in p-blocks for a prime p. In both of these theorems (Theorems (2 B) and (2 D) of [8]), it is assumed that the order of H is not divisible by p.

In section 1, we generalize Fong's first result for arbitrary blocks B and normal subgroups H, by defining an inertial subgroup S in G for each block $\tilde{B}$ of H. We show in Theorem 1 that if the restriction to H of any character of B contains a character of $\tilde{B}$, then the structure of B is identical with that of a corresponding block B' of S, and the characters of B are induced by those of B'. In effect, this reduces the whole problem to the case where all the constituents of the restrictions of the characters of B lie in a single block of H.

In sections 2 to 4, we study the second question. The group S/H is no longer appropriate when p divides the order of H, since reduction to S/H alters the defect group of the block and thereby destroys the block structure; and we must produce a specially constructed group which will have simpler

Received January 17, 1962.

A portion of this work was done at the Summer Institute for Finite Groups, California Institute of Technology, August, 1960.

structure than S, while retaining the structure of our block. We have succeeded in doing this in the case where the defect group D of B is normal in G and H is its centralizer. (The study of such blocks was begun by Brauer in [3], section 9.) The group M which we construct has a normal p-Sylow group, isomorphic to D (see Theorem 6).

In section 5, we apply this reduction to settle, in the case of a normal defect group, some questions which were raised by Brauer in [2]: Theorem 9 gives a connection between the degrees of the characters of B and the structure of D, and Theorem 11 gives a bound on the Cartan invariants of B.

Theorem 9 was independently proved by M. Suzuki; I wish to thank him for communicating his proof to me.

Notation. G will always be a group of finite order g, and p will be a fixed prime number. We denote by ν the exponential valuation of the rational field determined by p, normalized so that $\nu(p)=1$. Ω is a finite algebraic number field containing the g-th roots of unity; Ω^* is the residue class field with respect to a prime divisor $\mathfrak{p}$ of p in Ω. The (distinct nonequivalent) absolutely irreducible representations of G, ordinary and modular, can be written in Ω and Ω^* respectively; we shall suppose them so written, in definite matricial forms. We shall often suppress the adjective "irreducible".

Each p-block B of G is considered here as a set of ordinary irreducible characters, although we shall also make use of the irreducible modular characters of B. If $\chi_j \in B$, the **height** of χ_j is the non-negative integer $\nu(\chi_j(1)) - \nu(G: D)$, where the p-group D is a defect group of B (see [3]). We shall pass freely between a representation and its character: for example, we shall sometimes consider B as a set of representations, and speak of the heights of these representations.

1. Blocks and Induced Characters

Let H be any normal subgroup of G. Then each element x of G permutes the elements y of H by

$$y \to y^x = x^{-1}yx, \tag{1}$$

and permutes the irreducible characters ζ of H by $\zeta \to \zeta^x$, where

$$\zeta^x(y) = \zeta(xyx^{-1}). \tag{2}$$

Characters ζ and ζ^x so related are called *G-associates.* For each ζ, $\{x \in G : \zeta^x = \zeta\}$ is a subgroup of G, containing H, called the *inertial group* of ζ in G.

The permutations $\zeta \to \zeta^x$ induce a representation of G by permutations $B \to B^x$ of the p-blocks of H, by section 1 of [8]. Denote the systems of transitivity of this permutation representation by $\tilde{\mathscr{T}}_i$, and let $\tilde{T}_i$ be the union of those blocks which are elements of $\tilde{\mathscr{T}}_i$; thus $\tilde{T}_i$ is a set of irreducible characters of H.

For each i, let T_i consist of those irreducible characters χ_j of G such that some irreducible constituent of the restriction $\chi_j | H$ of χ_j to H is in $\tilde{T}_i$. By Lemma (1 B) of [8], the sets T_i are disjoint: that is, all the constituents of $\chi_j | H$ are in the same $\tilde{T}_i$; and each T_i is a union of p-blocks of G. We denote the set of those blocks whose union is T_i by $\mathscr{T}_i$; the same lemma shows that this agrees with Fong's definition of $\mathscr{T}_i$.

Now restrict attention to a fixed $\tilde{T}_i = \tilde{T}$ (dropping subscripts i), and to a fixed $\tilde{B} \subseteq \tilde{T}$. We call the group

$$S = \{x \in G : \tilde{B}^x = \tilde{B}\} \tag{3}$$

the *inertial group* of $\tilde{B}$ in G; its index $(G : S)$ is the number of blocks in $\tilde{\mathscr{T}}$.

Take a maximal set $\{\zeta_m\}$ of characters of $\tilde{B}$ such that no two are G-associates. By Theorem 1 of [7] we can write T as a disjoint union $\cup F_m$, where F_m consists of all irreducible characters χ_j of G such that $\chi_j | H$ contains ζ_m. Let S_m be the inertial group of ζ_m in G. In section 2 of [7], Clifford sets up a 1-1 correspondence between the characters χ_j of F_m and all the irreducible characters ξ_j of S_m such that $\xi_j | H$ contains ζ_m, in which $\chi_j = \xi_j^G$. Here ξ_j^G denotes the character of G induced by the character ξ_j of the subgroup S_m of G.

For any $x \in G$ such that $\zeta_m^x \in \tilde{B}$, $\tilde{B}^x = \tilde{B}$. This implies that $S_m \subseteq S$, and that $\{\zeta_m\}$ is also a maximal set of non-S-associate characters of $\tilde{B}$. If we define T' to consist of the irreducible characters χ'_j of S such that $\chi'_j | H$ contains some character of $\tilde{B}$, then we have $T' = \cup F'_m$ analogously to the above, and there is a 1-1 correspondence between the $\chi'_j \in F'_m$ and the same ξ_j as before, in which $\chi'_j = \xi_j^S$.

We can combine these 1-1 correspondences to obtain a 1-1 correspondence $\chi_j \leftrightarrow \chi'_j$ between F_m and F'_m, in which $(\chi'_j)^G = (\xi_j^S)^G = \xi_j^G = \chi_j$. We have, in fact, established a 1-1 correspondence between T and T' in which $\chi_j = (\chi'_j)^G$. Let $\mathcal{T}'$ be the set of those blocks which make up T'.

The modular characters of the blocks of $\mathcal{T}$ may be called the modular characters of T; they are the modular characters ϕ_k such that $\phi_k | H$ contains some modular character of $\tilde{B}$. Since Clifford's results hold in the modular case, our argument adapts to this case to give a 1-1 correspondence $\phi_k \leftrightarrow \phi'_k$ between the modular characters of T and those of T', in which $\phi_k = (\phi'_k)^G$.

The part (d'_{jk}) of the decomposition matrix (see [3] and [5]) of S belonging to the blocks in $\mathcal{T}'$ is expressed by the equations

$$\chi'_j = \sum_k d'_{jk} \phi'_k,$$

where the χ'_j and ϕ'_k are the ordinary and modular characters of T', and where the relation holds for p-regular elements. Inducing to G yields

$$\chi_j = (\chi'_j)^G = (\sum_k d'_{jk} \phi'_k)^G = \sum_k d'_{jk} (\phi'_k)^G = \sum_k d'_{jk} \phi_k.$$

This means that (d'_{jk}) is also the part (d_{jk}) of the decomposition matrix of G belonging to the blocks in $\mathcal{T}$. The corresponding statement for the Cartan matrix (c_{kl}) follows, since $c_{kl} = \sum_j d_{jk} d_{jl}$.

Since the Cartan matrices determine the blocks by section 8 of [5], the characters (both ordinary and modular) of each block $B_\tau \in \mathcal{T}$ correspond to those of a block $B'_\tau \in \mathcal{T}'$, and conversely. Since $\chi_j(1) = (G : S)\chi'_j(1)$, B_τ and B'_τ have the same defect. In fact, the argument of Lemma (2 A) of [8] applies almost verbatim to show that each defect group of B'_τ is also one for B_τ. Finally, the height of χ_j in B_τ is the same as the height of χ'_j in B'_τ.

We summarize in the following theorem, which generalizes Fong's Theorem (2 B).

THEOREM 1. *Let B_τ be a block of G, and let H be a normal subgroup of G. Then there exists a group S, $H \subseteq S \subseteq G$, and a block B'_τ of S, such that:*

(*a*) *the irreducible characters, both ordinary and modular, of B_τ are in* 1-1 *correspondence with those of B'_τ, the correspondence being obtained by induction from S to G;*

(*b*) *B_τ and B'_τ have the same decomposition matrix, and the same Cartan matrix;*

(*c*) *B_τ and B'_τ have a defect group in common;*

(*d*) *all the irreducible constituents of the restrictions to H of the characters of B'_τ lie in a single block $\tilde{B}$ of H.*

We can also show that B'_τ and B_τ stand in the relation defined by Brauer in section 2 of [4]. This is included in the following general result, which is independent of the assumptions of this section.

THEOREM 2. *Let B' be a block of a subgroup S of a group G, such that B' contains some character χ' such that $\chi = (\chi')^G$ is irreducible. Then $(B')^G$ is defined (in the notation of [4]), and $\chi \in (B')^G$.*

Proof. If ω' and ω are the characters of the class-algebras of S and G over Ω corresponding to χ' and χ respectively, it follows from the definitions that

$$\omega(K_\alpha) = \sum_{K'_\beta \subseteq K_\alpha} \omega'(K'_\beta),$$

where K_α and K'_β are arbitrary conjugate classes of G and S respectively. Taking this relation modulo $\mathfrak{p}$, we get the result.

2. Defect Group in Center

We begin our study of blocks with normal defect group by considering the case where the defect group is in the center of the group. (In this section, the field Ω in which all elements and representations lie is to contain the $(H:1)$-roots of unity.)

Assume that B is a p-block of a group H such that the defect group A of B is contained in the center of H. The normal p-subgroup A lies in the kernel of every irreducible modular representation $\mathfrak{F}_k$ of H (by (9 D) of [3]), so that these can be considered as modular representations of H/A. They are partitioned into p-blocks in the same way for H as for H/A, as can be seen by making use of (2 G) of reference [4]. Now B contains some $\mathfrak{F}$ such that $\nu(\deg \mathfrak{F}) = \nu(H) - \nu(A)$ where $\deg \mathfrak{F}$ is the degree of $\mathfrak{F}$, by (6 A) of [3]; taken on H/A, $\mathfrak{F}$ belongs to a block of defect 0, by (2 G) of [4]. By the properties of such blocks, this is the only modular representation in B; and there is an ordinary representation $\mathfrak{Z}$ in B such that $\mathfrak{Z}^* = \mathfrak{F}$. Here we write $\mathfrak{Z}$ with matrix coefficients which are locally integral at $\mathfrak{p}$, and $\mathfrak{Z}^*$ is obtained by reducing these coefficients modulo $\mathfrak{p}$. Denote the characters of $\mathfrak{Z}$ and $\mathfrak{F}$, on H, by ζ and ϕ respectively.

Since ζ belongs to a block of defect 0 of H/A, it vanishes for all p-singular elements of H/A. In terms of H, this means that

$$\zeta(x)=0 \qquad \text{if } x\in H,\ x_p\notin A, \tag{4}$$

where we write the factorization of x into a p-element x_p and a p-regular element x_r as $x=x_p x_r=x_r x_p$.

For the character ζ_j of any representation $\mathfrak{Z}_j$ in B (taking $\zeta=\zeta_1$), we have

$$\zeta_j(x)=d_j\phi(x)=d_j\zeta(x) \qquad \text{if } x \text{ is } p\text{-regular}, \tag{5}$$

since ϕ is the only modular character in B; the positive integers d_j are the decomposition numbers of B. For x such that $x_p\in A$, $\mathfrak{Z}_j(x)=\mathfrak{Z}_j(x_p)\mathfrak{Z}_j(x_r)$ $=\omega_j(x_p)\mathfrak{Z}_j(x_r)$, where ω_j is a character of the abelian group A, depending on $\mathfrak{Z}_j$. By (5),

$$\zeta_j(x)=\omega_j(x_p)\zeta_j(x_r)=d_j\omega_j(x_p)\zeta(x_r)=d_j\omega_j(x_p)\zeta(x)$$

whenever $x_p\in A$.

Now apply the orthogonality relations on H, setting $(H:1)=h$:

$$1=h^{-1}\sum_{x\in H}|\zeta_j(x)|^2=h^{-1}\sum_{x:\,x_p\in A}|d_j\omega_j(x_p)\zeta(x)|^2+h^{-1}\sum_{x:\,x_p\notin A}|\zeta_j(x)|^2.$$

If we denote the two terms on the right by s_{j1} and s_{j2}, (4) implies that $0<s_{j1}$ $=d_j^2h^{-1}\sum_x|\zeta(x)|^2=d_j^2=1-s_{j2}$, hence $d_j=1$, $s_{j2}=0$. That is,

$$\zeta_j(x)=\begin{cases}\omega_j(x_p)\zeta(x) & \text{if } x_p\in A,\\ 0 & \text{if } x_p\notin A.\end{cases} \tag{6}$$

If we denote all the characters of A by ω_j, $1\le j\le (A:1)$, (6) shows that each ω_j is associated with at most one character of B. Let J consist of those indices j such that the function ζ_j on H defined by (6) is actually a character of B. Then the principal indecomposable character Φ corresponding to ϕ is given by $\Phi=\sum_{j\in J}d_j\zeta_j=\sum_{j\in J}\zeta_j$, whence $\Phi|A=\zeta(1)\sum_{j\in J}\omega_j$ (see section 3 of [3]). But Φ vanishes for all p-singular elements, so that $\Phi|A$ is a multiple of the regular representation of A. Comparing these expressions, we see that all indices from 1 to $(A:1)$ are in J. This completes the proof of the following theorem.

THEOREM 3. *Let B be a block of a finite group H, whose defect group A is contained in the center of H. Then B contains just one modular character and $(A:1)$ ordinary characters, all of the same degree. Each of the ordinary characters is associated with a character of A by the equations* (6).

Since $\zeta_j(x)=\phi(x)$ for p-regular x, we can write $\mathfrak{Z}_j$ with locally integral coefficients in such a way that $\mathfrak{Z}_j^*=\mathfrak{Z}^*=\mathfrak{F}$.

We shall need the following form of Theorem 3 for projective representations. (See [1], [11], [12] for background on projective representations.)

THEOREM 4. *Let A_0 be a p-subgroup of the center of a finite group X, and let ε be a factor set of X/A_0 whose values are roots of unity of orders prime to p. Let $\mathfrak{Y}$ be an irreducible projective representation of X/A_0 with factor set ε, written with locally integral coefficients, such that $\nu(\deg Y)=\nu(X:A_0)$. Then for each linear character ω_{0j} of A_0, there exists a projective representation $\mathfrak{Y}_j$ of X for ε, with locally integral coefficients, such that $\mathfrak{Y}^*=\mathfrak{Y}_j^*$ and $\mathfrak{Y}_j|A_0$ is a multiple of ω_{0j}.*

Observe that we regard $\mathfrak{Y}$ as a representation of X, and ε as a factor set of X by inflation, without special mention. For the trivial factor set this theorem becomes a restatement of Theorem 3 in a form which avoids mentioning blocks. We can assume that $\varepsilon_{x,y}=1$ whenever either x or y lies in A_0.

Proof. We reduce this to Theorem 3 by the classical method of Schur (see [1], and also page 274 of [8]). Let E be the character group of the multiplicative cyclic group generated by the factor set ε; E is cyclic of order prime to p. For any elements x, y of X, let $m_{x,y}\in E$ be the character such that $\varepsilon^i\to(\varepsilon_{x,y})^i$; then $m=\{m_{x,y}\}$ is a factor set of X with values in E. Using extension theory (see [13]), let H be the extension of E by X with this factor set and trivial action; the elements of H may be written (e, x), where $e\in E$, $x\in X$. The elements $(1, a)$, $a\in A_0$, form a subgroup A of the center of H (since ε is trivial on A_0), isomorphic to A_0. A character ω_j of A is defined by $\omega_j(1,a)=\omega_{0j}(a)$; and the equation $\mathfrak{Z}(e,x)=e(\varepsilon)\mathfrak{Y}(x)$ defines an ordinary representation $\mathfrak{Z}$ of H, which can be regarded as belonging to a block of H/A of defect 0. Now Theorem 3 provides us with a representation $\mathfrak{Z}_j$ of H such that $\mathfrak{Z}_j^*=\mathfrak{Z}^*$ and $\mathfrak{Z}_j|A$ is a multiple of ω_j. The required $\mathfrak{Y}_j$ is then defined by $\mathfrak{Z}_j(e,x)=e(\varepsilon)\mathfrak{Y}_j(x)$.

3. Normal Defect Group: Analysis

In this section, we shall analyze the representations in a block whose defect group is normal. The information so obtained will be the basis for the constructions of the following section.

Let D be a normal p-subgroup of G. Denote the centralizer of D by H, and the center of D by A; then $A = D \cap H$, and H and A are normal in G. By (11 B) of [3], there is a 1-1 correspondence between the blocks B of G with defect group D and the families of G-associates of these characters ζ in blocks of defect 0 of H/A such that $p \nmid (S : DH)$, where S/A is the inertial group of ζ in G/A. (This restatement uses the isomorphism of DH/D with H/A.) In addition, D must be the maximal normal p-subgroup of G in order for any blocks with normal defect group D to exist, since this subgroup is contained in the defect group of every block of G.

Henceforth, we assume that we have a block B with normal defect group D; we study it along with a corresponding ζ and S, as just defined. We can regard ζ as a character of H. Since A is contained in the center of H, (2 G) of [4] tells us that ζ belongs to a block $\tilde{B}$ of H with defect group A, and Theorem 3 shows that the characters of $\tilde{B}$ have the form ζ_l, one for each character ω_l of A. Here $\zeta = \zeta_1$ corresponds to the 1-character of A.

Choose a maximal set $\{\omega_m\}$ of non-G-associate characters of A. Then by (6), $\{\zeta_m\}$ is a maximal set of non-G-associate characters of $\tilde{B}$. Let S_m be the inertial group of ζ_m in G, as in section 1. Then S_m is the intersection of $S = S_1$ with the inertial group of ω_m in G. Since ζ yields the only modular character of $\tilde{B}$, S is the inertial group of $\tilde{B}$ in G, defined in (3).

In the terminology of section 1, $B \in \mathscr{T}$ where $\tilde{B} \in \tilde{\mathscr{T}}$; this follows from equation (11. 11) of [3], and it enables us to study the corresponding block $B' \in \mathscr{T}'$ instead of B. B is actually the only block in $\mathscr{T}$, by the following argument. For any $B_\tau \in \mathscr{T}$, let χ'_j be an arbitrary character of B'_τ. Since $\chi'_j | H$ contains some ζ_m, $\zeta(1) = \zeta_m(1) | \chi'_j(1)$ by [7], whence $\nu(\chi'_j(1)) \geq \nu(\zeta(1)) = \nu(DH : D) = \nu(S : D)$, so that the defect of B'_τ does not exceed $\nu(D : 1)$. But since D is contained in all defect groups for blocks of S, D is the defect group for B'_τ and hence for B_τ also. But we have previously accounted for all blocks with defect group D, and none of them except B is in $\mathscr{T}$; therefore

$$\mathscr{T} = \{B\}. \tag{7}$$

Accordingly, $\mathscr{T}' = \{B'\}$, $B = \cup F_m$, and $B' = \cup F'_m$.

Since S/H has normal Sylow subgroup DH/H, there exists a subgroup K of S with $S = (DH)K$, $H = DH \cap K$, by a theorem of Schur (see [13], p. 132). Then $S = DK$, and $D \cap K = D \cap DH \cap K = D \cap H = A$.

Let us regard S as an extension of the abelian group A by $\overline{S} = S/A$. For each $\sigma \in \overline{S}$, choose a representative $s_\sigma \in S$. Since $\overline{S} = \overline{D}\overline{K}$, $1 = \overline{D} \cap \overline{K}$ where $\overline{D} = D/A$, $\overline{K} = K/A$, we can suppose that

$$s_{\delta\kappa} = s_\delta s_\kappa \tag{8}$$

when $\delta \in \overline{D}$, $\kappa \in \overline{K}$, and that $s_1 = 1$. For any $a \in A$ and $\sigma \in \overline{S}$ set

$$a^\sigma = a^{s_\sigma}. \tag{9}$$

Then for any σ, $\tau \in \overline{S}$,

$$s_\sigma s_\tau = r_{\sigma,\tau} s_{\sigma\tau}, \tag{10}$$

where $r = \{r_{\sigma,\tau}\}$ is a factor set of $\overline{S}$ with values in A: that is,

$$r_{\sigma,\tau} r_{\sigma\tau,\upsilon} = r_{\tau,\upsilon}^{\sigma^{-1}} r_{\sigma,\tau\upsilon} \qquad \text{for } \sigma,\ \tau,\ \upsilon \in \overline{S}. \tag{11}$$

For σ, $\tau \in \overline{S}_m = S_m/A$, set $(\rho_m)_{\sigma,\tau} = \omega_m(r_{\sigma,\tau})$. Since S_m is contained in the inertial group of ω_m, (11) yields $(\rho_m)_{\sigma,\tau}(\rho_m)_{\sigma\tau,\upsilon} = (\rho_m)_{\tau,\upsilon}(\rho_m)_{\sigma,\tau\upsilon}$ for σ, τ, $\upsilon \in \overline{S}_m$. That is, $\rho_m = \{(\rho_m)_{\sigma,\tau}\}$ is a factor set of $\overline{S}_m$ with trivial action, the values being roots of unity in Ω whose orders are powers of p. For $a \in A$, $\sigma \in \overline{S}_m$, set $\psi_m(as_\sigma) = \omega_m(a)$. Then $\psi_m(as_\sigma bs_\tau) = \psi_m(ab^{\sigma^{-1}} r_{\sigma,\tau} s_{\sigma\tau}) = \omega_m(ab^{\sigma^{-1}} r_{\sigma,\tau}) = \omega_m(a)\,\omega_m(b)\,\omega_m(r_{\sigma,\tau}) = (\rho_m)_{\sigma,\tau}\psi_m(as_\sigma)\,\psi_m(bs_\tau)$, so that ψ_m is a projective representation of $\overline{S}_m$ of degree 1, with factor set ρ_m^{-1}. Here ρ_m is regarded as a factor set of S_m, by inflation; as usual, we do not indicate this inflation explicitly.

From now on, we shall work in terms of representations rather than the corresponding characters. By the remark following Theorem 5, we suppose that $\mathfrak{Z}_m^* = \mathfrak{Z}^*$, where $\mathfrak{Z}_m$ is the representation of H corresponding to ζ_m.

By section 3 of [7] (cf. [10]), there must exist[1] a projective representation of S_m whose restriction to H is $\mathfrak{Z}_m$, and whose factor set, which we shall call ε_m^{-1}, is inflated from S_m/H. In order to compute ε_m in terms of $\varepsilon = \varepsilon_1$ (see (14)), we first restrict attention to K. We can assume that the values taken on by $\varepsilon | K$ are roots of unity in Ω, of orders prime to p (see [1]).

[1] At this point, and later on, we may have to replace Ω by a finite extension of Ω, since [7] uses an algebraically closed field. But in fact these extensions can be avoided, as the addendum at the end of this paper shows.

Let $\mathfrak{Y}$ be a projective representation of K, whose factor set is the restriction $\varepsilon^{-1}|K$ of ε^{-1} to K, such that $\mathfrak{Y}|H = \mathfrak{Z}$, and such that $\mathfrak{Y}$ has locally integral coefficients. Let $J_m \subseteq A$ be the kernel of ω_m, and let $K_m = S_m \cap K$. Then all commutators $a^{-1}s^{-1}as$, $a \in A$, $s \in K_m$, are in J_m, so that A/J_m is contained in the center of K_m/J_m. Since $\nu(\deg \mathfrak{Y}) = \nu(K_m : A)$, we can apply Theorem 4 to $\mathfrak{Y}|K_m$, regarded as a projective representation of $X = K_m/J_m$, to find that K_m has a projective representation $\mathfrak{Y}_m$ with factor set $\varepsilon^{-1}|K_m$, such that $\mathfrak{Y}_m|A$ is a multiple of ω_m, and such that $\mathfrak{Y}_m^* = \mathfrak{Y}^*|K_m$. Then $\mathfrak{Y}_m|H$ is equivalent to $\mathfrak{Z}_m$, and we may suppose that $\mathfrak{Y}_m|H = \mathfrak{Z}_m$.

We extend $\mathfrak{Y}_m$ to S_m by the following steps. First, write $\mathfrak{Y}_m = (\psi_m|K_m)\bar{\mathfrak{Y}}_m$, where $\bar{\mathfrak{Y}}_m$ is a projective representation of K_m with factor set $(\varepsilon^{-1}|K_m)(\rho_m|K_m)$ which can also be regarded as a projective representation of $\bar{K}_m = K_m/A$. This can be seen by section 8 of [10], or directly. (For finite groups, the topological assumptions of [10] are vacuously satisfied, and the restriction to unitary representations is unnecessary.) Secondly, since $\bar{S}_m = \bar{D}\bar{K}_m$, $1 = \bar{D} \cap \bar{K}_m$, we have a natural isomorphism of $\bar{S}_m/\bar{D}$ with $\bar{K}_m$; use this to carry over $\bar{\mathfrak{Y}}_m$ to $\bar{S}_m/\bar{D}$. Thirdly, inflate to $\bar{S}_m$; this yields an extension of the original $\bar{\mathfrak{Y}}_m$, for which we retain the same symbol. Finally, set $\mathfrak{Y}_m = \psi_m\bar{\mathfrak{Y}}_m$ on S_m. This gives a projective representation $\mathfrak{Y}_m$ of S_m, related to its restriction on K_m by $\mathfrak{Y}_m(as_{\delta\kappa}) = \mathfrak{Y}_m(as_\kappa)$ where $a \in A$, $\delta \in \bar{D}$, $\kappa \in \bar{K}_m$. Modularly, $\mathfrak{Y}_m^* = \mathfrak{Y}^*|S_m$.

The factor set of $\mathfrak{Y}_m$ is readily seen to be $(\varepsilon^{-1}|K_m)\mu_m^{-1}$, where μ_m can be defined as a factor set on $\bar{S}_m$ by

$$(\mu_m)_{\delta\kappa,\gamma\lambda} = (\rho_m)_{\delta\kappa,\gamma\lambda}(\rho_m)_{\kappa,\lambda}^{-1} = \omega_m(r_{\delta\kappa,\gamma\lambda}r_{\kappa,\lambda}^{-1})$$

for $\delta, \gamma \in \bar{D}$, $\kappa, \lambda \in \bar{K}_m$. Now for any $\delta, \gamma \in \bar{D}$, $\kappa, \lambda \in \bar{K}$, we have by (8) and (10) $r_{\delta\kappa,\gamma\lambda} = s_{\delta\kappa}s_{\gamma\lambda}s_{\delta\gamma_1\kappa\lambda}^{-1} = s_\delta s_\kappa s_\gamma s_\lambda s_{\kappa\lambda}^{-1}s_{\delta\gamma_1}^{-1}$, where we set $\gamma_1 = \gamma^{\kappa^{-1}}$. Since $r_{\kappa,\lambda} \in A$ and $s_{\delta\gamma_1} \in D$,

$$\begin{aligned} q_{\delta\kappa,\gamma\lambda} = r_{\delta\kappa,\gamma\lambda}r_{\kappa,\lambda}^{-1} &= s_\delta s_\kappa s_\gamma s_\lambda s_{\kappa\lambda}^{-1} r_{\kappa,\lambda}^{-1} s_{\delta\gamma_1}^{-1} \\ &= s_\delta(s_\kappa s_\gamma s_\kappa^{-1})(s_\kappa s_\lambda s_{\kappa\lambda}^{-1} r_{\kappa,\lambda}^{-1})s_{\delta\gamma_1}^{-1} = s_\delta(s_\gamma)^{s_\kappa^{-1}} s_{\delta\gamma_1}^{-1}. \end{aligned}$$

Since D commutes elementwise with H, the last expression shows that $q_{\delta\kappa,\gamma\lambda}$ does not change if we multiply κ and λ by elements of $\bar{H} = H/A$. Therefore $q = \{q_{\sigma,\tau}\}$, which is a factor set of $\bar{S}$ with values in A under the same action as for r (since $\bar{D}$ acts trivially on A), is inflated from a factor set of $\bar{S}/\bar{H}$ (or S/H), under the action of $\bar{S}/\bar{H}$ on A defined by

$$a^{\sigma\overline{H}} = a^{\sigma} \tag{12}$$

(cf. (9)). Since

$$(\mu_m)_{\sigma,\tau} = \omega_m(q_{\sigma,\tau}), \qquad \sigma, \tau \in \overline{S}_m, \tag{13}$$

μ_m is inflated from a factor set of S_m/H with values in Ω and trivial action. Therefore

$$\varepsilon_m = (\varepsilon \mid S_m)\mu_m \tag{14}$$

is inflated from S_m/H, and $\mathfrak{Y}_m$ is an extension of $\mathfrak{Z}_m$, with factor set ε_m^{-1}.

Theorem 3 of [7] sets up a 1-1 correspondence between the representations $\mathfrak{X}_j'$ corresponding to all the $\chi_j' \in F_m'$ (that is, all representations of S whose restrictions to H contain $\mathfrak{Z}_m$) and all projective representations $\mathfrak{A}_j$ of S_m/H with factor set ε_m, such that

$$\mathfrak{X}_j' = (\mathfrak{Y}_m \times \mathfrak{A}_j)^S. \tag{15}$$

Here we treat $\mathfrak{A}_j$ as a projective representation of S_m, so that the tensor product $\mathfrak{Y}_m \times \mathfrak{A}_j$ is an ordinary representation of S_m, which is induced to S to yield $\mathfrak{X}_j'$. We can suppose that $\mathfrak{A}_j$ has locally integral coefficients. By Theorem 1, we also have $\mathfrak{X}_j = (\mathfrak{Y}_m \times \mathfrak{A}_j)^G$, for $\mathfrak{X}_j \in F_m$.

Now consider the situation in Ω^*. $\mathfrak{A}_j^*$ is a modular projective representation of S_m/H, whose factor set ε_m^* is obtained by reducing ε_m modulo $\mathfrak{p}$. By (14), $\varepsilon_m^* = \varepsilon^* \mid S_m$, since the values of μ_m are roots of unity of p-power orders. By (15),

$$(\mathfrak{X}_j')^* = (\mathfrak{Y}_m^* \times \mathfrak{A}_j^*)^S = ((\mathfrak{Y}^* \mid S_m) \times \mathfrak{A}_j^*)^S \sim \mathfrak{Y}^* \times (\mathfrak{A}_j^*)^S,$$

where the meaning of and reason for the last relation are as follows. $(\mathfrak{A}_j^*)^S$ is the modular projective representation of S with factor set ε^* induced by $\mathfrak{A}_j^*$ (see [10], [11], [12] for definition). The symbol $\sim$ indicates that the modular representations of S which it joins have the same irreducible constituents and multiplicities. That this is true is basically a consequence of the Frobenius reciprocity theorem and the orthogonality relations; a proof can be constructed in the following way. Express everything in terms of modular projective *characters*, with the factor sets ε and $\varepsilon \mid S_m$, whose values are roots of unity in Ω of orders prime to p. Extend each such character by defining its values for p-singular elements to be zero, and express it as a rational linear combination of irreducible projective characters in Ω with the same factor set. By linearity, it now suffices to prove the corresponding statement for irreducible projective characters in Ω. But this follows from Theorem 4. 6 of [10]. (Cf. [11], and equation (4) of [6].)

The irreducible modular representations $\mathfrak{F}'_k$ which are in B' are precisely those such that $\mathfrak{F}'_k|H$ contains $\mathfrak{Z}^*$. Since the inertial group of the character of $\mathfrak{Z}^*$ is S and since Clifford's results still hold in the modular case, we have a 1-1 correspondence between these $\mathfrak{F}'_k$ and all the irreducible modular projective representations $\mathfrak{B}_k$ of S/H with factor set ε^* (or, as we may say, the ε^*-representations of S/H) in which $\mathfrak{F}'_k = \mathfrak{Y}^* \times \mathfrak{B}_k$. In B, the corresponding equation is $\mathfrak{F}_k = (\mathfrak{Y}^* \times \mathfrak{B}_k)^G$.

Since $\mathfrak{A}_j^*$ can be regarded as a projective representation of S_m/H, we can write formally

$$(\mathfrak{A}_j^*)^S \sim \sum_k d_{jk} \mathfrak{B}_k \tag{16}$$

for some non-negative integers d_{jk}. Combining our results, we find that

$$(\mathfrak{X}'_j)^* \sim \mathfrak{Y}^* \times \sum_k d_{jk} \mathfrak{B}_k \sim \sum_k d_{jk}(\mathfrak{Y}^* \times \mathfrak{B}_k) = \sum_k d_{jk} \mathfrak{F}'_k.$$

Using all values of m, this shows that the numbers d_{jk} are precisely the decomposition numbers for $B' = \cup F'_m$, and hence also for B, in agreement with the notation of section 1.

4. Normal Defect Group: Construction

In the previous section, we defined a factor set q of $\overline{S}/\overline{H}$ with values in A, under the action defined by (12). Let U be the extension of A by $\overline{S}/\overline{H}$ defined by this action and factor set. We may write the elements of U as ordered pairs $u = (a, \sigma\overline{H})$, where $a \in A$ and $\sigma \in \overline{S}$. Denote the subgroup $\{(a, \overline{H})\}$ by A^0; and let i be the natural isomorphism of $\overline{S}/\overline{H}$ onto U/A^0. The elements $(a, \delta\overline{H})$, $\delta \in \overline{D}$, form a subgroup D^0 of U isomorphic to D under the mapping $(a, \delta\overline{H}) \to as_\delta$, since $q|D = s|D$. Since $i(\overline{DH}/\overline{H}) = D^0/A^0$, we have the isomorphisms $U/D^0 \cong \overline{S}/\overline{DH} \cong S/DH$, so that D^0 is a normal p-Sylow subgroup of U. A p-complement in U is then given by $L^0 = \{(1, \kappa\overline{H}) : \kappa \in \overline{K}\} \cong K/H$: thus $U = D^0L^0$, $1 = D^0 \cap L^0$. The action of L^0 on D^0 by conjugation can be computed straightforwardly; it corresponds in the natural way to the action of K/H on D in S. Hence it follows that D^0 contains its own centralizer in U, namely A^0. Thus the structure of U is analogous to that of S, but is more definite; the analogues of H and K are A^0 and $K^0 = A^0L^0$ respectively.

The factor set ε, regarded as a factor set of $\overline{S}/\overline{H}$, can be carried over by i to a factor set ε^0 of U/A^0, which can then be inflated to U. In the same way, we can use ε_m and μ_m to define factor sets ε_m^0 and μ_m^0 of S_m^0, where $i(\overline{S}_m/\overline{H}) = S_m^0/A^0$ (cf. (13)). The equation $\psi_m^0(a, \sigma\overline{H}) = \omega_m(a)$ defines a projective representation of S_m^0 with factor set $(\mu_m^0)^{-1}$. Then $\psi_m^0 | A^0 = \omega_m^0$ corresponds to ω_m under the natural isomorphism of A and A^0.

We now study the ε^0-representations $\mathfrak{X}_j^0$ of U in analogy with the study of the representations of B in section 3. Since $\varepsilon^0 | A^0$ is trivial, the $\mathfrak{X}_j^0$ can be distributed among disjoint sets F_m^0 according to which ω_m^0 is contained in $\mathfrak{X}_j^0$; note that there is no question of blocks in this definition. The inertial group of ω_m^0 in U is S_m^0, even with respect to ε^0 in the sense of Mackey ([10], Theorem 8.1). Then Theorem 8.4 of [10] gives a 1-1 correspondence between the $\mathfrak{X}_j^0 \in F_m^0$ and all the representations $\mathfrak{A}_j^0$ of S_m^0/A^0 for $\varepsilon_m^0 = (\varepsilon^0 | S_m^0)\mu_m^0$, in which

$$\mathfrak{X}_j^0 = (\psi_m^0 \mathfrak{A}_j^0)^U, \tag{17}$$

the induction being with respect to ε^0. We can suppose that $\mathfrak{A}_j^0$ has been obtained from the $\mathfrak{A}_j$ of (15) by means of i. Thus we have a 1-1 correspondence $\mathfrak{X}_j \leftrightarrow \mathfrak{X}_j^0$ between F_m and F_m^0, hence between the ordinary representations of B and all the ε^0-representations of S.

The $(\varepsilon^0)^*$-representations $\mathfrak{F}_k^0$ of U all have kernels containing A^0, as we see by generalizing (9 D) of [3] to the projective case—cf. the construction of M below. Thus we can suppose that $\mathfrak{F}_k^0$, as an $(\varepsilon^0)^*$-representation of U/A^0, corresponds to $\mathfrak{B}_k$ under i. This gives a 1-1 correspondence $\mathfrak{F}_k \leftrightarrow \mathfrak{F}_k^0$.

Applying i to (16) yields $((\mathfrak{A}_j^0)^*)^U \sim \sum_k d_{jk}\mathfrak{F}_k^0$. Since the values of ψ_m^0 are all p-power roots of unity, $(\psi_m^0)^* = 1^*$, so that $(\mathfrak{X}_j^0)^* = ((\mathfrak{A}_j^0)^*)^U \sim \sum_k d_{jk}\mathfrak{F}_k^0$, where the d_{jk} are again the decomposition numbers of B. We have proved the following theorem.

THEOREM 5. *Let B be a block of a group G, with normal defect group D. Then there exists a group U and a factor set ε^0 of U, whose values are roots of unity of orders prime to p, such that:*

(*a*) *the ordinary representations of B are in* 1-1 *correspondence with the*

ε^0-representations of U, the modular representations of B are in 1-1 *correspondence with the $(\varepsilon^0)^*$-representations of U, and corresponding representations have proportional degrees and equal heights;*

(*b*) *the decomposition matrix and Cartan matrix for B are the same as those for the ε^0-representations of U;*

(*c*) *U has a normal Sylow subgroup D^0 isomorphic to D, and D^0 contains its own centralizer;*

(*d*) *$U/D^0 \cong S/DH$, where S and H were defined in section* 3.

This theorem can be interpreted as meaning that just as the part of G outside S can be removed without changing the structure of B, so the "part between A and H" can be removed if we allow the introduction of a factor set.

We now reformulate Theorem 5 so as to eliminate the use of projective representations, just as in the proof of Theorem 4. Let E be the character group of the cyclic group generated by ε^0. For any elements u, v of U, let $n_{u,v} \in E$ be the character such that $(\varepsilon^0)^i \to (\varepsilon^0_{u,v})^i$; then $n = \{n_{u,v}\}$ is a factor set of U in E. Let M be the extension of E by U with this factor set and trivial action; the elements of M may be written $(e, u) = (e, (a, \sigma\overline{H}))$, where $e \in E$. E is in the center of U, and is cyclic of order prime to p. By defining $\mathfrak{X}_j''(e, u) = e(\varepsilon^0)\mathfrak{X}_j^0(u)$, we set up a 1-1 correspondence between the ε^0-representations $\mathfrak{X}_j^0$ of U and a certain set T'' of ordinary irreducible representations $\mathfrak{X}_j''$ of M; and $\mathfrak{X}_j''$ is in T'' if and only if the linear character of E contained in $\mathfrak{X}_j''|E$ can be identified with ε^0 under the canonical isomorphism between an abelian group and the character group of its character group. (For consistent notation, T'' should really be defined as a set of characters.) There is a similar correspondence for modular representations. Together these correspondences preserve decomposition numbers; so T'' is actually a block B'' of M.

To study the structure of M, let π be the natural homomorphism of M onto U. Let $H'' = \pi^{-1}(A^0)$. Since $n|D$ is trivial, $\pi^{-1}(D^0)$ is a direct product $D'' \times E = D''H''$, where D'' is a normal Sylow subgroup of M. $D'' \cap H''$ is the center A'' of D'', and $\pi^{-1}(L^0) = L''$ is a p-complement; $\pi^{-1}(K^0) = K'' = A''L''$. Since B'', like every block of M, has the normal Sylow group D'' as its defect group, the theory of section 3 applies to B''. We need not go into detail, but simply remark that M plays the role of S as well as of G, that our notation for subgroups indicates their roles, and that the group corresponding to U is U itself (up to isomorphism). This completes the proof of the following reduction theorem for the structure of blocks with normal defect group.

THEOREM 6. *Let B be a block of a group G, with normal defect group D. Then there exists a group M and a block B'' of M such that:*

(*a*) *the representations, both ordinary and modular, of B are in a* 1-1 *correspondence with those of B'', and corresponding representations have proportional degrees and equal heights;*

(*b*) *the decomposition matrix and Cartan matrix for B are the same as for B'';*

(*c*) *D is isomorphic to the defect group D'' of B'', D'' is a normal Sylow subgroup of M, and the centralizer of D'' has form $H'' = A'' \times E$, where A'' is the center of D'' and E is a cyclic subgroup of the center of M;*

(*d*) $M/D''H'' \cong S/DH$.

5. Applications of the Construction

The following theorem treats a situation midway between that of section 2 and that of sections 3 and 4.

THEOREM 7. *Let G be a group of form DH, where D is the defect group of a block of G, and H is the centralizer of D in G. Then there is a* 1-1 *correspondence between the representations $\mathfrak{B}_l$ of B and all the representations $\mathfrak{B}_l^0$ of D, in which corresponding representations have proportional degrees. B contains just one modular representation.*

Proof. D is normal in G, so that we can use the terminology of sections 3 and 4. Here $S = DH$, $K = H$, and ε is trivial. Therefore $U = D^0 \cong D$, and we can identify U with D. Theorem 5 then gives the result.

Explicitly, a short computation based on equations (15) and (17) shows that $\mathfrak{B}_l(as_\delta s_\kappa) = \mathfrak{Z}_m(s_\kappa) \times \mathfrak{B}_l^0(as_\delta)$ in our usual notations. From this it is not hard to show that for $x \in DH$ we have, as a generalization of (6),

$$\theta_l(x) = \begin{cases} \theta_l^0(x_p)\zeta(x) & \text{if } x_p \in D, \\ 0 & \text{if } x_p \notin D, \end{cases} \tag{18}$$

where θ_l and θ_l^0 are the characters of $\mathfrak{B}_l$ and $\mathfrak{B}_0^l$ respectively. Theorem 7 is less deep than Theorems 5 and 6, since the proofs of these collapse considerably in the case $G=DH$. An alternate proof of Theorem 7 can be constructed along the lines of the proof of Theorem 3, considering DH as a homomorphic image of the abstract direct product of the groups D and H. We now return to the general case.

THEOREM 8. *Let B be a block of G with normal defect group D. Then the height of each representation $\mathfrak{X}_j$ in B is equal to $\nu(\deg \mathfrak{V}_l^0)$, where $\mathfrak{V}_l^0$ is any irreducible constituent of $\mathfrak{X}_j \mid D$.*

Proof. If $\mathfrak{X}_j=(\mathfrak{X}_j')^G$, if $\mathfrak{V}_l$ is a constituent of $\mathfrak{X}_j' \mid DH$, and if R_l is the inertial group in S of the character of $\mathfrak{V}_l$, then by [7] $\mathfrak{X}_j'$ is induced from a representation of R_l which is a tensor product of two projective representations, one of them being an extension of $\mathfrak{V}_l$, and the other being a projective representation of R_l/DH. A theorem of Schur (see [9]) tells us that the degree of the last divides $(R_l : DH)$, whence $\nu(\deg \mathfrak{X}_j')=\nu(\deg \mathfrak{V}_l)$. Since $\mathfrak{X}_j$ and $\mathfrak{X}_j'$ have the same height, the result follows from Theorem 7.

As an immediate corollary, we have the following result.

THEOREM 9. *If B has normal defect group D, then B contains representations of positive height if and only if D is non-abelian.*

THEOREM 10. *If the block B of G has normal defect group D, the representations of G/D in B are all modularly irreducible and of height* 0. *Taken modularly, they all remain distinct, and they yield all the irreducible modular representations of B.*

Proof. We can reduce at once to the situation in S. Every irreducible modular representation $\mathfrak{F}_k'$ in B' has D in its kernel, and hence is a modular constituent of some $\mathfrak{X}_j'$ in B' whose kernel contains D. Every such representation $\mathfrak{X}_j'$ has height 0 by Theorem 8; this means that $\nu(\deg \mathfrak{X}_j')=\nu(S:D)$, so that $\mathfrak{X}_j'$ is in a block of S/D of defect 0. Then all such $\mathfrak{X}_j'$ are modularly irreducible, and yield distinct modular representations, as required.

The above theorems actually depend only on Theorems 1 and 7. We conclude with a result whose proof uses Theorem 6 more fully.

THEOREM 11. *If B has normal defect group D, then each Cartan invariant c_{kl} of B satisfies $c_{kl} \leq (D:1)$.*

Proof. By Theorem 6, we need only prove this for the group M. But M contains the p-complement L'', and therefore the result is given by Theorems 8 and 9 of [5].

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Addendum: It is possible to carry out the constructions of sections 3 and 4 in any finite algebraic number field Ω containing the g-th roots of unity without enlarging this field. The proof of this depends upon the following theorem, whose proof I shall publish elsewhere.

THEOREM. *Let H be a normal subgroup of a group G of finite order g, and let ε be a complex-valued factor set on G/H. Then ε is equivalent to a factor set ε' on G/H whose values are g-th roots of unity, and such that for each group G_1, $H \subseteq G_1 \subseteq G$, every $(\varepsilon' | G_1)$-representation of G_1 can be written in the field of the g-th roots of unity. If furthermore p is a prime which does not* *divide* $(G : H)$, *and if* $g = p^a g_r$ *with* $(p, g_r) = 1$, *then we can suppose that the values of* ε' *are* g_r*-th roots of unity.*

If we apply this theorem to K at the point in section 3 where the $\mathfrak{Y}_m | K_m$ have just been constructed in an extension field of Ω, we can replace ε by ε' and then write $\mathfrak{Y}_m | K_m$ in Ω. Note however that this replacement may increase the multiplicative order of ε, and with it the order of M in section 4. Standard methods (see in particular p. 223 of N. Jacobson, Lectures on Abstract Algebra, vol. 2, New York, 1953) then let us take $\mathfrak{Y}_m | K_m$ locally integral in Ω and $\mathfrak{Y}_m^* | K_m = \mathfrak{Y}^* | K_m$. The construction of $\mathfrak{Y}_m$ on S_m then proceeds in Ω.

The representation $\mathfrak{Y}_m \times \mathfrak{A}_j$ in (15) is similar to a representation in Ω whose restriction to H is $\mathfrak{Z}_m \times I$; the latter representation has form $\mathfrak{Y}_m \times \mathfrak{A}'_j$ with $\mathfrak{A}'_j$ in Ω, and we can replace $\mathfrak{A}_j$ by $\mathfrak{A}'_j$ without any further change in the factor sets. A similar argument works for $\mathfrak{B}_k$. Then the constructions of sections 3 and 4 can be completed with all representations written in Ω and its residue class field Ω^*.

Tufts University